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GROUP ACTIONS AND MODULAR REPRESENTATION THEORY

Ph.D. Thesis

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Contents

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Introduction

As we can find out from [29] as well as from [15] the ordinary representation theory of finite groups dates back in the late nineteenth century. That was the period when F. G. Frobenius came up with his first idea on this theory while working with certain homogeneous polynomial associated with a finite group. Among others , Burnside was Frobenius's most important rival at that time and contributed to the developing of the character theory of finite groups. Basically, the idea of a representation was to use homomorphisms of a finite group G into the group of nonzero complex numbers. This lead to the "birth " of the so-called complex characters which played a significant role in the theory. Schur's contribution followed when he gave a new introduction to the theory based on facts from linear algebra. Schur is mainly known for his fundamental work on the representation as well as his work in number theory. One of the most fundamental results which he discovered at this time is today called Schur's Lemma. He also introduced the concept now known as the "Schur multiplier". Forty years later this multiplier turned out to be the second cohomology group with coefficients in the nonzero complex numbers.

At this point we are forced to go deeper into the details of some of the mathematical elements mentioned above. Following [26, 8.5] we see that for a finitely generated module M over the field of complex numbers $\mathbb{C}$ a representation of a finite group G is the same as giving a $\mathbb{C}G$-module structure on M. The representation in matter is called irreducible if the associated module is simple. As far as representations can be added, a representation is completely reducible if the corresponding $\mathbb{C}G$-module is semisimple. By Maschke's theorem the group algebra $\mathbb{C}G$ is semisimple. Reformulated in language of modules over the group algebra $\mathbb{C}G$ this theorem states that any such module can be built from its irreducible subrepresentations, i.e. simple direct summands. Actually this statement holds for any field, not necessarily of characteristic zero, but of positive characteristic that does not divide the order of G. Leonard Eugene Dickson was the one to prove that when the characteristic is positive and does not divide the order of the group then the representation theory is similar to that in characteristic zero.

Richard Brauer initiated from about 1940 onwards the study of linear representation of finite groups over a field that has a positive characteristic. The results developed by R. Brauer led to a significant progress in the classification of finite groups. Explicitly, (see [40]), Brauer suggested a direction to classify all finite nonabelian groups. In the theory initially developed by Brauer, the link between ordinary representation theory and modular representation theory is best exemplified by considering the group algebra of the group G over a complete discrete valuation ring $\mathcal{O}$ with residue field k of positive characteristic p and field of fractions $\mathcal{K}$ of characteristic 0. The structure of $\mathcal{O}G$ is closely related both to the structure of the group algebra kG and to the structure of the semisimple group algebra $\mathcal{K}G$. Further details can be found in [25]. Here we only mention that when Maschke's theorem does not hold, that is when p divides the order of G, the group algebra may be decomposed as the direct sum of block algebras. Each block algebra corresponds to a unique primitive central idempotent. For each indecomposable $\mathcal{O}G$-module, there is only one such primitive idempotent that does not annihilate it, and the module is said to belong to that block. In particular, each simple module belongs to a unique block. This connection between blocks of the group algebra and irreducible representations of the group are all described in Brauer's three main theorems. Brauer's first main theorem concerning the blocks of the group algebra together with their defect groups is often used in the paper.

Hidden between the lines of the proof of Brauer's first main is the information that the action of the finite group G on itself provides more than only conjugacy classes representing a $\mathcal{O}$-basis in the center of the group algebra. From [23] we find out that the conjugation action of G on itself gives $\mathcal{O}G$ a structure of a G-algebra. Moreover, this concept can be generalized to obtain results similar to Brauer's first theorem on more general G-algebras.

The paper is divided in six chapters, each of them is shortly presented in the following:

The first chapter is devoted to some of most useful results for this thesis. We only work with finitely generated modules over a discrete valuation ring $\mathcal{O}$ having the residual field k of positive characteristic. There are only few situations in which we assume k to be algebraically closed. For a finite group G we start by giving the definition of a *G-algebra* and of some other mathematical objects such as the *relative trace map*, the *Brauer quotient*, the *Brauer map, pointed groups* together with the relations between them and the *defect pointed groups* of a pointed group. All of the above are given in order to state one of the most important result in the modular representation theory, i.e. the *Lifting idempotents theorem.* In the second paragraph of the first chapter we explicitly give the structures of group graded algebras and of the particular cases of the crossed product and of the twisted group algebra. Next, we simultaneously deal with G-algebras and *G-interior* algebras and introduce two well-known types of induction, one is due to Puig and the other to Turull. The fifth paragraph contains basic correspondence such as *The Green correspondence, The Burry-Carlson-Puig Theorem, The Brauer correspondence* for the blocks of the group algebra and *The Harris - Knörr Theorem.* For the last result we actually provide a different proof than the ones found in the literature. Further we shortly characterize the defect pointed groups of a block in terms of the Brauer pairs associated to the same block. In the seventh part we give the general construction of a Clifford extension that can be done with a group graded algebra and a block of its identity component. This is a situation often found in the work of E.C. Dade. Finally at the end of the first chapter we introduce the *fusions* on G-interior algebras. These elements were originally introduced in [35], here we only expose them such that they best fulfill our purposes.

In the second chapter we first deal with the group algebra. Theorem 2.2.1 shows that for a finite group G, a subgroup L and an L-algebra B, the induction to G in the sense of Puig of the skew group algebra $S = B * L$ is isomorphic to the skew group algebra of the induction to G in the sense of Turull of B. Moreover, the obtained isomorphism is one of G-interior algebras.

In the first part of the third chapter we work with a G-algebra A, where G is a finite group. We state Definition 3.1.1 which gives the conditions of a point of A^G to cover a point of A^N, where N is normal in G. We then fix β, a point of A^N having defect group Q, and δ, a point of $A^{N_N(Q)}$, the Green correspondent of β. In Theorem 3.1.2 we show that the Green correspondence for points induces a defect group preserving bijection between the points of A^G covering β and the points of $A^{N_G(Q)}$ covering δ. When A is the endomorphism algebra of an $\mathcal{O}G$-module M, this result matches the main result of [2]. Working with divisors on an inductively complete G-algebra A, in Theorem 3.1.4 we managed to give a completion of the Green correspondence.

The second part of chapter three employs p-permutation G-algebras, as general setting. We give another definition for covering points, namely Definition 3.2.4, and with this definition Theorem 3.2.7 proves that the main result of [24] can be generalized to any p-permutation G-algebra.

In the end of the third chapter we give some connections between the Green correspondence and the Brauer correspondence.

The following two chapters deal with *Clifford extensions* associate with blocks and with points. The main result of Chapter 4 is actually an adaptation of the main result of [17] in the case of an arbitrary ground field k. More explicitly, we consider a normal subgroup K of the finite group H. We denote by $G = H/K$ and we choose a block b of the identity component of the G-graded centralizer

$$C_{\mathcal{O}H}(\mathcal{O}K) = (\mathcal{O}H)^K.$$

Then we let P denote a defect group of b in K and we consider $\bar{b}$, the Brauer correspondent block of b also having defect group P. It is well-known that $\bar{b}$ is a block of the identity component of the $C_H(P)/C_K(P)$-graded centralizer

$$C_{kC_H(P)}(kC_K(P))^{N_K(P)} = kC_H(P)^{N_K(P)}.$$

Due to the properties of the Brauer quotient in the case of group algebra we construct two Clifford extensions, one associated with b and the other with a block e of $kC_H(P)$ that is associated with $\bar{b}$. In Theorem 4.3.1 we compute the elements that characterize the first extension and show that the mentioned Clifford extensions are isomorphic.

Note that the group algebra $\mathcal{O}K$ is a K-interior H-algebra provided that K is normal in H. The idea of the fifth chapter is to replace $\mathcal{O}K$ with an arbitrary K-interior H-algebra A. In this case $\mathcal{O}H$ is replaced by $\hat{A} = \bigoplus_{g\in G} \hat{A}_g$, where for all $g \in G$ we have $\hat{A}_g = A \otimes x$, for some representative $x \in g$. This situation only gives the inclusion

$$C_{\hat{A}}(A) \subseteq \hat{A}^K,$$

and for this matter we are forced to use the lager algebra $\hat{A}^K$. Hence in this case β is a point of A^K, that is a $(A^K)^*$-conjugacy class of a primitive idempotent of A^K. A series of complications arise in this situation since we can not work with the whole point β, for we are forced to chose an element $j \in \beta$. The stabilizer G_j is not the same with the normalizer $N_H(K_\beta)$ and then we must introduce some infinite groups containing $N_H(K_\beta)$. These infinite groups make it difficult to use the same techniques as in the previous chapter in order to prove that the Clifford extensions of β and of $\bar{\beta} = \mathrm{Br}_P(\beta)$ are isomorphic. Indeed, Chapter 4 uses group graded algebra basic theory while in the proof of Theorem 5.5.1, and in Chapter 5 we turn to the use of *fusions* in G-interior algebras and to the results on lifting idempotents. At the end of Chapter 5 we reconsider de situation, but for a slight particularization. More exactly we assume that β is generated be a central element, i.e. a singleton. This particularization serves for obtaining an extended main result and also for using other techniques in proving it.

In the last chapter we return to the isomorphic Clifford extension of the Brauer correspondent blocks b and $\bar{b}$. We observe that the isomorphism between the corresponding Clifford extensions of b and of $\bar{b}$ induces a bijection between the blocks covering b and $\bar{b}$ respectively. Moreover we prove in Theorem 6.4.1 that this bijection preserves the defect groups and coincides with the Brauer correspondence. So that the main result of Chapter 4 implies the main result of [24].

I would like to thank Professor Andrei Marcus for his well received guidance and support, I am grateful for his patience and his involvement. I would also like to acknowledge all of the members of the Department of Algebra who always offered useful advice. Last but not least, I thank my family and all of my friends who stood beside me throughout this entire period of time.

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1. General Results in Modular Representation Theory of Finite Groups

Let p be a prime, and let $\mathcal{O}$ be a discrete valuation ring with residue field k of characteristic p. We make no assumptions on the size of $\mathcal{O}$ and k, also allowing $\mathcal{O} = k$. In order to avoid unending field extensions, in some situations we will assume that k is algebraically closed.

1.1. **Green Theory and Defect Theory.**

Let G be a finite group and let A be an $\mathcal{O}$-algebra.

Definition 1.1.1. The $\mathcal{O}$-algebra A is a G-algebra whenever there is a group morphism

$$\varphi : G \to \mathrm{Aut}_{\mathcal{O}}(A),$$

between the group G and the group of algebra automorphisms of A.

The action of $g \in G$ on $a \in A$ is denoted by

$$\varphi(g)(a) =: a^g.$$

Definition 1.1.2. If A_1 and A_2 are two G-algebras a map $f : A_1 \to A_2$ is a homomorphism of G-algebras if it is a morphism of $\mathcal{O}$-algebras and satisfies

$$f(a^g) = f(a)^g$$

for all $a \in A_1$ and all $g \in G$.

1.1.3. On an G-algebra, for any subgroup L in G we denote by A^L the subalgebra of A consisting of elements fixed under the action of L. If L' is another subgroup of G such that $L \subseteq L'$ and if $[L/L']$ denotes a system of representative of the classes of L' in L, then the map

$$\mathrm{Tr}_{L'}^{L} : A^{L'} \to A^L,$$
$$\mathrm{Tr}_{L'}^{L}(a) = \sum_{g \in [L/L']} a^g$$

for any $a \in A^{L'}$, is the relative trace map. Obviously it is a module homomorphism. Moreover the image $A_{L'}^L := \mathrm{Tr}_{L'}^L(A^{L'})$ is an ideal of A^L.

Therefore a G-algebra allows the construction of the quotient

$$A(H) := A^H / \sum_{L<H} A_L^H,$$

for any subgroup H in G. Here the sum runs trough all the proper subgroups of H. Employing a Sylow p-subgroup Q of H one easily checks, see [41, Lemma 11.7], that if H is not a p-group then $A(H) = 0$. This leaves us with few choices of subgroups in G such that $A(H)$ is non-zero. So let P denote a p-subgroup of the finite group G. We introduce the Brauer map:

$$\mathrm{Br}_P : A^P \to A(P),$$

sending each $a \in A^P$ to

$$\bar{a} = a + \sum_{Q<P} A_Q^P.$$

Obviously the Brauer map is a morphism of $N_H(P)$-algebras.

1.1.4. Let e be a primitive idempotent of A^G.

Definition 1.1.5. A p-subgroup P of G is a defect group of e if P is a minimal subgroup such that $e \in A^G_P$.

For any primitive idempotent e in A^G such p-subgroups of G always exist. Indeed, this follows from [23, 4i. Theorem]. Moreover any other defect group of e is G-conjugate to P.

We intend to use a generalization of this definition. For that we introduce the notion of a point. As it is used in [41] or in [34], a point of G on A is a A^*-conjugacy class of a primitive idempotent $j \in A^G$. Clearly this definition can be given for any subgroup of G. So for any subgroup of G, hence for G itself we introduce the notation $\mathcal{P}(A^G)$. This denotes the set of points of A^G. We denote such a conjugacy class by β and then the pair $G_\beta := (G, \beta)$ is called a pointed group. If L and K are subgroups of G such that $K \subseteq L$ and if α and β are points of K on A and of L on A respectively, then $K_\alpha \leq L_\beta$ if and only if for any $j \in \beta$ there exists $i \in \alpha$ appearing in a decomposition of j in A^K, that is $ji = ij = i$. Moreover the group G acts on the set of pointed groups of a G-algebra and the action is compatible with the inclusion between pointed groups. That is, if $g \in G$ and $K_\alpha \leq L_\beta$ as above then $(K_\alpha)^g \leq (L_\beta)^g$ is equivalent to $K^g_{\alpha^g} \leq L^g_{\beta^g}$. Also let $r^L_K : A^L \to A^K$ denote the standard inclusion. We shortly present all of the above statements in the next proposition

Proposition 1.1.6. *On a G-algebra A the following hold.*

i) For any subgroup L of G there is a bijection between the points of A^L and the maximal ideals of A^L. If $\beta \in \mathcal{P}(A^L)$ then the corresponding maximal ideal m_β verifies $\beta \nsubseteq m_\beta$. Moreover the quotient A^L/m_β is a simple $N_G(L_\beta)$-algebra over k.

ii) If K_α and L_β are pointed groups on A then the following are equivalent.
(1) $K_\alpha \leq L_\beta$;
(2) $(r^L_K)^{-1}(m_\alpha) \subseteq m_\beta$;
(3) $m_\alpha \cap A^L \subseteq m_\beta$.

Definition 1.1.7. A local pointed group P_γ (local means $\mathrm{Br}_P(\gamma) \neq 0$) is a defect pointed group of G_β if P is a minimal subgroup such that $\beta \in A^G_P$.

1.1.8. Instead of working with this definition sometimes we will quote [41, Proposition 18.5]. This result gives equivalent statements with Definition 1.1.7 above. Most frequently we will verify that P is a defect group of the point β if and only if $\beta \in A^G_P$ and $\mathrm{Br}_P(\beta) \neq 0$.

The following theorem is a mix of [41, Theorem 3.2] and [34, Proposition 3.23]. Actually, these are the general properties of lifting idempotents in $\mathcal{O}$-algebras.

Theorem 1.1.9. *Let $f : A \to B$ be a surjective $\mathcal{O}$-algebra homomorphism. The following statements hold.*

i) The map $A^* \to B^*$ is surjective.
ii) If α is a point of A such that $\alpha \notin \mathrm{Ker}(f)$, then $f(\alpha)$ is a point of B.
iii) There is a bijection between the sets $\mathcal{P}(A \setminus \mathrm{Ker}(f))$ and $\mathcal{P}(B)$.
iv) Let $\mathcal{I}$ be an ideal of A. The point α belongs to $\mathcal{I}$ if and only if $f(\alpha)$ belongs to $f(\mathcal{I})$.

Remark 1.1.10. The Mackey formula is also widely used in this paper. Let $[L\setminus G/K]$ denote a set of representatives of the double cosets LgK of any two subgroups L and K in G. Then we have

$$\mathrm{Tr}_K^G(a)=\sum_{g\in[L\setminus G/K]}\mathrm{Tr}_{L\cap K^g}^L(a^g).$$

1.2. **Group graded algebras and crossed products.** Let G denote a finite group. A G-graded $\mathcal{O}$-algebra is a direct sum of $\mathcal{O}$-modules

$$\hat{A}=\bigoplus_{g\in G}\hat{A}_g,$$

where for all $g,h\in G$ we have

$$\hat{A}_g\cdot\hat{A}_h\subseteq\hat{A}_{gh}.$$

Note that we usually have $1_{\hat{A}}\in\hat{A}_1$ and that in most of the times, since usually it is a group acted algebra, we will give up the hat and the index on $\hat{A}_1$. Whenever the equality

$$\hat{A}_{gh}=\hat{A}_g\cdot\hat{A}_h$$

holds for all $g,h\in G$, we say that $\hat{A}$ a strongly G-graded algebra. Instead of checking this equality one can verify

$$\hat{A}_g\cdot\hat{A}_{g^{-1}}=\hat{A}_1,$$

for all $g\in G$.

Then, if $A:=\hat{A}_1$ is a G-algebra we construct the so-called skew group algebra of A and G. We denote this by $\hat{A}=A*G$. The product is as follows: for any $a\cdot g$ and $c\cdot h$ in $A*G$ we have

$$(a\cdot g)(c\cdot h)=ac^g\cdot gh.$$

An example of all the structures that we introduced above is the twisted group algebra. Consider the central extension of G by k^*

$$1\to k^*\to\hat{G}\to G\to 1.$$

To this extension we can associate the twisted group algebra denoted $k\hat{G}$, which is the usual group algebra of the infinite group $\hat{G}$ having a basis indexed by the elements of G. This basis is obtained by lifting the elements of G, explicitly $\{\hat{x}\mid x\in G\}$. The multiplication is given by $\hat{x}\hat{y}=\alpha(x,y)\widehat{xy}$, where $\alpha(x,y)\in k^*$ is the image of a 2-cocycle associated with this central extension.

1.3. **G-algebras and interior G-algebras.** Let L be a subgroup of a finite group G, and consider an L-algebra A. We use the definition of the induction of A as in [43, Section 8]. The induction of A from L to G is

$$\mathrm{Ind}_L^G(A)=\mathcal{O}G\otimes_{\mathcal{O}L}A,$$

where an element $g\otimes a\in\mathcal{O}G\otimes_{\mathcal{O}L}A$ is denoted by ${}^ga$, and for $b\in\mathrm{Ind}_L^G(A)$ and $g\in G$ the element ${}^gb$ is the result of G acting on b. If $a,b\in A$ and $g_1,g_2\in G$, the multiplication in this algebra is given by:

$$({}^{g_1}a)({}^{g_2}b)=\begin{cases}{}^g(ab) & \text{if } g=g_1=g_2;\\ 0 & \text{if } g_1L\neq g_2L.\end{cases}$$

As noted in [31, 4.3], this is a particular case of the induction of crossed products introduced in [27].

Definition 1.3.1. The $\mathcal{O}$-algebra A is called G-interior if there is a morphism of groups

$$\phi : G \to A^*,$$

between the group G and the group of units of A.

In this case we denote $a \cdot g = a\phi(g)$ and $g \cdot a = \phi(g)a$. Also note that any G-interior structure gives rise to a G-algebra structure on A. Indeed, the map sending G to the algebra automorphism $a \mapsto g^{-1} \cdot a \cdot g$ for any $a \in A$ is a morphism of groups.

Definition 1.3.2. If A and B are G-interior algebras, a map $f : A \to B$ is a morphism of G-interior algebras if it is a morphism of $\mathcal{O}$-algebras and satisfies

$$f(g \cdot a \cdot h) = g \cdot f(a) \cdot h,$$

for all $a \in A$ and $g, h \in G$.

Now let A denote an L-interior algebra. There is another type of induction which is due to Puig and which can be applied to the interior L-algebra A. Explicitly we can construct

$$\mathcal{O}G \otimes_{\mathcal{O}L} A \otimes_{\mathcal{O}L} \mathcal{O}G.$$

Recall that its algebra structure is given by

$$(g \otimes a \otimes g^{'})(g_1 \otimes a_1 \otimes g_1^{'}) = \begin{cases} g \otimes a \cdot g^{'}g_1 \cdot a_1 \otimes g_1^{'} & \text{if } g^{'}g_1 \in G \\ 0 & \text{if } g^{'}g_1 \notin G, \end{cases}$$

where $g, g^{'}, g_1, g_1^{'} \in G$ and $a, a_1 \in A$. The interior G-algebra structure is given by $g \cdot (x \otimes a \otimes y) = gx \otimes a \otimes y$ and $(x \otimes a \otimes y) \cdot g = x \otimes a \otimes yg$ for all $g, x, y \in G$ and $a, a_1 \in A$.

At last, if A is an G-interior $\mathcal{O}$-algebra then $A * G$ denotes the skew group algebra of A. Recall that the multiplication is given by $(a_1 \cdot g_1)(a_2 \cdot g_2) = a_1 a_2^{g_1} \cdot g_1 g_2$ for any $a_1, a_2 \in A$ and $g_1, g_2 \in G$. Note that since A is G-interior it is an G-algebra, in this context $a_2^{g_1}$ means ${g_1}^{-1} \cdot a_2 \cdot g_1$.

1.4. **Basic Correspondences.**

Theorem 1.4.1 (The Green correspondence)**.** *Let A be a G-algebra, let P_γ be a local pointed group on A and let H be a subgroup of G containing*

$$N_G(P_\gamma) = \{g \in G \mid g \in N_G(P) \text{ and } \gamma^g = \gamma\}.$$

There is a bijective correspondence between the sets

$$\{\alpha \mid \alpha \in \mathcal{P}(A^G) \text{ such that } P_\gamma \text{ is a defect of } G_\alpha\}$$

and

$$\{\beta \mid \beta \in \mathcal{P}(A^H) \text{ such that } P_\gamma \text{ is a defect of } H_\beta\}.$$

Moreover, if m_α and m_β denote the corresponding maximal ideals of A^G and of A^H associated with the correspondent points α and β respectively, then

$$m_\alpha = (r_H^G)^{-1}(m_\beta) = A^G \cap m_\beta.$$

Even more we have $P_\gamma \leq H_\beta \leq G_\alpha$.

Closely related to The Green Correspondence is the following result known as The Burry-Carlson-Puig Theorem.

Theorem 1.4.2. *Let A be an G-algebra, P_γ a local pointed group on A and H a subgroup of G containing $N_G(P_\gamma)$. Take $\alpha \in \mathcal{P}(A^G)$ and $\beta \in \mathcal{P}(A^H)$ such that $P_\gamma \leq H_\beta \leq G_\alpha$. Then P_γ is a defect pointed group of H_β if and only if P_γ is a defect pointed group of G_α. In these conditions β and α are two Green-correspondent points.*

Theorem 1.4.3 (The Brauer correspondence)**.** *There is a bijective correspondence between the set of blocks of $\mathcal{O}G$ having defect group P and the set of blocks of $\mathcal{O}H$ having the same defect group P.*

Consider a normal subgroup N of the finite group G and let b denote a block of $\mathcal{O}N$. Note that $\mathcal{O}N$ and $\mathcal{O}G$ are both G-invariant algebras. Denote by G_b the stabilizer of b in G under the conjugation action of G on $\mathcal{O}N$. The element

$$s = \sum_{g \in [G/N]} b^g$$

is an idempotent of $Z(\mathcal{O}N)$ that also lies in $Z(\mathcal{O}G)$. A block B of $\mathcal{O}G$ is said to *cover* b if we have

$$Bs = sB = B.$$

Let D be a p-subgroup of N representing a defect group of b. T

Theorem 1.4.4 (Harris-Knörr)**.** *The Brauer map determines a defect group preserving bijective correspondence between the blocks of $\mathcal{O}G$ covering b and the blocks of $\mathcal{O}N_G(D)$ covering b_1. Moreover this correspondence induced by the Brauer map coincides with the Brauer correspondence.*

1.5. **Brauer pairs on p - permutation algebras.** Let G be a finite group. A Brauer pair (P, e) consists of a p-subgroup P of G and of a block e of $kC_G(P)$. Let γ be a point of $\mathcal{O}G^P$ such that P_γ is a local pointed group. Then for any $i \in \gamma$, using the Brauer map

$$\mathrm{Br}_P : \mathcal{O}G^P \to kC_G(P),$$

the primitive idempotent $\mathrm{Br}_P(i)$ is different from zero in $kC_G(P)$. The block e decomposes in $kC_G(P)$ as a sum of primitive idempotents. If $\mathrm{Br}_P(i)e = \mathrm{Br}_P(i)$ we say that P_γ is associated with e.

The next result is a mix between [41, Lemma 40.12] and [41, Proposition 40.13]. Its purpose is to characterize the relation between the defect pointed groups of a block and the Brauer pairs associated with them.

Proposition 1.5.1. *Let b be a block of $\mathcal{O}G$ and let (P, e) be a Brauer pair. Then P is a defect group of b that satisfies $\mathrm{Br}_P(b)e = e$ if and only if there exists a unique defect pointed group P_γ of b which is associated with (P, e). Moreover, this relation induces a bijection between the set of defect pointed groups of b and the set of Brauer pairs (P, e) verifying $\mathrm{Br}_P(b)e = e$ such that P is a defect of b.*

1.6. **Group graded algebras and Clifford extensions.** In this section k is algebraically closed . Consider a G-graded algebra $\hat{A}$ indexed by the finite group G. This $\mathcal{O}$-algebra satisfies

$$\hat{A} = \bigoplus_{g\in G} \hat{A}_g.$$

Also let b denote a block of $\hat{A}_1$. Then b lies in the center of

$$C_{\hat{A}}(\hat{A}_1) = \{a \in \hat{A} \mid aa_1 = a_1 a \text{ for all } a_1 \in \hat{A}_1\}.$$

By [18, Paragraph 2] the centralizer

$$C_{\hat{A}}(\hat{A}_1) = \bigoplus_{g\in G} C_{\hat{A}}(\hat{A}_1)_g$$

has a natural structure of a G-algebra. We denote by G_b the stabilizer of b in G and then $bC := C_{b\hat{A}b}(b\hat{A}_1) = bC_{\hat{A}}(\hat{A}_1)$ has the following structure

$$bC = \bigoplus_{g\in G_b} bC_g,$$

where $bC_g = bC \cap \hat{A}_g$ for all $g \in G_b$. The set

$$G[b] = \{g \in G_b \mid bC_g \cdot bC_{g^{-1}} = bC_1\}$$

is a normal subgroup of G_b (see [17, Proposition 2.17]). In this situation

$$C[b] := \bigoplus_{g\in G[b]} bC_g$$

is a strongly $G[b]$-graded G_b-invariant algebra. But $bC_1 = bZ(\hat{A}_1) = Z(b\hat{A}_1)$ is a local ring and then by applying [39, Lemma 1.1] we see that $C[b]$ is actually a crossed product of $\hat{A}_1$ with $G[b]$. Taking the quotient

$$C[b]/J_{gr}(C[b]) = \bigoplus_{g\in G[b]} bC_g/bC_g J(C[b]_1)$$

we obtain a twisted group algebra which is a crossed product of $k \simeq C[b]_1/J(C[b]_1)$ with $G[b]$ and in the same time a G_b-algebra. This twisted group algebra corresponds uniquely to the *Clifford extension*

$$1 \to k^* \to \mathrm{hU}(C[b]/J_{gr}(C[b])) \to G[b] \to 1.$$

Here $\mathrm{hU}(C[b]/J_{gr}(C[b]))$ denotes the homogeneous units of $C[b]/J_{gr}(C[b])$.

1.7. **Fusions on interior algebras.** Throughout this section N denotes a normal subgroup of the finite group G. Fusions on G-interior algebras were first introduce by Puig in [35]. Later on, dealing with N-interior G-algebras the concept was generalized as we can see in [34].

1.7.1. Although in [34, §8] and in [35] the author deals with the so-called *exomorphisms*, here we try to avoid them and simply use group homomorphisms to define the *fusions* as it is done in [38, 2.2].

Let A be a G-interior algebra and let K_β, H_α be two pointed groups on A. If $i \in \alpha$ then iAi is an H-interior algebra while for $j \in \beta$ the algebra jAj is K-interior.

Definition 1.7.2. A group isomorphism $\varphi : K \simeq H$ is an A-fusion from K_β to H_α if there exists $a \in A^*$ such that

$$(y \cdot j)^a = \varphi(y) \cdot i,$$

for all $y \in K$.

We denote by $\mathrm{F}_A(K_\beta, H_\alpha)$ the set of A-fusions from K_β to H_α. Note that the A-fusions can be composed, meaning that if L_γ is another pointed group on A then for $\varphi \in \mathrm{F}_A(K_\beta, H_\alpha)$ and $\psi \in \mathrm{F}_A(H_\alpha, L_\gamma)$ we get $\psi \circ \varphi \in \mathrm{F}_A(K_\beta, L_\gamma)$. If $K_\beta = H_\alpha$ then we denote $\mathrm{F}(K_\beta) := \mathrm{F}(K_\beta, K_\beta)$. The following results follow immediately from Definition 1.7.2.

Proposition 1.7.3. *The set $\mathrm{F}_A(K_\beta)$ together with the composition of morphisms forms a group.*

Proposition 1.7.4. *If the pointed groups K_β and H_α are fusion-related via Definition 1.7.2 then, for any $j \in \beta$ and $i \in \alpha$ there exists $a \in A^*$ such that $(K \cdot j)^a = H \cdot i$, hence $(jAj)^a = iAi$.*

1.7.5. Now let A denote an N-interior G-algebra. As in the case of G-interior algebras we intend to give the characterization of fusions that relies on group isomorphisms. So let K and H be any two subgroups of G and let β and α be two points of K and of H on A respectively.

Definition 1.7.6. Let $j \in \beta$ and $i \in \alpha$. A group isomorphism $\varphi : K \to H$ that verifies $\varphi(y) \in yN$ for all $y \in K$ is an A-fusion from K_β to H_α if there exists $a \in A^*$ such that for any $y \in K$ we have

$$j^a = i, \qquad (ja)^y \cdot y^{-1}\varphi(y) = ja.$$

Proposition 1.7.7. *If the pointed groups K_β and H_α are fusion-related via Definition 1.7.6 then for any $j \in \beta$ and $i \in \alpha$ there exists $a \in A^*$ such that $(jAj)^a = iAi$.*

1.7.8. The group $\mathrm{F}(K_\beta)$ has the following property.

Proposition 1.7.9. *Let $j \in \beta$. The automorphisms $\varphi : K \to K$ that satisfies $\varphi(y) \in yN$ for all $y \in K$ is an A-fusion of K_β if and only if the exists $a \in (jAj)^*$ such that $a^y \cdot y^{-1}\varphi(y) = a$ for all $y \in K$.*

2. Induction and Skew Group Algebras

In section 1.3 we introduced the structure of a skew group algebra and the notions of induction in the sense of Puig and in the sense of Turull. In this chapter we give a connection between them.

2.1. **Puig and Turull induction for G-algebras and interior G-algebras.** Since we work with two types of induction we will modify the notation from [30] by replacing "Ind" with "IndP", and that from [43] by replacing "Ind" with "IndT".

For a finite group G and for a subgroup L of G we consider an L-interior algebra C.

We connect the two types if inductions in the following result.

Theorem 2.1.1. *There is a surjective homomorphism of interior G-algebras from* $\mathrm{IndT}_{\mathcal{O}G}(C) * G$ *onto* $\mathrm{IndP}_{\mathcal{O}G}(C * L)$*.*

2.2. **An isomorphism between two types of induction.** As in the second part of [12] we consider a subgroup L of a finite group G. Now let B be an L-algebra over $\mathcal{O}$ and consider the skew group algebra $S := B * L$ of B and L. Let $A = \mathrm{Ind}_L^G(B)$ be the above induced algebra in the sense of Turull, and denote by $R := A * G$.

We have the following result.

Theorem 2.2.1. *The map*

$$\varphi : \mathcal{O}G \otimes_{\mathcal{O}L} S \otimes_{\mathcal{O}L} \mathcal{O}G \to R, \quad g \otimes s \otimes f \mapsto g \cdot s \cdot f,$$

where $g, f \in G$ and $s \in S$, is an isomorphism of G-graded G-interior algebras, and the diagram

$$\begin{array}{ccc} \mathcal{O}G \otimes_{\mathcal{O}L} S \otimes_{\mathcal{O}L} \mathcal{O}G & \longrightarrow & R \\ \uparrow & \nearrow & \\ \mathcal{O}G & & \end{array}$$

of G-graded G-interior algebras is commutative.

3. Correspondences In G-algebras

Harris and Knörr proved in [24] that there is a defect group preserving correspondence between the covering blocks of two Brauer correspondent blocks. A module theoretical version of this result exists and it is due to Alperin [2]. Here we present the content of [9] showing that these two results still hold in a more general setting, that is the case of points on some G-algebras over a discrete valuation ring.

3.1. The Green correspondence for covering points.

Let N be a normal subgroup of G and let $\alpha \in \mathcal{P}(A^G)$ and $\beta \in \mathcal{P}(A^N)$ such that $N_\beta \leq G_\alpha$. Suppose P_γ is a defect pointed group of G_α.

Definition 3.1.1. If $N \cap {}^gP$ is the minimal subgroup of N with the property $\beta \in A^N_{N\cap {}^gP}$, then we say that α *covers* β. In this case there exists a point $\gamma' \in \mathcal{P}(A^Q)$, $Q = N \cap {}^gP$, such that $Q_{\gamma'}$ is a defect pointed group of N_β.

Let $\beta \in \mathcal{P}(A^N)$ such that N_β has defect pointed group $Q_{\gamma'}$ for some point $\gamma' \in \mathcal{P}(A^Q)$. Since $N_N(Q_{\gamma'}) \subseteq N_N(Q)$, there is a unique point $\delta \in \mathcal{P}(A^{N_N(Q)})$ corresponding to β under the Green Correspondence, moreover $N_N(Q)_\delta$ has $Q_{\gamma'}$ as defect pointed group (see 1.4.1).

Theorem 3.1.2. *There is a one-to-one correspondence between points of A^G covering β and points of $A^{N_G(Q)}$ covering δ. Moreover, if $Q_{\gamma'}$ is a defect pointed group of $N_N(Q)_\delta$, hence a defect pointed group of N_β, and P_γ is a defect pointed group of $N_G(Q)_\epsilon$, hence of G_α then ${}^g(Q_{\gamma'}) \leq P_\gamma$ for some $g \in N_G(Q)$.*

Remark 3.1.3. Let M be a kG module. By applying the above theorem to the G-algebra $A := \mathrm{End}_k(M)$ one obtains Alperin's result on modules. Indeed, by [41, Example 13.4], we see that in this case, to any point it corresponds an indecomposable direct summand of M. Moreover, the definition of the covering points from this paragraph applied to A yields the definition from [2].

In the case of a inductively complete G-algebra, a more precise version of the Green correspondence holds.

Theorem 3.1.4 (The Green Correspondence)**.** *Let A be an inductively complete G-algebra and let P_γ be a local pointed group on A, also let H be a subgroup of G containing $N_G(P_\gamma)$. Then, if α is a point of G on A with defect pointed group P_γ there exists a unique point β of H on A with the same defect pointed group such that $\beta \subset \mathrm{res}^G_H(\alpha)$ or equivalently $\alpha \subset \mathrm{ind}^G_H(\beta)$.*

3.2. The Harris-Knörr correspondence for covering points of permutation algebras.

3.2.1. As before, let N be a normal subgroup of the finite group G.

Let D be a p-subgroup of N and consider the set

$$\mathcal{Q} = \{Q \leq G \mid Q \text{ is a } p\text{-subgroup with } Q \cap N = D\}.$$

In what follows we use the "bar" notation for the image under the Braur morphism determined by D. With this setting we have:

Proposition 3.2.2. *The Brauer morphism*

$$\mathrm{Br}_D : A^D \to A(D)$$

induces defect group preserving bijection between the points of G on A with defect group in $\mathcal{Q}$ and the points of $N_G(D)$ on $A(D)$ with defect group in $\mathcal{Q}$. Moreover, if α and $\bar{\alpha}$ correspond via this bijection then, Q_ϵ is a defect pointed group of α if and only if $Q_{\bar{\epsilon}}$ is a defect pointed group of $\bar{\alpha}$.

3.2.3. We consider an G-invariant subalgebra C of the p-permutation G-algebra A such that C is a direct summand of A as $\mathcal{O}$-modules and contains 1_A. Let β be a point of C^N having defect group D in N and let $\bar{\beta} := \mathrm{Br}_D(\beta)$ denote the correspondent point of β lying in $C(D)^{N_N(D)}$, also having defect group D.

Definition 3.2.4. We say that a point α of G on A covers β if α has defect group in $\mathcal{Q}$, and for any $i \in \alpha$ there is an idempotent $j_1 \in A^N$ that lies in the conjugacy class of β and there is a primitive idempotent $f \in A^N$ belonging to a point with defect group D such that $j_1 f = f j_1 = f$ and $if = fi = f$.

3.2.5. In fact, the same definition can be given in the case of points of $N_G(D)$ on $A(D)$ that cover $\bar{\beta}$.

Lemma 3.2.6. *Let $B \in A^G$ and $b \in C^N$ be two primitive idempotents. Then B covers b as in Definition 3.2.4 if and only if B covers b as blocks of the group algebra.*

We are now ready to state the main result of this section:

Theorem 3.2.7. *The Brauer morphism with respect to D induces a bijective correspondence preserving the defect pointed groups in $\mathcal{Q}$ between the points of G on A that cover β and the points of $N_G(D)$ on $A(D)$ that cover $\bar{\beta}$.*

As a corollary we have [24, Theorem].

4. Clifford extensions for blocks of group algebras

4.1. **The Clifford extension of a block.** We do not make any assumption on the size of $\mathcal{O}$ and k. Let K be a normal subgroup of the finite group H, and denote $G = H/K$. Consider the group algebra $\mathcal{O}H$. This is a strongly G-graded algebra, where for each $g \in G$, $\mathcal{O}H_g = \mathcal{O}g$.

Let b a block of $\mathcal{O}K$; this primitive central idempotent remains central in the G-graded algebra

$$C_{\mathcal{O}H}(\mathcal{O}K) = (\mathcal{O}H)^K = \bigoplus_{g\in G} (\mathcal{O}H)_g^K,$$

where $(\mathcal{O}H)_g^K = (\mathcal{O}g)^K$ for all $g \in G$. Then

$$b\mathcal{O}Hb = \bigoplus_{g\in G_b} b\mathcal{O}g = b\mathcal{O}H_b$$

is a strongly G_b-graded algebra.

Define

$$G[b] = \{g \in G \mid b(\mathcal{O}H)_g^K \cdot b(\mathcal{O}H)_{g^{-1}}^K = b(\mathcal{O}H)_1^K\}.$$

It is easy to see that $G[b]$ is a normal subgroup of G_b, and that

$$\hat{A} := \bigoplus_{g\in G[b]} b(\mathcal{O}H)_g^K$$

is a strongly $G[b]$-graded G_b-acted subalgebra of $b\mathcal{O}Hb$. Note that the identity component of $\hat{A}$ is the H-algebra $A := b(\mathcal{O}K)^K$.

Because $A = b(\mathcal{O}K)^K = bZ(\mathcal{O}K)$ is a local ring,

$$\hat{k}_1 := A/J(A)$$

is a finite extension of the field k. Consider the strongly $G[b]$-graded algebra

$$\bar{\hat{A}} := \hat{A}/\hat{A}J(A);$$

for all $g \in G[b]$, we have $\bar{\hat{A}}_g = \hat{A}_g/\hat{A}_gJ(A)$. By definition, the *Clifford extension* of the block b is the group extension

$$1 \to \hat{k}_1^* \to hU(\bar{\hat{A}}) \to G[b] \to 1 \tag{1}$$

associated to the crossed product $\bar{\hat{A}}$ of $\hat{k}_1$ and $G[b]$.

4.2. **The Second Clifford extension of a block.** Now let P_γ denote a defect pointed group of b in K, so γ is a local point of $(\mathcal{O}K)^P$ determining a unique Brauer pair (P, e).

We use the Brauer homomorphism

$$\mathrm{Br}_P : (\mathcal{O}H)^P \to kC_H(P)$$

and obtain $\bar{b} := \mathrm{Br}_P(b)$ that satisfies $\bar{b}e = e$. We use the centralizer

$$C_{kC_H(P)^{N_K(P)}}(kC_K(P)^{N_K(P)}) = kC_H(P)^{N_K(P)},$$

in place of $\mathcal{O}H^K$ and the block e in place of b to obtain another extension:

$$1 \to \hat{k}_2^* \to \mathrm{hU}(\bar{\hat{B}}) \to C_H(P)_0/C_K(P) \to 1. \tag{2}$$

Here B is the subalgebra of $ekC_H(P)^{N_K(P)}$ that is strongly graded by $C_H(P)_0/C_K(P)$.

4.3. **A proof of Dade's result for group algebras over an arbitrary field.** We are now ready to give an alternative proof of [17, Corollary 12.6].

Theorem 4.3.1. *With the above notations, the following statements hold.*

1) G_b *equals* $N_H(P)_eK/K$.
2) *The group* $G[b]$ *equals* $C_H(P)_0K/K$.
3) *The extensions* (1) *and* (2) *are isomorphic.*
4) *The isomorphism between the extensions* (1) *and* (2) *is compatible with the natural isomorphism*

$$G[b] \to C_H(P)_0/C_K(P), \qquad g \mapsto g \cap C_H(P)_0,$$

and preserves the conjugation action of $G_b \simeq N_H(P)_e/N_K(P)_e$ *on the two extensions.*

5. Clifford extension for points of K-interior H-algebras

5.1. **Introduction.** Our purpose is to further generalizer the constructions and the main result of Chapter 4. In order to accomplish this purpose we consider K, a normal subgroup of a finite group H, and a unitary K-interior H-algebra A over $\mathcal{O}$. We take a point $\beta \in \mathcal{P}(A^K)$ having defect group P in K and its Brauer correspondent point $\bar{\beta} := \mathrm{Br}_P(\beta) \in \mathcal{P}(A(P)^{N_K(P)})$ also having defect group P in K. We intend to replace the central primitive idempotent b of A^K from the previous chapter with the point β.

5.2. **The Clifford extension of a point.**

5.2.1. As in the introduction, let K be a normal subgroup of the finite group H, and let $G = H/K$. Let A be an unitary K-interior H-algebra over the $\mathcal{O}$. As in [20, 2.1], there exists a strongly G-graded algebra

$$\hat{A} := A \otimes_{\mathcal{O}K} \mathcal{O}H = \bigoplus_{x \in [H/K]} A \otimes x$$

with structural homomorphism

$$\mathcal{O}H \to \hat{A}$$

of G-graded algebras, where for any $a, b \in A$ and $x, y \in H$ we have

$$(a \otimes x)(b \otimes y) = ab^{x^{-1}} \otimes xy.$$

The homomorphism that sends any $x \in H$ to $1 \otimes x$ endows $\hat{A}$ with the structure of an H-interior algebra.

5.2.2. Let A^K denote the subalgebra of A consisting of elements fixed under the conjugation action of K. For the pointed group K_β we denote the normalizer

$$N_H(K_\beta) = \{x \in H \mid \beta^x = \beta\}.$$

5.2.3. We can obtain an action of $(A^K)^* \otimes H$ on $\hat{A}$. If $a \otimes x \in (A^K)^* \otimes H$ then for any generator $b \otimes h \in \hat{A}$ we have

$$(b \otimes h)^{a \otimes x} = (a \otimes x)^{-1}(b \otimes h)(a \otimes x) = (a^{-1}b)^x a^{h^{-1}x} \otimes x^{-1}hx.$$

Thus $\hat{A}$ is an $(A^K)^* \otimes H$-algebra.

5.2.4. Let $j \in \beta$. For any $x \in N_H(K_\beta)$ we have

$$(j \otimes 1)^{1 \otimes x} = j^x \otimes 1 = j^{a_x} \otimes 1$$

for a suitable $a_x \in (A^K)^*$. We introduce the stabilizers

$$(A^K)^*_j = \{a \in (A^K)^* \mid ja = aj\}$$

and

$$((A^K)^* \otimes N_H(K_\beta))_j = \{a \otimes x \in (A^K)^* \otimes N_H(K_\beta) \mid (j \otimes 1)^{a \otimes x} = j \otimes 1\}.$$

Then, setting

$$\widehat{K} = (A^K)^*_j \otimes K, \widehat{N_H(K_\beta)} = ((A^K)^* \otimes N_H(K_\beta))_j \text{ and } \bar{N}_H(K_\beta) = N_H(K_\beta)/K$$

we have:

Lemma 5.2.5. *The following sequence*

$$1 \to \widehat{K} \to \widehat{N_H(K_\beta)} \to \bar{N}_H(K_\beta) \to 1$$

is exact.

5.2.6. The above choice of $j \in \beta$ allows to denote $A_\beta := jAj$. This $\mathcal{O}$-algebra is independent of the choice of the idempotent j in β. Moreover, the map

$$\widehat{K} \to (A_\beta)^*, \text{ where } a \otimes x \mapsto j \cdot a \cdot x$$

is a group homomorphism and A_β becomes a $\widehat{K}$-interior $\widehat{N_H(K_\beta)}$-algebra. The $\widehat{N_H(K_\beta)}$-interior algebra

$$\hat{A}_\beta := A_\beta \otimes_{\widehat{K}} \widehat{N_H(K_\beta)}$$

is strongly $\bar{N}_H(K_\beta)$-graded and can be identified via Lemma 5.2.5 with a $N_H(K_\beta)$-interior subalgebra of $\hat{A}$.

Now we can consider the following subset of $\bar{N}_H(K_\beta)$:

$$G[\beta] = \{\bar{x} \in \bar{N}_H(K_\beta) \mid (A_\beta \otimes \hat{x})^K \cdot (A_\beta \otimes \hat{x}^{-1})^K = (A_\beta \otimes 1)^K\}$$

where $\hat{x} \in \widehat{N_H(K_\beta)}$ is a lifting of $\bar{x}$.

Proposition 5.2.7. *The subset $G[\beta]$ is a normal subgroup of $\bar{N}_H(K_\beta)$.*

We denote by $N_H^A(K_\beta)$ the subgroup of $N_H(K_\beta)$ consisting of elements x such that the automorphism of K induced by the conjugation action of x is an A-fusion.

Proposition 5.2.8. *The subgroups $G[\beta]$ and $N_H^A(K_\beta)/K$ coincide.*

Let us return to the algebra $\hat{A}_\beta$. Using Lemma 5.2.5 the subgroup $G[\beta]$ is isomorphic to a subgroup of $\widehat{N_H(K_\beta)}/\widehat{K}$. Then letting $\widehat{N_H^A(K_\beta)}$ represent the converse image of $\bar{N}_H^A(K_\beta) := N_H^A(K_\beta)/K$ in $\widehat{N_H(K_\beta)}$ we obtain the strongly $G[\beta]$-graded $\widehat{N_H(K_\beta)}$-algebra

$$\hat{A}_\beta^K := (A_\beta \otimes_{\widehat{K}} \widehat{N_H^A(K_\beta)})^K.$$

Taking the quotient $\hat{\hat{A}}_\beta = \hat{A}_\beta^K / J_{gr}(\hat{A}_\beta^K)$, where J_{gr} denotes the graded Jacobson radical, we obtain the strongly $G[\beta]$-graded $\widehat{N_H(K_\beta)}$-algebra that corresponds to the unique Clifford extension (see [Paragraph 2][17])

$$(1\text{''}) \qquad 1 \to A_\beta(K_\beta)^* \to \hat{\hat{N}}_H^A(K_\beta) \to \bar{N}_H^A(K_\beta) \to 1.$$

Here $A_\beta(K_\beta) := jA^Kj/J(jA^Kj)$ is a skew field whose center is a finite extension of k, while $\hat{\hat{N}}_H^A(K_\beta)$ stands for the homogeneous units of $\hat{\hat{A}}_\beta$. We should quote [39, Lemma 1.1] which shows that $\hat{\hat{A}}_\beta$ is a crossed product of $A_\beta(K_\beta)$ with $G[\beta]$.

5.3. **An attempt of constructing the second extension of a point.** While trying to work in a similar way with $\bar{\beta}$ in place of β we see that the property of $A(P)$ being only $C_K(P)$-interior is not enough.

5.4. **The second Clifford extension of a point.**

5.4.1. As mentioned, the elements considered in Remark 5.3 above are not sufficient for the construction of the second extension of a point. We need to employ another exact sequence

$$1 \to \widehat{N_K(P)} \to N_{N_H(P)}\widehat{(N_K}(P)_{\bar{\beta}}) \to \bar{N}_{N_H(P)}(N_K(P)_{\bar{\beta}}) \to 1,$$

where we denoted

$$\widehat{N_K(P)} = (A(P)^{N_K(P)})^*_{\bar{j}} \otimes N_K(P) \text{ and}$$
$$\bar{N}_{N_H(P)}(N_K(P)_{\bar{\beta}}) = N_{N_H(P)}(N_K(P)_{\bar{\beta}})/N_K(P).$$

Using the Brauer quotient $\hat{A}(P) = \bigoplus_{x\in[H/K]}(A \otimes x)(P)$ we make the identification

$$\hat{A}(P)_{\bar{\beta}} := \bigoplus_{\hat{x}} \bar{j}(A \otimes \hat{x})(P)\bar{j}$$

where $\hat{x} \in \widehat{N_H(K_\beta)}$ lifts a system of representatives for $\bar{N}_H(K_\beta)$.

5.4.2. Now consider the elements $\hat{x}$ such that

$$(A_\beta \otimes \hat{x})(P)^{N_K(P)} \cdot (A_\beta \otimes \hat{x}^{-1})(P)^{N_K(P)} = (A_\beta \otimes 1)(P)^{N_K(P)}.$$

Any $\hat{x}$ having this property lifts an element $\bar{x}$ which lies in a subgroup of $\bar{N}_H(K_\beta)$. We denote this subgroup by $G[\bar{\beta}]$. So by applying [39, Lemma 1.1] the algebra

$$\hat{A}(P)^{N_K(P)}_{\bar{\beta}} := \bigoplus_{\hat{x}} (A_\beta \otimes \hat{x})(P)^{N_K(P)},$$

where $\hat{x}$ runs through a set of representatives for $G[\bar{\beta}]$, is a crossed product whose identity component is the local ring $A(P)^{N_K(P)}_{\bar{\beta}}$.

5.4.3. The quotient

$$\hat{\bar{A}}(P)^{N_K(P)}_{\bar{\beta}} := \hat{A}(P)^{N_K(P)}_{\bar{\beta}}/J_{gr}(\hat{A}(P)^{N_K(P)}_{\bar{\beta}})$$

is a crossed product, strongly $G[\bar{\beta}]$-graded algebra. Moreover $\hat{\bar{A}}(P)_{\bar{\beta}}$ corresponds uniquely to the Clifford extension

$$1 \to A(P)_{\bar{\beta}}(N_K(P)_{\bar{\beta}})^* \to \hat{\bar{N}}^{A(P)}_{N_H(P)}(N_K(P)_{\bar{\beta}}) \to G[\bar{\beta}] \to 1. \qquad (2")$$

5.5. **The isomorphism between the two extensions.** We keep the notations of the previous sections of this chapter.

Theorem 5.5.1. *The following statements hold.*

(*i*) *The extensions* (1*"*) *and* (2*"*) *are isomorphic.*

(*ii*) *The crossed products they correspond to are isomorphic as* $\widehat{H_1}/N_K(P) \simeq \widehat{N_H(K_\beta)}/K$*-algebras.*

5.6. **The group algebra case.** In this section we simply apply the above theory on the group algebra.

5.7. **Clifford extensions for blocks of K-interior H-algebras.** Making use of a primitive idempotent of A^K the lies in $Z(A)$ we give a deferent treatment of the above theory.

6. Correspondences for covering blocks

6.1. **Introduction.** Here we prove that the isomorphism of Clifford extensions induces a defect group preserving bijective correspondence between the blocks of H covering b and the blocks of $N_H(D)$ covering b_1, which coincides with the Harris-Knörr correspondence.

6.2. **Preliminaries.**

6.2.1. We use exactly the elements of Chapter 4. So we assume that the residual field k is not algebraically closed. Then we let K be a normal subgroup of the finite group H, denote $G = H/K$, and consider the group algebra $\mathcal{O}H$ regarded as a strongly G-graded algebra

$$A := \mathcal{O}H = \bigoplus_{g\in G} \mathcal{O}g,$$

which is also an H-algebra under the conjugation action of H. We fix a block b of the identity component $A_1 := \mathcal{O}K$ of A. We denote by D a defect group in K of the block b. The centralizers bC and $C[b]$ satisfy:

Lemma 6.2.2. *The algebras $(bC)^{H_b}$ and $C[b]^{H_b}$ have the same primitive idempotents.*

6.3. **Remarks on defect groups.**

6.3.1. We have the isomorphism

$$(*)\qquad Z(s\mathcal{O}H) \simeq Z(b\mathcal{O}Hb) = Z(b\mathcal{O}H_b) = (bC)^{H_b}.$$

We denote by B a block that cover b and by B' the correspondent of B through the isomorphism (*).

If Q is a defect group in H_b of the block B', then Q is a defect group of the block B and satisfies $Q \cap K = D$.

6.4. **The Harris-Knörr correspondence.** Let b_1 denote the Brauer correspondent block of b.

Theorem 6.4.1. *The isomorphic Clifford extensions of b and of b_1 define a defect group preserving bijective correspondence between blocks of $\mathcal{O}H$ covering b and blocks of $\mathcal{O}N_H(D)$ covering b_1. Moreover the Clifford-Dade correspondence between the blocks covering b and b_1 coincides with the Brauer correspondence.*

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