Babeş-Bolyai University Cluj-Napoca
Faculty of Mathematics and Computer Science

Summary of Doctoral Thesis

VARIATIONAL APPROACH TO STACKELBERG EQUILIBRIA

Scientific advisor
Prof. Dr. Alexandru Kristály

Ph.D. Student
Szilárd Nagy

Cluj-Napoca, 2015

Contents

Introduction

The Stackelberg competition model is a game in which the leader player moves first and then the follower player moves sequentially. In order to solve such a game, the so-called *backward induction method* is applied. The first step is to find the best strategy/response for the follower player by considering the strategy action of the leader player as a parameter; then, having in our mind this parameter-depending response, the choice of the best strategy of the leader player concludes the problem. Consequently, the leader in the Stackelberg model has an advantage while the follower has to react to the leader's action, otherwise the game reduces to the usual Nash competition model. In the usual Nash model the two players are competing at the same level, while in the Stackelberg model the players are subordinated to each other. For some comparison results, we refer the reader to the papers of R. Amir and I. Grilo [2], A.J. Novak, G. Feichtinger and G. Leitmann [53], and W. Stanford [63]. For instance, in [53] the authors show that the Stackelberg model describes efficiently the combat against terror activities (the terrorists being the leaders while officialities are the followers).

The purpose of the present thesis is to provide a variational approach to Stackelberg equilibria, focusing mainly on the behavior of the follower player by using various elements from Calculus of Variations. Since the follower's strategy is to *minimize his loss* (which depends on the strategy of the leader), we shall apply several results from the theory of variational inequalities together with deep results from nonsmooth analysis and Riemannian geometry. This approach is clearly motivated by:

- *Variational inequalities.* Since we are interested to find minimum points of certain payoff functions, it is natural to consider (nonsmooth) critical points and variational inequalities. Simple examples support this approach; indeed, if $f(x) = x$ for $x \in [0, 1]$, no critical points exist in the usual sense but the variational inequality $f'(x_0)(x - x_0) \geq 0$ for every $x \in [0,1]$ has the unique solution $x_0 = 0$ which is indeed a minimum point of f over $[0,1]$. The appropriate concept to handle these phenomena is provided by the *Motreanu-Panagiotopoulos-type functionals*, which is the perturbation of a locally Lipschitz function by a proper, convex and lower semicontinuous functional (e.g. the indicator function of a convex set).

- *Riemannian geometry.* There are certain cases when the strategy sets are not convex in the usual (Euclidean) sense but are geodesic convex with respect to some other metric. A special class of manifolds where we can elaborate Stackelberg and Nash equilibria is the so-called *Hadamard manifolds* (which are simply connected, complete Riemannian manifolds with non-positive sectional curvature).

The thesis is divided into four chapters.

In the first chapter we recall those theoretical results which are needed to elaborate the thesis itself, making this work as self-contained as possible. In particular, we recall Ekeland's variational principle, fixed point theorems, elements from set-valued and nonsmooth analysis, results from nonsmooth critical point theory and basic elements from Riemannian geometry.

In the second chapter we study the existence and location of Stackelberg equilibria for two players by using appropriate variational inequalities and fixed point arguments; here, the payoff functions need not be smooth. When the strategy sets are *compact*, we establish an existence result for the so-called *Stackelberg variational response* by using the set-valued Begle fixed point theorem. Here we also construct a payoff function for which Stackelberg equilibria exist but the set of Nash equilibria is empty. In the case when the strategy sets are *non-compact*, we prove a uniqueness result by using a discrete and a continuous projective dynamical system, both converging exponentially to the unique element of the Stackelberg variational response set. This chapter is based on the paper written by Sz. Nagy [49] (where the results were presented in the smooth setting).

The third chapter extends the results from the previous chapter to the case when the strategy sets are curved. More precisely, inspired by A. Kristály [36, 37], we assume that the strategy sets are geodesic convex subsets of certain finite-dimensional Riemannian manifolds. In order to achieve our aims, we consider the class of Hadamard manifolds which possesses two crucial properties: the non-expansiveness of the metric projection map and the so-called Moskovitz-Dines property (called also the obtuse-angle property) of the projection. We consider both *compact* and *non-compact* strategy sets in order to prove existence, location and uniqueness of the *Stackelberg variational response*. We notice that the presence of Hadamard manifolds plays a crucial role in our investigations not only from the analytical point of view (certain estimates are based on the Rauch comparison theorem and fine properties of the metric projections) but also from geometric point of view; indeed, we know after A. Kristály [37] that the aforementioned properties of the metric projection map characterize the Hadamard manifolds among complete, simply connected Riemannian manifolds. The results of this chapter are based on the paper by A. Kristály and Sz. Nagy [38].

The fourth chapter is devoted to the study of multiplicity of the Stackelberg variational responses. The existing literature usually provides results where the latter set is a singleton, i.e., the variational response of the follower is uniquely determined. However, as we can expect, there are situations where the uniqueness of the Stackelberg response fails. The objective of this chapter is to provide a whole class of payoff functions to show that the Stackelberg variational response set may contain at least three different elements. In fact, the payoff functions we are working on depend on a real parameter which plays a crucial role in the number of responses. In this way, once the parameter is small enough, we prove that the follower has only the null response (no reason for him to participate actively to the game), while for large values of the parameter, he has at least three variational possibilities to choose his strategy. To prove this result, we explore the nonsmooth critical point theory for Motreanu-Panagiotopoulos-type functionals (global minimization, nonsmooth Mountain Pass theorem, nonsmooth Palais-Smale condition). Some numerical examples are also provided to support the sharpness of our results. These results are based on the paper by Sz. Nagy [50].

The thesis is based on three papers as follows:

- A. Kristály, **Sz. Nagy**, Followers strategy in Stackelberg equilibrium problems on curved strategy sets, *Acta Polytech. Hung.* 10 (2013), no. 7, 69–80. ISI Journal (IF: 0.471)
- **Sz. Nagy**, Stackelberg equilibria via variational inequalities and projections. *J. Global Optim.* 57 (2013), no. 3, 821–828. ISI Journal (IF: 1.355)
- **Sz. Nagy**, Multiple Stackelberg variational responses, *Stud. Univ. Babeş-Bolyai Math.*, accepted, 2015.

We mention another paper which also contains original results that are not included into the body of the thesis in order to keep the unity of the presentation:

- Cs. Farkas, A.É. Molnár, **Sz. Nagy**, A generalized variational principle in $b-$metric spaces. *Matematiche* (Catania) 69 (2014), no. 2, 205–221.
  In this paper we establish a generalized variational principle for $b-$metric spaces. As a consequence, we obtain a weak Zhong-type variational principle in $b-$metric spaces and we show its applicability by presenting a Caristi-type fixed point theorem and an extension of the main result for bifunctions.

In the sequel, we list our original contributions within the thesis:

- Chapter 1: some remarks.
- Chapter 2: Theorems 2.2.1, 2.3.1, 2.3.2, 2.3.3; Propositions 2.2.1, 2.2.2; Remarks 2.2.1, 2.2.2, 2.3.1, 2.3.2; Examples 2.3.1, 2.3.2; Figures 2.1, 2.2.
- Chapter 3: Theorems 3.2.1, 3.3.1, 3.3.2, 3.3.3; Lemma 3.2.1; Remark 3.2.1; Examples 3.3.1, 3.3.2, 3.3.3.
- Chapter 4: Theorem 4.1.1; Lemma 4.3.1; Propositions 4.3.1, 4.4.1, 4.4.2; Remarks 4.1.1, 4.1.3, 4.2.1, 4.3.1, 4.4.1; Example 4.1.1; Figure 4.1.

Keywords: Stackelberg equilibrium point, Stackelberg variational response set, Nash equilibrium point, variational arguments, fixed points, Ekeland variational principle, critical points, nonsmooth Mountain Pass theorem, Palais-Smale condition, minimization, locally Lipschitz functionals, Motreanu-Panagiotopoulos-type functionals.

Acknowledgements

I would like to mention that completing the PhD and writing this thesis was an amazing journey for me, because there is a long and difficult road to take from the field of economics to the fields of applied mathematics and Riemannian geometry.

First, I would like to express my sincere gratitude to my supervisor Professor Alexandru Kristály who provided all the support and constantly built my confidence during my PhD studies. It has been a great honor to work beside him. I would also like to express my gratitude to Professor Csaba Varga and Árpád Baricz and thank them for all the useful comments, advice and encouragement during my studies in the fields of applied mathematics.

Second, I would like to extend my appreciation to the scientific committee members, Professor Adrian Petruşel, Professor Radu Precup and Professor Csaba Varga, for their advice and help to understand how to model economic processes by using mathematical tools.

Special thanks go to my friend Csaba Farkas who was always there when I needed some help using mathematical programs.

Finally, I would like to express my thanks and gratefulness to my family for their unique support and love, they stood by me not only in great moments of life but in difficult moments too. I know that I owe them much more than this and a lifetime will be not enough to thank them. Let me simply but profoundly tell them: Thank you for your belief in me, for your hope and love!

Chapter 1

Preliminaries

In order the thesis to be self-contained as much as possible, we recall in this chapter several results from nonsmooth analysis, variational inequalities, subdifferential calculus, fixed points and critical points for not necessarily smooth functionals, and Riemannian geometry. We mainly followed the works of H. Brezis [13], F.H. Clarke [20], M.P. do Carmo [22], D. Motreanu and P.D. Panagiotopoulos [48] and A. Kristály, V. Rădulescu and Cs. Varga [39]. In the sequel we shall summarize these results, keeping the sections from the thesis.

1.1 Variational principles

- Metric spaces (and $b-$metric spaces) with comments on Riemannian and Finsler manifolds.
- Lower semicontinuous and convex functions.
- Variational principles: Theorem of K.T.W. Weierstrass, Ekeland variational principle, Borwein-Preiss variational principle.

1.2 Subdifferentials and critical points

1.2.1 Subdifferential calculus

- Locally Lipschitz functions.
- Properties of the generalized directional derivative and generalized gradient.

- Regularity results.

1.2.2 Critical point theory for nonsmooth functionals

- Motreanu-Panagiotopoulos-type functionals. Nonsmooth Palais-Smale condition.
- Global minimization. Nonsmooth version of the Mountain Pass theorem.

1.3 Fixed points

- Brouwer, Begle, Banach and Caristi fixed point theorems.

1.4 Elements from Riemannian geometry

1.4.1 Geodesics, exponential map, and curvature

- Distance function. Geodesics. Levi-Civita connection.
- Exponential map.
- Differential of functions on manifolds.
- Curvature. The parallelogramoid of Levi-Civita.
- Theorem of Hopf-Rinow. Hadamard manifolds.
- Rauch comparison theorem.

1.4.2 Metric projections

- Definition of the metric projection.
- Non-expansiveness of the metric projection.
- Moskovitz-Dines property.
- Characterization of Hadamard manifolds.

Chapter 2

Stackelberg equilibria in Euclidean spaces

In this chapter the existence and location of Stackelberg equilibria is studied for two players by using appropriate variational inequalities and fixed point arguments. Both compact and non-compact strategy sets are considered in Euclidean spaces. We follow the paper by Sz. Nagy [49] (where the results were presented in the smooth setting).

2.1 Formulation of the problem

As we already pointed out in the Introduction, the Stackelberg competition can be handled by the *backward induction method*, i.e., to find the best response for the follower player (the strategy action of the leader player being a parameter at this stage) and then, to choose the best strategy of the leader player. From above it becomes clear that the primordial purpose is to handle the response set of the follower player.

Without loss of generality, we assume that both players strategies are certain sets K_1, K_2 in $\mathbb{R}^m$. Let $l : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the payoff/loss function of the *l*eader, while $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is the payoff function of the *f*ollower player.

The first step is to determine the *Stackelberg equilibrium response set*, defined by

$$R_{SE}(x_1) = \{x_2 \in K_2 : f(x_1, y) - f(x_1, x_2) \geq 0,\ \forall y \in K_2\}$$

for every fixed $x_1 \in K_1$. Now, assuming that $R_{SE}(x_1) \neq \emptyset$ for every $x_1 \in K_1$, the concluding step (for the leader player) is to minimize the map $x \mapsto l(x, r(x))$ on K_1 where r is a suitable selection of the set-valued map R_{SE}; more precisely, the *Stackelberg equilibrium leader set* is

$$S_{SE} = \{x_1 \in K_1 : l(x, r(x)) - l(x_1, r(x_1)) \geq 0,\ \forall x \in K_1\}.$$

Our primary aim is to locate the elements of the Stackelberg equilibrium response set. To complete this purpose, we define a slightly larger set than the Stackelberg equilibrium response set by means of variational inequalities. Let us assume that for the follower payoff function $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ we have that $f(x_1, \cdot)$ is a locally Lipschitz function for every $x_1 \in K_1$. We introduce the so-called *Stackelberg variational response set* defined by

$$R_{SV}(x_1) = \left\{x_2 \in K_2 : f^0_{x_2}((x_1, x_2); y - x_2) \geq 0,\ \forall y \in K_2\right\},$$

where $f^0_{x_2}((x_1, x_2); v)$ is the generalized directional derivative of $f(x_1, \cdot)$ at the point $x_2 \in K_2$ in the direction $v \in \mathbb{R}^m$.

We notice first that we are able to compute the Stackelberg variational response set more easier than $R_{SE}(x_1)$, thus, we can locate the elements of the Stackelberg equilibrium response set among these points. Second, we may characterize the elements of the Stackelberg variational response set by the fixed points of a suitable function which involves the metric projection map into the set K_2. Due to the latter fact, we are able to guarantee not only existence but also location results (via projective dynamical systems) of the Stackelberg competition model whenever the strategy sets are compact or non-compact. Recently, projection-like methods for Nash equilibria have been developed by E. Cavazzuti, M. Pappalardo, M. Passacantando [16], A. Kristály [37], J.-S. Pang and M. Fukushima [55], Y.S. Xia and J. Wang [68], J. Zhang, B. Qu and N. Xiu [71], and references therein. For generic equilibrium results via variational and non-variational methods, we refer the reader to the volumes [19], [29].

Assume further that the payoff function $l : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ of the leader is a locally Lipschitz function. If $R_{SV}(x_1) \neq \emptyset$ for every $x_1 \in K_1$ and once we are able to choose a suitably regular selection $r : K_1 \to K_2$ of the set-valued map R_{SV}, we may introduce the *Stackelberg variational leader set*

$$S_{SV} = \left\{x_1 \in K_1 : l^0_{x_1}((x_1, r(x_1)); y - x_1) \geq 0,\ \forall y \in K_1\right\}.$$

As expected, the set S_{SV} contains the best strategies of the first player, i.e., the minimizers for the map $x \mapsto l(x, r(x))$ on K_1.

The chapter is structured as follows. In Section 2.2 we present some basic results concerning the relationship between the Stackelberg variational response set and fixed points of a suitable projection map. In Section 2.3 we present the main results of this chapter by considering both the compact and non-compact cases for the strategy sets of the players.

2.2 Stackelberg variational response set

In this section we state the basic properties of the Stackelberg variational response set.

Proposition 2.2.1 (Sz. Nagy [49]) *Let $K_i \subset \mathbb{R}^m$ be two convex sets $(i = 1, 2)$, and let $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the follower payoff function such that $f(x_1, \cdot)$ is locally Lipschitz for every $x_1 \in K_1$. Then, we have the following assertions:*

(a) *$R_{SE}(x_1) \subseteq R_{SV}(x_1)$ for every $x_1 \in K_1$.*

(b) *If $f(x_1, \cdot)$ is convex on K_2 for some $x_1 \in K_1$, then $R_{SE}(x_1) = R_{SV}(x_1)$.*

Remark 2.2.1 Note that (a) can be proved via a critical point argument. Indeed, if $f(x_1, y) \geq f(x_1, x_2)$ for all $y \in K_2$, then $x_2 \in K_2$ is a global minimum point for the function $f(x_1, \cdot) + \delta_{K_2}$, where δ_{K_2} is the indicator function of the set K_2, i.e.,

$$\delta_{K_2}(y) = \begin{cases} 0 & \text{if} \quad y \in K_2; \\ +\infty & \text{if} \quad y \notin K_2. \end{cases}$$

Now, x_2 is a critical point of $f(x_1, \cdot) + \delta_{K_2}$, which implies that

$$f^0_{x_2}((x_1, x_2); y - x_2) \geq 0, \ \forall y \in K_2,$$

i.e., $x_2 \in R_{SV}(x_1)$.

We present an important observation which makes a connection between the Stackelberg variational response set and the fixed point of the set-valued map $A^{x_1}_\alpha : K_2 \to 2^{K_2}$ defined by

$$A^{x_1}_\alpha(x) = P_{K_2}\left(x - \alpha \partial_{x_2} f(x_1, x)\right), \tag{2.2.1}$$

where $x_1 \in K_1$ and $\alpha > 0$ are fixed.

Theorem 2.2.1 (Sz. Nagy [49]) *Let $K_i \subset \mathbb{R}^m$ be two convex sets $(i = 1, 2)$, and let $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the follower payoff function such that $f(x_1, \cdot)$ is locally Lipschitz for every $x_1 \in K_1$. Let $x_1 \in K_1$ be arbitrarily fixed. The following statements are equivalent:*

(a) $x_2 \in R_{SV}(x_1)$;

(b) $x_2 \in A_\alpha^{x_1}(x_2)$ *for all* $\alpha > 0$;

(c) $x_2 \in A_\alpha^{x_1}(x_2)$ *for some* $\alpha > 0$.

We conclude this section by a result concerning the Stackelberg variational leader set; more precisely, the definitions imply

Proposition 2.2.2 *Let $l : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be a function of class C^1. Assume that $x \mapsto R_{SV}(x)$ is a single-valued function of class C^1 on K_1. Then $S_{SE} \subseteq S_{SV}$.*

Remark 2.2.2 In Chapter 4 we shall see that usually the map $x \mapsto R_{SV}(x)$ is not single-valued; thus we have to choose a suitable selection from the set-valued map $x \mapsto R_{SV}(x)$ in order to apply Proposition 2.2.2.

2.3 Existence and location of Stackelberg variational responses

Due to Theorem 2.2.1, to find elements in $R_{SV}(x_1)$ is equivalent to find fixed points of $A_\alpha^{x_1}$, $\alpha > 0$. To complete this aim, we distinguish two cases: compact and non-compact strategy sets.

2.3.1 Compact strategies in Euclidean spaces

Theorem 2.3.1 (Sz. Nagy [49]) *Let $K_i \subset \mathbb{R}^m$ be two convex sets $(i = 1, 2)$, K_2 be compact, and let $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the follower payoff function such that $f(x_1, \cdot)$ is locally Lipschitz for every $x_1 \in K_1$. Then the following statements hold:*

(a) $\emptyset \neq R_{SE}(x_1) \subseteq R_{SV}(x_1)$ *for every* $x_1 \in K_1$;

(b) *If* $\mathrm{card}(R_{SV}(x_1)) = 1$ *for every $x_1 \in K_1$ and the map $x \mapsto R_{SV}(x)$ and the leader payoff function l are of class C^1, then $S_{SV} \neq \emptyset$.*

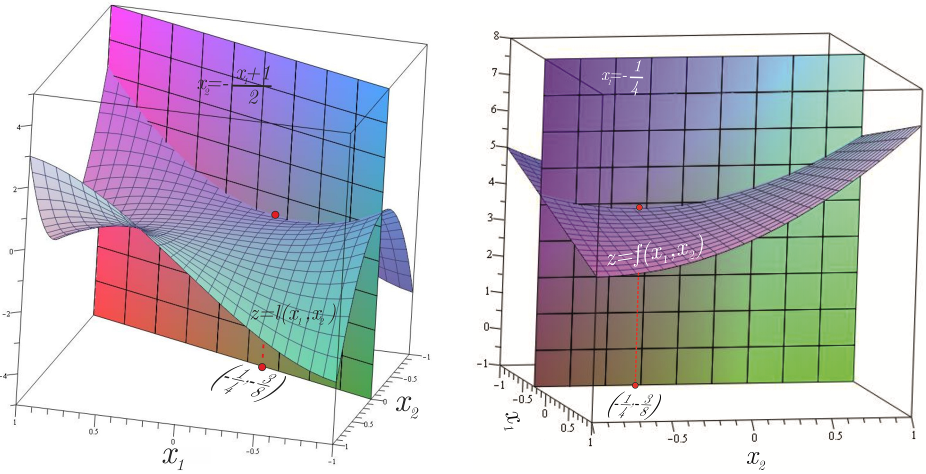

Figure 2.1: Minimization of the function $x_1 \mapsto l(x_1, R_{SV}(x_1))$, $x_1 \in K_1$, obtaining $x_1 = -\frac{1}{4}$ (left). The response of the follower to the action $x_1 = -\frac{1}{4}$ is $R_{SE}(-\frac{1}{4}) = R_{SV}(-\frac{1}{4}) = \left\{-\frac{3}{8}\right\}$ (right).

Example 2.3.1 Let $l, f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two payoff functions defined by

$$l(x_1, x_2) = 4x_1x_2^2 - x_1^3;$$
$$f(x_1, x_2) = x_2^2 + x_2(x_1 + 1) + 4,$$

and the sets $K_1 = K_2 = [-1, 1]$. It is clear that Theorem 2.3.1(a) can be applied, and a simple computation yields (as a first step in the backward induction method) that

$$R_{SV}(x_1) = \left\{-\frac{x_1 + 1}{2}\right\}, \; x_1 \in K_1.$$

Note that $f(x_1, \cdot)$ is convex on K_2 for every $x_1 \in K_1$; thus, on account of Proposition 2.2.1(a), $R_{SE}(x_1) = R_{SV}(x_1)$ for every $x_1 \in K_1$. Moreover, since $\mathrm{card}(R_{SV}(x_1)) = 1$ for every $x_1 \in K_1$, and the map $x \mapsto R_{SV}(x)$ is of class C^1, then one has that $S_{SV} \neq \emptyset$, see Theorem 2.3.1 (b). A simple calculation also yields that

$$S_{SV} = \left\{-\frac{1}{4}\right\}.$$

Now, by using Proposition 2.2.2, we can check that the Stackelberg equilibrium leader set is $S_{SE} = \left\{-\frac{1}{4}\right\}$, while the Stackelberg equilibrium/variational response set is $R_{SE}(-\frac{1}{4}) = R_{SV}(-\frac{1}{4}) = \left\{-\frac{3}{8}\right\}$, see Figure 2.1.

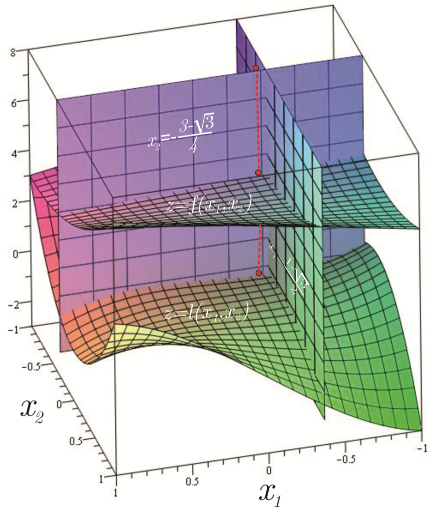

Figure 2.2: Although the point $\left(\frac{1-\sqrt{3}}{2}, -\frac{3-\sqrt{3}}{4}\right)$ is a Nash-Stampacchia equilibrium point of l and f on $K_1 \times K_2$, it is not a usual Nash equilibrium point.

Remark 2.3.1 For the same functions and sets as in Example 2.3.1, we state that the set of Nash equilibrium points is empty. This fact can be seen by following the arguments from the paper of A. Kristály [36] where a very general framework is discussed for nonsmooth functions on finite-dimensional Riemannian manifolds. More precisely, the first step is to determine the set of Nash-Stampacchia equilibrium points, i.e., the solutions for the system

$$\begin{cases} \left\langle l'_{x_1}(x_1, x_2), x - x_1 \right\rangle \geq 0 & \text{for all} \quad x \in K_1; \\ \left\langle f'_{x_2}(x_1, x_2), y - x_2 \right\rangle \geq 0 & \text{for all} \quad y \in K_2. \end{cases}$$

This system has the solution

$$N_S = \left\{ \left(\frac{1-\sqrt{3}}{2}, -\frac{3-\sqrt{3}}{4} \right) \right\}.$$

The set of Nash equilibrium points is a subset of N_S, cf. A. Kristály [37]. Note however that the above point does not fulfil the system for Nash equilibria

$$\begin{cases} l(x, x_2) \geq l(x_1, x_2) & \text{for all} \quad x \in K_1; \\ f(x_1, y) \geq f(x_1, x_2) & \text{for all} \quad y \in K_2, \end{cases}$$

see Figure 2.2.

2.3.2 Non-compact strategies in Euclidean spaces

If $l, f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the functions defined by

$$l(x_1, x_2) = f(x_1, x_2) = e^{-x_1 - x_2},$$

and $K_1 = K_2 = [0, \infty)$, then for every $x_1 \in K_1$, one has

$$R_{SE}(x_1) = R_{SV}(x_1) = \emptyset.$$

Consequently, in order to guarantee existence of elements from the Stackelberg equilibrium/variational response set in the non-compact case, further (growth) assumptions are needed beside the regularity of the functions.

To complete the latter problem, we introduce two dynamical systems by assuming that both payoff functions are of class C^1 on $\mathbb{R}^m \times \mathbb{R}^m$. Let $x_1 \in K_1$ and $\alpha > 0$ be fixed elements.

(a) Let $(DDS)_{x_1}^{\mathbb{R}^m}$ be the *discrete dynamical system* in the form

$$\begin{cases} y_{n+1} = A_\alpha^{x_1}(P_{K_2}(y_n)), \quad n \geq 0, \\ y_0 \in \mathbb{R}^m. \end{cases}$$

(b) Let $(CDS)_{x_1}^{\mathbb{R}^m}$ be the *continuous dynamical system* in the form

$$\begin{cases} \frac{dy}{dt} = A_\alpha^{x_1}(P_{K_2}(y(t))) - y(t), \\ y(0) = y_0 \in \mathbb{R}^m. \end{cases}$$

The main theorem of the present section reads as follows:

Theorem 2.3.2 (Sz. Nagy [49]) *Let $K_i \subset \mathbb{R}^m$ be two convex (not necessarily compact) sets ($i = 1, 2$), and let $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be a C^1-class payoff function of the follower. Let $x_1 \in K_1$ be fixed and assume that $f'_{x_2}(x_1, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is an $L-$Lipschitz and $\kappa-$strongly monotone function. Then* $\mathrm{card}(R_{SV}(x_1)) = 1$*; moreover, the orbits of both dynamical systems, $(DDS)_{x_1}^{\mathbb{R}^m}$ and $(CDS)_{x_1}^{\mathbb{R}^m}$, exponentially converge to the unique element of $R_{SV}(x_1)$.*

Remark 2.3.2 The argument based on projective dynamical systems has been exploited in the papers of E. Cavazzuti, M. Pappalardo, M. Passacantando [16], Y.S. Xia and J. Wang [68], J. Zhang, B. Qu and N. Xiu [71] for Nash-type equilibria. Note that the present result for Stackelberg equilibria is slightly general than those in the above works. A systematic approach to this topic can be found also in A. Kristály, V. Rădulescu and Cs. Varga [39, Chapter III] in the context of Nash equilibria on curved spaces.

Example 2.3.2 Fix $n \geq 2$. Let $M_n(\mathbb{R})$ be the set of symmetric $n \times n$ matrices with real entries. The standard inner product on $M_n(\mathbb{R})$ is defined as

$$\langle U, V \rangle = \operatorname{tr}(UV).$$

Here, $\operatorname{tr}(Y)$ denotes the trace of $Y \in M_n(\mathbb{R})$. It is well-known that $(M_n(\mathbb{R}), \langle \cdot, \cdot \rangle)$ is an Euclidean space, the unique segment between $X, Y \in M_n(\mathbb{R})$ is

$$\gamma_{X,Y}(s) = (1-s)X + sY, \quad s \in [0,1]. \tag{2.3.1}$$

Let us consider the functions $l, f : \mathbb{R} \times M_n(\mathbb{R}) \to \mathbb{R}$ defined by

$$l(t, X) = t^3 - t\det(X), \;\; f(t, X) = \operatorname{tr}((X - tA)^2),$$

where $A \in M_n(\mathbb{R})$ is fixed, and

$$K_1 = [0, \infty), \;\; K_2 = \{X \in M_n(\mathbb{R}) : \operatorname{tr}(X) \geq 1\}.$$

It is clear that both sets are non-compact and convex. Moreover, one has that for every $t \in K_1$, the function

$$X \mapsto f'_{x_2}(t, X) = 2(X - tA)$$

is $2-$Lipschitz and $2-$strongly monotone. Then, on account of Theorem 2.3.2, $\operatorname{card}(R_{SV}(t)) = 1$ for every $t \in K_1$, and both dynamical systems, $(DDS)_t^{\mathbb{R}}$ and $(CDS)_t^{\mathbb{R}}$, exponentially converge to the unique element of $R_{SV}(t)$. In this particular example, one can see that

$$R_{SV}(t) = \{P_{K_2}(tA)\}, \;\; \forall t \in K_1.$$

In order to obtain the Stackelberg equilibrium leader set S_{SE}, it remains to minimize the function $t \mapsto l(t, P_{K_2}(tA)) = t^3 - t\det(P_{K_2}(tA))$ on $K_1 = [0, \infty)$.

We conclude this chapter with a Caristi-type result concerning the existence of Stackelberg equilibria:

Theorem 2.3.3 (Sz. Nagy) *Let $K_i \subset \mathbb{R}^m$ be two convex (not necessarily compact) sets $(i = 1, 2)$, and let $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be a C^1-class payoff function of the follower. Let $x_1 \in K_1$ be fixed. If there exists a lower semicontinuous function $g : \mathbb{R}^m \to \mathbb{R}_+$ and $\alpha > 0$ such that*

$$\|x - A_\alpha^{x_1}(x)\| \leq g(x) - g(A_\alpha^{x_1}(x)), \;\; \forall x \in \mathbb{R}^m,$$

then $R_{SV}(x_1) \neq \emptyset$.

Chapter 3

Stackelberg equilibria on Riemannian manifolds

In this chapter we present the Riemannian extension of the results from Chapter 2; in order to avoid technicalities we shall work in the smooth setting, following the paper written by A. Kristály and Sz. Nagy [38].

3.1 Riemannian approach to the Stackelberg variational response set

In the previous chapter we presented some existence and location results for the Stackelberg variational response set in the Euclidean framework.

The purpose of the present chapter is to extend the analytical results from Chapter 2 to games defined on strategy sets which are *embedded in a geodesic convex manner into certain Riemannian manifolds.* The motivation is that some strategy sets may be not convex in the Euclidean setting. The idea to embed geodesic convexly such sets into Riemannian/Finsler manifolds originates from T. Rapcsák [59] who applied this approach to solve various nonlinear optimization problems. Similar studies can be found in the literature, where certain variational arguments are applied to study equilibrium problems on Riemannian manifolds, see G.C. Bento and J.G. Melo [7], A. Kristály [36, 37], X. Li, N. Huang [42], C. Li, G. López, V. Martín-Márquez [43], S.Z. Németh [52] and references therein.

For simplicity, we shall consider only two players although our arguments can be extended to several players as well. Let $K_1 \subset M_1$

and $K_2 \subset M_2$ be two sets in the Riemannian manifolds (M_1, g_1) and (M_2, g_2), respectively, and let $l, f : M_1 \times M_2 \to \mathbb{R}$ be the payoff functions for the two players. As we already know from the backward induction method, the first step (for the follower) is to find the response set

$$\mathcal{R}_{SE}(x_1) = \{x_2 \in K_2 : f(x_1, y) - f(x_1, x_2) \geq 0, \ \forall y \in K_2\}$$

for every fixed $x_1 \in K_1$. If $\mathcal{R}_{SE}(x_1) \neq \emptyset$ for every $x_1 \in K_1$, the next step is to minimize the map $x \mapsto l(x, r(x))$ on K_1 where r is a selection function of the set-valued map $x \mapsto \mathcal{R}_{SE}(x)$; thus, the objective of the first player is to determine the set

$$\mathcal{S}_{SE} = \{x_1 \in K_1 : l(x, r(x)) - l(x_1, r(x_1)) \geq 0, \ \forall x \in K_1\}.$$

Similarly to Chapter 2, we shall introduce sets related to the above ones by variational inequalities defined on Riemannian manifolds. For simplicity, let us assume throughout of this chapter that $f : M_1 \times M_2 \to \mathbb{R}$ is a C^1-class function[1]; for every $x_1 \in K_1$, we introduce the set

$$\mathcal{R}_{SV}(x_1) = \left\{x_2 \in K_2 : g_2\left(f'_{x_2}(x_1, x_2), \exp_{x_2}^{-1}(y)\right) \geq 0, \ \forall y \in K_2\right\},$$

where $f'_{x_2}(x_1, x_2)$ is the differential of $f(x_1, \cdot)$ with respect to the metric g_2 at the point $x_2 \in K_2$. According to A. Kristály [36, 37], it is more easier to determine the set $\mathcal{R}_{SV}(x_1)$ than $\mathcal{R}_{SE}(x_1)$. Moreover, we usually have that $\mathcal{R}_{SE}(x_1) \subseteq \mathcal{R}_{SV}(x_1)$, thus we shall choose the appropriate Stackelberg equilibrium candidates from the elements of the latter set. Finally, by imposing further curvature assumptions on the Riemannian manifolds we are working on, we are able to characterize the elements of the set $\mathcal{R}_{SV}(x_1)$ by the fixed points of a suitable set-valued map which involves the metric projection map into the set K_2. In fact, we shall assume that the strategy sets are embedded into non-positively curved Riemannian manifolds where two basic properties of the metric projection will be deeply exploited; namely,

- the non-expansiveness of the metric projection map, and
- the Moskovitz-Dines property.

[1] A similar study can be done also for locally Lipschitz functions following the nonsmooth analysis on manifolds developed by D. Azagra, J. Ferrera and F. López-Mesas [3] and Yu. S. Ledyaev and Q.J. Zhu [41].

The optimal geometric framework to develop the theory of Stackelberg equilibrium theory in the Riemannian framework is provided by the class of Hadamard manifolds. Having the fixed point characterization, we will be able to apply various fixed point theorems on (acyclic) metric spaces in order to find elements of the set $\mathcal{R}_{SV}(x_1)$.

We assume finally that the payoff function $l : M_1 \times M_2 \to \mathbb{R}$ for the leader is of class C^1 and for every $x_1 \in K_1$ we have that $\mathcal{R}_{SV}(x_1) \neq \emptyset$. If we are able to choose a C^1-class selection $r : K_1 \to K_1$ of the set-valued map $\mathcal{R}_{SV}$, we also introduce the set

$$\mathcal{S}_{SV} = \left\{x_1 \in K_1 : g_1\left(l'(x_1, r(x_1)), \exp_{x_1}^{-1}(y)\right) \geq 0,\ \forall y \in K_1\right\}.$$

In particular, $\mathcal{S}_{SV}$ contains the optimal strategies of the leader, i.e., the minimizers for the map $x \mapsto l(x, r(x))$ on K_1.

3.2 Basic properties of the response sets

In the sequel we shall establish some basic properties of the response sets by using some elements from the theory of variational inequalities on Riemannian manifolds. The notations are from Section 3.1.

Lemma 3.2.1 (A. Kristály and Sz. Nagy [38]) *Let (M_i, g_i) be Riemannian manifolds, $l, f : M_1 \times M_2 \to \mathbb{R}$ be payoff functions of class C^1, and $K_i \subset M_i$ be closed, geodesic convex sets, $i = 1, 2$. Then the following assertions hold:*

(a) $\mathcal{R}_{SE}(x_1) \subseteq \mathcal{R}_{SV}(x_1)$ *for every* $x_1 \in K_1$;

(b) $\mathcal{R}_{SE}(x_1) = \mathcal{R}_{SV}(x_1)$ *when* $f(x_1, \cdot)$ *is convex on* K_2 *for some* $x_1 \in K_1$;

(c) $\mathcal{S}_{SE} \subseteq \mathcal{S}_{SV}$ *when* $x \mapsto \mathcal{R}_{SV}(x)$ *is a single-valued function which has a* C^1-*extension to an arbitrary open neighborhood* $D_1 \subset M_1$ *of* K_1.

For a fixed $x_1 \in K_1$ and $\alpha > 0$, let $\mathcal{A}_\alpha^{x_1} : K_2 \to K_2$ be defined by

$$\mathcal{A}_\alpha^{x_1}(x) = P_{K_2}\left(\exp_x\left(-\alpha f'_{x_2}(x_1, x)\right)\right). \tag{3.2.1}$$

Note that $\mathcal{A}_\alpha^{x_1}$ is a single-valued function whenever (M_2, g_2) is a Hadamard manifold (since $P_{K_2}(x)$ is a Chebyshev set for every $x \in K_2$).

Theorem 3.2.1 (A. Kristály and Sz. Nagy [38]) *Let (M_1, g_1) be a Riemannian manifold, and (M_2, g_2) be a Hadamard manifold. Let $f : M_1 \times M_2 \to \mathbb{R}$ be a C^1-class payoff function of the follower and $K_i \subset M_i$ be closed, geodesic convex sets, $i = 1, 2$. Let $x_1 \in K_1$ be fixed. The following statements are equivalent:*

(a) $x_2 \in \mathcal{R}_{SV}(x_1)$;

(b) $\mathcal{A}_\alpha^{x_1}(x_2) = x_2$ *for all* $\alpha > 0$;

(c) $\mathcal{A}_\alpha^{x_1}(x_2) = x_2$ *for some* $\alpha > 0$.

Remark 3.2.1 Note that

$$\mathcal{R}_{SV}(x_1) = \left\{x_2 \in K_2 : P_{K_2}\left(\exp_{x_2}\left(-\alpha f'_{x_2}(x_1, x_2)\right)\right) = x_2\right\}.$$

3.3 Follower strategy: existence of equilibria

3.3.1 Compact strategies on manifolds

Theorem 3.3.1 (A. Kristály and Sz. Nagy [38]) *Let (M_i, g_i) be Hadamard manifolds, $l, f : M_1 \times M_2 \to \mathbb{R}$ be C^1-class payoff functions and $K_i \subset M_i$ be compact, geodesic convex sets, $i = 1, 2$. Then the following statements hold:*

(a) $\emptyset \neq \mathcal{R}_{SE}(x_1) \subseteq \mathcal{R}_{SV}(x_1)$ *for every* $x_1 \in K_1$;

(b) $\mathcal{S}_{SV} \neq \emptyset$ *whenever* $\mathcal{R}_{SV}(x_1)$ *is a singleton for every* $x_1 \in K_1$ *and the map* $x \mapsto \mathcal{R}_{SV}(x)$ *has a* C^1-*extension to an arbitrary open neighborhood* $D_1 \subset M_1$ *of* K_1.

3.3.2 Non-compact strategies on manifolds

We first assume that for some $x_1 \in K_1$ one has
$(H^f_{x_1})$ There exists $x_2 \in K_2$ such that

$$L_{x_1,x_2} = \limsup_{\substack{d_{g_2}(x,x_2)\to\infty \\ x\in K_2}} \frac{g_2\left(f'_{x_2}(x_1, x), \exp_x^{-1}(x_2)\right) + g_2\left(f'_{x_2}(x_1, x_2), \exp_{x_2}^{-1}(x)\right)}{d_{g_2}(x, x_2)} <$$

$$< -\left\|f'_{x_2}(x_1, x_2)\right\|_{g_2}.$$

The first main result of the present section is the following theorem.

Theorem 3.3.2 (A. Kristály and Sz. Nagy [38]) *Let (M_1, g_1) be a Riemannian manifold and (M_2, g_2) be a Hadamard manifold. Let $f : M_1 \times M_2 \to \mathbb{R}$ be a C^1-class payoff function of the follower and $K_i \subset M_i$ be closed, geodesic convex sets, $i = 1, 2$. Let $x_1 \in K_1$ and assume that hypothesis $(H^f_{x_1})$ holds true. Then $\mathcal{R}_{SV}(x_1) \neq \emptyset$.*

In the sequel, we are dealing with another class of functions, similar to the previous chapter. For a fixed $x_1 \in K_1$, $\alpha > 0$ and $0 < \rho < 1$ we introduce the hypothesis:

$$(H^{\alpha,\rho}_{x_1}) : \ d_{g_2}\left(\exp_x\left(-\alpha f'_{x_2}(x_1, x)\right), \exp_y\left(-\alpha f'_{x_2}(x_1, y)\right)\right) \leq (1-\rho) d_{g_2}(x, y)$$

for all $x, y \in K_2$. For fixed $x_1 \in K_1$ and $\alpha > 0$, we consider the following two dynamical systems:

(a) Let $(DDS)_{x_1}$ be the *discrete dynamical system* in the form

$$\begin{cases} y_{n+1} = \mathcal{A}^{x_1}_\alpha(P_{K_2}(y_n)), & n \geq 0, \\ y_0 \in M_2; \end{cases}$$

(b) Let $(CDS)_{x_1}$ be the *continuous dynamical system* in the form

$$\begin{cases} \frac{dy}{dt} = \exp^{-1}_{y(t)}(\mathcal{A}^{x_1}_\alpha(P_{K_2}(y(t)))), \\ y(0) = x_2 \in M_2. \end{cases}$$

By exploring the Rauch comparison principle and the properties of the metric projection, we can state the following theorem.

Theorem 3.3.3 (A. Kristály and Sz. Nagy [38]) *Let (M_1, g_1) be a Riemannian manifold and (M_2, g_2) be a Hadamard manifold. Let $f : M_1 \times M_2 \to \mathbb{R}$ be a C^1-class payoff function of the follower and $K_i \subset M_i$ be closed, geodesic convex sets, $i = 1, 2$. Let $x_1 \in K_1$ and assume that hypothesis $(H^{\alpha,\rho}_{x_1})$ holds true for some $\alpha > 0$ and $0 < \rho < 1$. Then $\mathcal{R}_{SV}(x_1)$ is a singleton and the orbits of both dynamical systems, $(DDS)_{x_1}$ and $(CDS)_{x_1}$, exponentially converge to the unique element of $\mathcal{R}_{SV}(x_1)$.*

In the sequel, inspired by A. Kristály [37], we shall provide some examples of Hadamard manifolds and geodesic convex sets where the previous results can be applied.

Example 3.3.1 (*Euclidean space*) Assume that $M_2 = \mathbb{R}^{m_2}$ and the function $f'_{x_2}(x_1, \cdot)$ is an $L-$Lipschitz and $\kappa-$strongly monotone function for some $x_1 \in K_1$. Then Theorem 3.3.3 reduces to Theorem 2.3.2. Indeed, hypothesis $(H^{\alpha,\rho}_{x_1})$ holds true with the constants $0 < \alpha < \frac{\kappa - \sqrt{(\kappa^2 - L^2)_+}}{L^2}$ and $\rho = 1 - \sqrt{1 - 2\alpha\kappa + \alpha^2 L^2} \in (0, 1)$, respectively.

Example 3.3.2 (*Hyperbolic space*) Let $\mathbb{H} = \{(u,v) \in \mathbb{R}^2 : v > 0\}$ be the Poincaré upper half-plane model endowed with the Riemannian metric defined for every $(u,v) \in \mathbb{H}$ by

$$g_{ij}(u,v) = \frac{1}{v^2}\delta_{ij}, \quad \text{for } i,j = 1,2.$$

The pair $(\mathbb{H}, g)$ is a Hadamard manifold with constant sectional curvature -1 and the geodesics in $\mathbb{H}$ are the semilines and the semicircles orthogonal to the line $v = 0$. The Riemannian distance between two points $(u_1,v_1),(u_2,v_2) \in \mathbb{H}$ is given by

$$d_{\mathbb{H}}\left((u_1,v_1),(u_2,v_2)\right) = \operatorname{arccosh}\left(1 + \frac{(u_2-u_1)^2 + (v_2-v_1)^2}{2v_1v_2}\right).$$

Let

$$K = \{(u,v) \in \mathbb{H} : u^2 + v^2 \leq 9 \leq (u-2)^2 + v^2\}. \tag{3.3.1}$$

Note that $K \subset \mathbb{R}^2$ is not convex in the usual sense, but it is in $(\mathbb{H}, g)$, see Figure **??**.

Example 3.3.3 (*Symmetric positive definite matrices*) As in Example 2.3.2, let $M_n(\mathbb{R})$ be the set of symmetric $n \times n$ matrices with real values, $\mathrm{P}(n,\mathbb{R}) \subset M_n(\mathbb{R})$ be the $\frac{n(n+1)}{2}$−dimensional cone of symmetric positive definite matrices. The set $\mathrm{P}(n,\mathbb{R})$ is endowed with the scalar product

$$\langle\langle U, V\rangle\rangle_X = \operatorname{tr}(X^{-1}VX^{-1}U)$$

for all $X \in \mathrm{P}(n,\mathbb{R}),\ U,V \in T_X(\mathrm{P}(n,\mathbb{R})) \simeq M_n(\mathbb{R})$.

The pair $(\mathrm{P}(n,\mathbb{R}), \langle\langle\cdot,\cdot\rangle\rangle)$ is a Hadamard manifold, see Lang [40, Chapter XII]. The unique geodesic segment connecting $X, Y \in \mathrm{P}(n,\mathbb{R})$ is defined by

$$\gamma^H_{X,Y}(s) = X^{1/2}(X^{-1/2}YX^{-1/2})^s X^{1/2}, \quad s \in [0,1]. \tag{3.3.2}$$

In particular, $\frac{d}{ds}\gamma^H_{X,Y}(s)|_{s=0} = X^{1/2}\ln(X^{-1/2}YX^{-1/2})X^{1/2}$; consequently, for each $X, Y \in \mathrm{P}(n,\mathbb{R})$, we have

$$\exp_X^{-1} Y = X^{1/2}\ln(X^{-1/2}YX^{-1/2})X^{1/2}.$$

The induced metric function on $\mathrm{P}(n,\mathbb{R})$ is given by

$$d_H^2(X,Y) = \langle\langle \exp_X^{-1} Y, \exp_X^{-1} Y\rangle\rangle_X = \operatorname{tr}(\ln^2(X^{-1/2}YX^{-1/2})). \tag{3.3.3}$$

Let $a \in (1, e]$, and

$$K = \{X \in \mathrm{P}(n, \mathbb{R}) : \mathrm{tr}(\ln^2 X) \leq 1 \leq \det X \leq a\}.$$

On one hand, we notice that the set K is not geodesic convex with respect to the metric $\langle \cdot, \cdot \rangle$ from Example 2.3.2. Indeed, let $X = \mathrm{diag}(a, 1, ..., 1) \in K$ and $Y = \mathrm{diag}(1, a, ..., 1) \in K$ and $\gamma_{X,Y}$ be the Euclidean geodesic connecting them, see (2.3.1); although $\gamma_{X,Y}(s) \in \mathrm{P}(n, \mathbb{R})$ and

$$\mathrm{tr}(\ln^2(\gamma_{X,Y}(s))) = \ln^2(1 + (a-1)s) + \ln^2(a + (1-a)s) \leq \ln^2 a \leq 1$$

for every $s \in [0, 1]$, one has that

$$\det(\gamma_{X,Y}(s)) = a + (a-1)^2 s(1-s) > a, \ \forall s \in (0, 1).$$

On the other hand, we claim that K is geodesic convex in the manifold $(\mathrm{P}(n, \mathbb{R}), \langle\langle \cdot, \cdot \rangle\rangle)$. To see this, let $I_n \in \mathrm{P}(n, \mathbb{R})$ be the identity matrix, and $\tilde{B}_H(I_n, 1)$ be the closed geodesic ball in $\mathrm{P}(n, \mathbb{R})$ with center I_n and radius 1. On account of the above facts, we observe that

$$K = \tilde{B}_H(I_n, 1) \cap \{X \in \mathrm{P}(n, \mathbb{R}) : 1 \leq \det X \leq a\}.$$

Indeed, for every $X \in \mathrm{P}(n, \mathbb{R})$, we have $d_H^2(I_n, X) = \mathrm{tr}(\ln^2 X)$. Since K is bounded and closed in $(\mathrm{P}(n, \mathbb{R}), \langle\langle \cdot, \cdot \rangle\rangle)$, due to the Hopf-Rinow theorem, K is compact. Moreover, being a geodesic ball in the Hadamard manifold $(\mathrm{P}(n, \mathbb{R}), \langle\langle \cdot, \cdot \rangle\rangle)$, the set $\tilde{B}_H(I_n, 1)$ is geodesic convex. Keeping the notation from (3.3.2), if $X, Y \in K$, one has for every $s \in [0, 1]$ that

$$\det(\gamma_{X,Y}^H(s)) = (\det X)^{1-s}(\det Y)^s \in [1, a],$$

i.e., K is geodesic convex in $(\mathrm{P}(n, \mathbb{R}), \langle\langle \cdot, \cdot \rangle\rangle)$.

Remark 3.3.1 Since the elements of the Stackelberg variational response sets are fixed points of $\mathcal{A}_\alpha^{x_1}$ for some $\alpha > 0$ and $x_1 \in K_1$, besides the aforementioned theorems, other fixed point arguments are expected to be applied, see D. O'Regan and R. Precup [54] and the papers of A. Petruşel [56], A. Petruşel, I.A. Rus and M. Şerban [57], and references therein.

Chapter 4

Multiplicity of Stackelberg variational responses

Contrary to the standard literature (where the Stackelberg response function is single-valued), we provide sufficient conditions for a whole class of functions to show that the Stackelberg variational response set contains at least three elements.

4.1 Setting of the problem

In this section we focus our attention to a specific payoff function for the follower player; namely, we assume that $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is given by

$$f_\lambda(x_1, x_2) := f(x_1, x_2) = \frac{1}{2}\|x_2\|^2 - \lambda \tilde{f}(x_1, x_2) + \delta_{K_2}(x_2), \qquad (4.1.1)$$

where $K_2 \subset \mathbb{R}^m$ is a nonempty, closed, non-compact set, $\lambda > 0$ is a parameter and $\tilde{f}(x_1, \cdot)$ is locally Lipschitz for every $x_1 \in \mathbb{R}^m$. As usual, δ_{K_2} denotes the indicator function of the set K_2.

Let $x_1 \in \mathbb{R}^m$ be arbitrarily fixed. On the locally Lipschitz function $\tilde{f}(x_1, \cdot)$ we assume:

$(H^1_{x_1})$ $\quad \max\{\|z\| : z \in \partial_{x_2} \tilde{f}(x_1, x_2)\} = o(\|x_2\|)$ whenever $\|x_2\| \to 0$;

$(H^2_{x_1})$ $\quad \max\{\|z\| : z \in \partial_{x_2} \tilde{f}(x_1, x_2)\} = o(\|x_2\|)$ whenever $\|x_2\| \to +\infty$;

$(H^3_{x_1})$ $\quad \tilde{f}(x_1, 0) = 0$ and there exists $\tilde{x}_2 \in K_2$ such that $\tilde{f}(x_1, \tilde{x}_2) > 0$.

Here, $o(\cdot)$ is the usual Landau symbol.

Remark 4.1.1 (a) Hypotheses $(H^1_{x_1})$ and $(H^2_{x_1})$ mean that $\partial_{x_2}\tilde{f}(x_1,\cdot)$ is *superlinear* at the origin and *sublinear* at infinity, respectively. Hypothesis $(H^3_{x_1})$ implies that $\tilde{f}(x_1,\cdot)$ is not identically zero.

(b) According to hypotheses $(H^1_{x_1})$ and $(H^2_{x_1})$, the number

$$\tilde{c} = \max_{x_2 \in \mathbb{R}^m \setminus \{0\}} \frac{\max\{\|z\| : z \in \partial_{x_2}\tilde{f}(x_1,x_2)\}}{\|x_2\|} \tag{4.1.2}$$

is well-defined, finite, and from the upper semicontinuity of $\partial_{x_2}\tilde{f}(x_1,\cdot)$ and hypothesis $(H^3_{x_1})$, we have $0 < \tilde{c} < \infty$.

(c) We also introduce the number

$$\tilde{\lambda} = \frac{1}{2} \inf_{\substack{\tilde{f}(x_1,x_2)>0 \\ x_2 \in K_2}} \frac{\|x_2\|^2}{\tilde{f}(x_1,x_2)}, \tag{4.1.3}$$

which is well-defined, finite and $0 < \tilde{\lambda} < \infty$.

It is known that the Stackelberg variational response set for the function f_λ in (4.1.1) is given by
$R^\lambda_{SV}(x_1) =$

$$= \left\{ x_2 \in K_2 : \langle x_2, y - x_2 \rangle + \lambda \tilde{f}^0_{x_2}((x_1,x_2); -y + x_2) \geq 0,\ \forall y \in K_2 \right\}.$$

The main theorem of the present chapter is the following.

Theorem 4.1.1 (Sz. Nagy [50]) *Let $K_i \subset \mathbb{R}^m$ be two convex sets $(i = 1, 2)$, and let $f_\lambda : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be the follower payoff function of the form* (4.1.1) *such that $\tilde{f}(x_1,\cdot)$ is locally Lipschitz for every $x_1 \in K_1$. Assume that K_2 is closed and non-compact such that $0 \in K_2$. Fix $x_1 \in K_1$ and assume that the hypotheses $(H^i_{x_1})$ hold true, $i \in \{1,2,3\}$. Then the following statements hold:*

(a) $0 \in R^\lambda_{SV}(x_1)$ *for every* $\lambda > 0$;

(b) $R^\lambda_{SV}(x_1) = \{0\}$ *for every* $\lambda \in (0, \tilde{c}^{-1})$, *where* $\tilde{c}$ *is from* (4.1.2);

(c) $\mathrm{card}(R^\lambda_{SV}(x_1)) \geq 3$ *for every* $\lambda > \tilde{\lambda} > 0$, *where* $\tilde{\lambda}$ *is from* (4.1.3).

Remark 4.1.2 (a) In fact, by using the three critical point theorem of B. Ricceri [60] (or one of its variants, e.g. G. Bonanno [10], S. Marano and D. Motreanu [45]) we could prove that the number of the

Stackelberg variational response set is *stable*, i.e., it is invariant with respect to small perturbations of the function $\tilde{f}$.

(b) The Stackelberg variational response set could have many elements whenever the nonlinear term $\tilde{f}$ has certain *oscillatory* behavior. Similar phenomena have been described in the theory of PDEs within the papers by F. Faraci and A. Kristály [25], P. Omari and F. Zanolin [44], J. Saint Raymond [61] and references therein. We notice that these solutions appeared as local minima of certain energy functionals; consequently, in the theory of Stackelberg equilibria one can expect to obtain some *local* (but perhaps not global) variational responses.

Remark 4.1.3 By Theorem 4.1.1 (b) and (c) it is clear that $\tilde{c}^{-1} \leq \tilde{\lambda}$. We postpone some discussions on this subject to Section 4.4.

In the sequel we provide an application.

Example 4.1.1 Let $K_2 = [0, \infty)$ and $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{f}(x_1, x_2) = (1 + |x_1|)\left(\min\left(8x_2^3, (|x_2| + 3)^{\frac{3}{2}}\right)\right)_+ .$$

A simple calculation shows that

$$\partial_{x_2}\tilde{f}(x_1, x_2) = \begin{cases} \{0\}, & \text{if } \ x_2 < 0; \\ \{24(1 + |x_1|)x_2^2\}, & \text{if } \ x_2 \in [0, 1); \\ [3(1 + |x_1|), 24(1 + |x_1|)], & \text{if } \ x_2 = 1; \\ \left\{\frac{3}{2}(1 + |x_1|)(x_2 + 3)^{\frac{1}{2}}\right\}, & \text{if } \ x_2 > 1. \end{cases}$$

We observe that hypotheses $(H_{x_1}^1)$, $(H_{x_1}^2)$ and $(H_{x_1}^3)$ are verified.

Let $x_1 \in \mathbb{R}$ be fixed. We notice that $\tilde{c} = 24(1 + |x_1|)$ and $\tilde{\lambda} = \frac{1}{16(1+|x_1|)}$. According to Theorem 4.1.1, only the zero solution is given for $\lambda \in (0, \frac{1}{24(1+|x_1|)})$, while for $\lambda > \frac{1}{16(1+|x_1|)}$ there are three solutions for the inclusion

$$x_2 \in \lambda\partial_{x_2}\tilde{f}(x_1, x_2), \ x_2 \geq 0, \tag{4.1.4}$$

which is equivalent to $x_2 \in R_{SV}^{\lambda}(x_1)$; Figure 4.1 supports geometrically these facts.

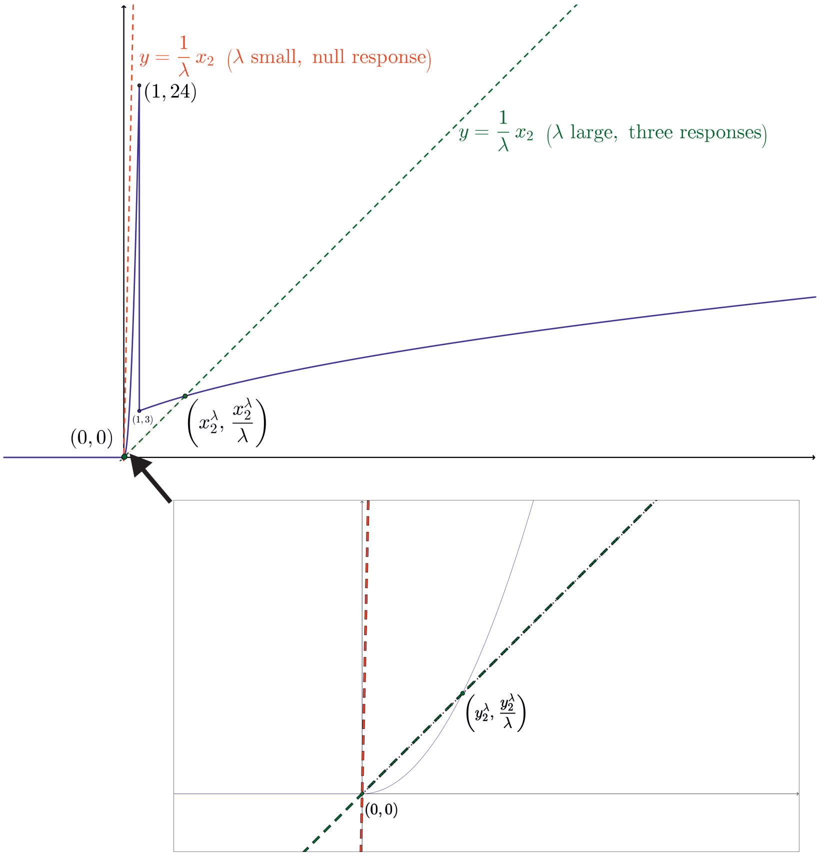

Figure 4.1: The graph of the set-valued map $\partial_{x_2}\tilde{f}(x_1, \cdot)$ (blue) and the lines $y = \frac{1}{\lambda}x_2$ for small (red) and large (green) parameters of λ. The intersections of $\partial_{x_2}\tilde{f}(x_1, \cdot)$ and the lines $y = \frac{1}{\lambda}x_2$ give the elements in the Stackelberg variational response set $R^\lambda_{SV}(x_1)$.

For λ large enough we solve the inclusion (4.1.4), obtaining that $R^\lambda_{SV}(x_1)$ contains exactly three elements; namely, $R^\lambda_{SV}(x_1) = \{0, x_2^\lambda, y_2^\lambda\}$ where $x_2^\lambda = \frac{9\lambda^2(1+|x_1|)^2+3\lambda(1+|x_1|)\sqrt{9\lambda^2(1+|x_1|)^2+48}}{8}$ and $y_2^\lambda = \frac{1}{24\lambda(1+|x_1|)}$. After a simple computation we conclude that the Stackelberg equilibrium response set is $R^\lambda_{SE}(x_1) = \{x_2^\lambda\}$ whenever λ is large. $\square$

In the rest of the chapter we sketch the proof of Theorem 4.1.1. From now one, without mentioning explicitly, we assume that the hypotheses of Theorem 4.1.1 are fulfilled.

4.2 Null Stackelberg response

Proof of Theorem 4.1.1 (a). By using nonsmooth analysis for locally Lipschitz functions, one can prove that $0 \in R^{\lambda}_{SV}(x_1)$ for every $\lambda > 0$.

Proof of Theorem 4.1.1 (b). By simple estimates we can prove that $R^{\lambda}_{SV}(x_1) = \{0\},\ \forall \lambda \in (0, \tilde{c}^{-1})$.

Remark 4.2.1 From game-theoretical point of view, Theorem 4.1.1 (a) means that the follower has always the possibility to choose the null strategy $x_2 = 0$ as a Stackelberg variational response. In fact, in this case, the follower refuses to participate at the game, his/her loss being $f_\lambda(x_1, 0) = 0$.

When the parameter is small enough, described in Theorem 4.1.1 (b), the Stackelberg variational response reduces to a unique element, which is the null strategy. In other words, he/she has no other reasonable strategy than the null solution. According to Proposition 2.2.1, one has that $R^{\lambda}_{SE}(x_1) \subseteq R^{\lambda}_{SV}(x_1)$, thus the Stackelberg equilibrium response set is either the null strategy or it is empty.

4.3 Geometry of Stackelberg responses

Let $x_1 \in K_1$ be fixed.

Lemma 4.3.1 (Sz. Nagy [50]) *Let $\lambda > 0$ be fixed. The functional $f_\lambda(x_1, \cdot)$ defined in* (4.1.1) *is bounded from below and coercive, i.e., $f_\lambda(x_1, x_2) \to +\infty$ whenever $\|x_2\| \to +\infty$. Moreover, $f_\lambda(x_1, \cdot)$ satisfies the Palais-Smale condition in the sense of Motreanu-Panagiotopoulos.*

Proposition 4.3.1 *The number $\tilde{\lambda}$ in* (4.1.3) *is well-defined and*

$$0 < \tilde{\lambda} < \infty.$$

Proof of Theorem 4.1.1 (c). Let us fix $\lambda > \tilde{\lambda}$.

Step 1. (First response) According to property (a), one has $0 \in R^{\lambda}_{SV}(x_1)$, which is the first (trivial) response.

Step 2. (Second response) Combining Lemma 4.3.1 and the global minimization argument, the validity of the Palais-Smale condition implies that the Motreanu-Panagiotopoulos-type functional $f_\lambda(x_1, \cdot)$ achieves its infimum at a point $x_2^{\lambda} \in \mathbb{R}^m$ which is a critical point in the sense of Motreanu-Panagiotopoulos. Moreover, $x_2^{\lambda} \neq 0$.

Step 3. (Third response) By the nonsmooth Mountain Pass theorem, it follows that the number

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f_\lambda(x_1, \gamma(t))$$

is a critical value for $f_\lambda(x_1, \cdot)$, where $\Gamma = \{\gamma \in C([0, 1], \mathbb{R}^m) : \gamma(0) = 0, \gamma(1) = x_2^\lambda\}$. Thus, if $y_2^\lambda \in K_2$ is the mountain pass-type critical point of $f_\lambda(x_1, \cdot)$ with $c_\lambda = f_\lambda(x_1, y_2^\lambda) > 0$, we clearly have that $y_2^\lambda \neq 0$ and $y_2^\lambda \neq x_2^\lambda$, which is the third response.

Summing up the above three steps, we conclude that

$$\{0, x_2^\lambda, y_2^\lambda\} \subset R^\lambda_{SV}(x_1),\ \forall\lambda > \tilde{\lambda}.$$

Remark 4.3.1 As we pointed out in Remark 4.2.1, the Stackelberg variational response set reduces to the null strategy whenever the parameter is small enough. However, when the parameter is beyond a threshold value (see Theorem 4.1.1 (c)), there are three possible Stackelberg variational responses; in this case, the follower enters actively into the game in order to minimize his loss. More precisely, besides the null strategy (see Step 1), he can choose the global minimum-type solution/response (see Step 2); in this case, his loss function takes a negative value, i.e., he is in a winning position. In the case when the player chooses the mountain pass-type minimax response (see Step 3), his payoff function takes a positive value.

4.4 The gap-interval

The aim of this section is twofold, formulated in the following two proposition.

Proposition 4.4.1 (Sz. Nagy [50]) *When $K_2 = \mathbb{R}^m$, we have $\tilde{c}^{-1} \leq \tilde{\lambda}$.*

Remark 4.4.1 In general, we have that $\tilde{c}^{-1} < \tilde{\lambda}$. Such a situation occurs e.g. when $m = 1$, $K_2 = [0, \infty)$ and the payoff function $\tilde{f} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is of class C^1 in the second variable.

Let $\eta > 1$ and $\tilde{f} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{f}(x_1, x_2) = (1 + |x_1|) \int_0^{x_2} \min\{(s - 1)_+, \eta - 1\} ds,$$

and $K_2 = \mathbb{R}$. For these choices we have

Proposition 4.4.2 (Sz. Nagy [50]) *The gap-interval $[\tilde{c}^{-1}, \tilde{\lambda}]$ can be arbitrarily small.*

Bibliography

[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications. J. Funct. Anal. 14 (1973), 349–381.

[2] R. Amir, I. Grilo, Stackelberg versus Cournot Equilibrium. Games and Economic Behavior. 26 (1999), 1–21.

[3] D. Azagra, J. Ferrera, F. López-Mesas, Non-smooth analysis and Hamilton-Jacobi equations on Riemannian manifolds. J. Funct. Anal. 220 (2005), 304–361.

[4] W. Ballmann, Manifolds of nonpositive curvature. Workshop Bonn 1984 (Bonn, 1984), 261268, Lecture Notes in Math., 1111, Springer, Berlin, 1985.

[5] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. 3(1922), 133–181.

[6] E. Begle, A fixed point theorem. Ann. of Math. (2)51 (1950), 544–550.

[7] G.C. Bento, J.G. Melo, Subgradient method for convex feasibility on Riemannian manifolds. J. Optim. Theory Appl. 152 (2012), no. 3, 773–785.

[8] C. Bereanu, P. Jebelean, Multiple critical points for a class of periodic lower semicontinuous functionals. Discrete Contin. Dyn. Syst. 33 (2013), no. 1, 47–66.

[9] V. Berinde, Generalized contractions in quasimetric spaces, Seminar on Fixed Point Theory (Preprint), Babeş-Bolyai University of Cluj-Napoca 3 (1993), 3-9.

[10] G. Bonanno, Some remarks on a three critical points theorem. Nonlinear Anal. 54 (2003), 651–665.

[11] J.M. Borwein, D. Preiss, A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions. Trans. Amer. Math. Soc. 303 (1987), 517–527.

[12] J.M. Borwein, Q.J. Zhu, Techniques of variational analysis, Springer, New York, 2005.

[13] H. Brezis, Analyse fonctionnelle, Théorie et applications, Masson, Paris, 1992.

[14] L.E.J. Brouwer, Über Abbildungen von Mannigfaltigkeiten. Math. Ann. 71(1911), 97–115.

[15] E. Cartan, Geometry of Riemannian spaces (Math Sci Press, Brookline, MA, 1983). Translated from French: Leçons sur la Géométrie des Espaces de Riemann, Paris, 1928.

[16] E. Cavazzuti, M. Pappalardo, M. Passacantando, Nash equilibria, variational inequalities, and dynamical systems. J. Optim. Theory Appl. 114 (2002), no. 3, 491–506.

[17] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions. Trans. Amer. Math. Soc. 215 (1976), 241–251.

[18] K.-C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations. J. Math. Anal. Appl. 80 (1981), 102–129.

[19] A. Chinchuluun, P.M. Pardalos, A. Migdalas, L. Pitsoulis (eds.): Pareto Optimality, Game Theory and Equilibria, Springer Optimization and Its Applications, vol. 17. Springer, New York, 2008.

[20] F.H. Clarke, Optimization and Non-smooth Analysis, John Wiley&Sons, New York, 1983.

[21] M. Degiovanni, M. Marzocchi, A critical point theory for nonsmooth functionals. Ann. Mat. Pura Appl. (4) 167 (1994), 73–100.

[22] M.P. do Carmo, Riemannian Geometry, Birkhäuser, Boston, 1992.

[23] I. Ekeland, On the variational principle. J. Math. Anal. App. 47 (1974), 324–353.

[24] I. Ekeland, Nonconvex minimization problems. Bull. Amer. Math. Soc. 1 (3) (1979), 443–474.

[25] F. Faraci, A. Kristály, One-dimensional scalar field equations involving an oscillatory nonlinear term. Discrete Contin. Dyn. Syst. 18 (2007), no. 1, 107–120.

[26] Cs. Farkas, A.É. Molnár, **Sz. Nagy**, A generalized variational principle in $b-$metric spaces. Matematiche (Catania) 69 (2014), no. 2, 205–221.

[27] M. Frigon, On a critical point theory for multivalued functionals and application to partial differential inclusions. Nonlinear Anal. 31 (1998), no. 5-6, 735–753.

[28] L. Gasiński, N.S. Papageorgiou, Nonsmooth critical point theory and nonlinear boundary value problems. Series in Mathematical Analysis and Applications, 8. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[29] F. Giannessi, A. Maugeri, P.M. Pardalos, P.M. (eds.), Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Methods, Nonconvex Optimization and its Applications, vol. 58. Kluwer, Dordrecht (2001).

[30] S. Grognet, Théorème de Motzkin en courbure négative. Geom. Dedicata 79 (2000), 219–227.

[31] H. Hopf, W. Rinow, Über den Begriff der vollständigen differentialgeometrischen Fläche. Comm. Math. Helvetici 3 (1)(1931) 209–225.

[32] P. Jebelean, G. Moroşanu, Mountain pass type solutions for discontinuous perturbations of the vector $p-$Laplacian. Nonlinear Funct. Anal. Appl. 10 (2005), no. 4, 591–611.

[33] P. Jebelean, G. Moroşanu, Ordinary $p-$Laplacian systems with nonlinear boundary conditions. J. Math. Anal. Appl. 313 (2006), no. 2, 738–753.

[34] J. Kobayashi, M. Ôtani, The principle of symmetric criticality for non-differentiable mappings. J. Funct. Anal. 214 (2004), no. 2, 428–449.

[35] A. Kristály, Detection of arbitrarily many solutions for perturbed elliptic problems involving oscillatory terms. J. Differential Equations 245 (2008), no. 12, 3849–3868.

[36] A. Kristály, Location of Nash equilibria: a Riemannian geometrical approach. Proc. Amer. Math. Soc. 138 (2010), 1803–1810.

[37] A. Kristály, Nash-type equilibria on Riemannian manifolds: a variational approach. J. Math. Pures Appl. (9) 101 (2014), no. 5, 660–688.

[38] A. Kristály, **Sz. Nagy**, Followers strategy in Stackelberg equilibrium problems on curved strategy sets. Acta Polytech. Hung. 10 (2013), no. 7, 69–80.

[39] A. Kristály, V. Rădulescu, Cs. Varga, Variational Principles in Mathematical Physics, Geometry, and Economics, Cambridge University Press, Encyclopedia of Mathematics and its Applications, No. 136, 2010.

[40] S. Lang, Fundamentals of Differential Geometry, Springer - Verlag, 1998.

[41] Yu. S. Ledyaev, Q.J. Zhu, Nonsmooth analysis on smooth manifolds. Trans. Amer. Math. Soc. 359 (2007), 3687–3732.

[42] X. Li, N. Huang, Generalized vector quasi-equilibrium problems on Hadamard manifolds. Optim. Lett. 9 (2015), no. 1, 155–170.

[43] C. Li, G. López, V. Martín-Márquez, Monotone vector fields and the proximal point algorithm on Hadamard manifolds. J. Lond. Math. Soc. (2) 79 (2009), no. 3, 663–683.

[44] P. Omari, F. Zanolin, Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential. Comm. Partial Differential Equations 21 (1996) 721–733.

[45] S. Marano, D. Motreanu, On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems. Nonlinear Anal. 48 (2002), no. 1, 37–52.

[46] J.F. McClendon, Minimax and variational inequalities for compact spaces. Proc. Amer. Math. Soc. 89 (4), 1983, 717–721.

[47] D. Moskovitz, L.L. Dines, Convexity in a linear space with an inner product. Duke Math. J. 5 (1939), 520–534.

[48] D. Motreanu, P.D. Panagiotopoulos, Minimax theorems and qualitative properties of the solutions of hemivariational inequalities. Nonconvex Optimization and its Applications, 29. Kluwer Academic Publishers, Dordrecht, 1999.

[49] **Sz. Nagy**, Stackelberg equilibria via variational inequalities and projections. J. Global Optim. 57 (2013), no. 3, 821–828.

[50] **Sz. Nagy**, Multiple Stackelberg variational responses, Studia Universitatis Babeş-Bolyai Mathematica, accepted (2015).

[51] J.F. Nash, Non-cooperative games. Ann. of Math. 54 (1951), 286–295.

[52] S.Z. Németh, Variational inequalities on Hadamard manifolds. Nonlinear Analysis 52 (2003), 1491–1498.

[53] A.J. Novak, G. Feichtinger, G. Leitmann, A differential game related to terrorism: Nash and Stackelberg strategies. J. Optim. Theory Appl. 144 (2010), no. 3, 533–555.

[54] D. O'Regan, R. Precup, Theorems of Leray-Schauder type and applications. Series in Mathematical Analysis and Applications, 3. Gordon and Breach Science Publishers, Amsterdam, 2001.

[55] J.-S. Pang, M. Fukushima, Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. CMS 2, (2005), 21–56.

[56] A. Petruşel, Cirić type fixed point theorems. Stud. Univ. Babeş-Bolyai Math. 59 (2014), no. 2, 233–245.

[57] A. Petruşel, I.A. Rus, M. Şerban, Fixed Points, Fixed Sets and Iterated Multifunction Systems for Nonself Multivalued Operators. Set-Valued Var. Anal. 23 (2015), no. 2, 223–237.

[58] R.R. Phelps, Convex sets and nearest points. Proc. Amer. Math. Soc. 8 (1957), 790–797.

[59] T. Rapcsák, Geodesic convexity in nonlinear optimization. J. Optim. Theory Appl. 69 (1991), no. 1, 169–183.

[60] B. Ricceri, A general variational principle and some of its applications, Fixed point theory with applications in nonlinear analysis. J. Comput. Appl. Math. 113 (2000), 401–410.

[61] J. Saint Raymond, On the multiplicity of the solutions of the equations $-\Delta u = \lambda f(u)$. J. Differential Equations 180 (2002) 65–88.

[62] H. von Stackelberg, Market Structure and Equilibrium: 1st Edition Translation into English, Bazin, Urch & Hill, Springer, 2011.

[63] W. Stanford, Pure strategy Nash equilibria and the probabilistic prospects of Stackelberg players. Oper. Res. Lett. 38 (2010), no. 2, 94–96.

[64] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems. Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986), no. 2, 77–109.

[65] M. Turinici, Zhong's variational principle is equivalent with Ekeland's. Fixed Point Theory 6 (1) (2005), 133–138.

[66] C. Udrişte, Convex Functions and Optimization Methods on Riemannian Manifolds, Mathematics and its Applications, 297. Kluwer Academic Publishers Group, Dordrecht, 1994.

[67] Y.S. Xia, Further results on global convergence and stability of globally projected dynamical systems. J. Optim. Theory Appl. 122 (2004), no. 3, 627–649.

[68] Y.S. Xia, J. Wang, On the stability of globally projected dynamical systems. J. Optim. Theory Appl. 106 (2000), no. 1, 129–150.

[69] L. Yongxing, S. Shuzhong, A generalization of Ekeland's ε-variational principle and its Borwein-Preiss variant. J. Math. Anal. Appl. 246(1) (2000), 308–319.

[70] E. Zeidler, Nonlinear functional analysis and its applications. III. Variational methods and optimization. Springer-Verlag, New York, 1985.

[71] J. Zhang, B. Qu, N. Xiu, Some projection-like methods for the generalized Nash equilibria. Comput. Optim. Appl. 45 (2010), no. 1, 89–109.