BABEŞ-BOLYAI UNIVERSITY
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STUDY OF ISOMETRY GROUPS

PhD thesis abstract

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Introduction

The concept of distance is fundamental to the human experience. In daily life we need to understand the closeness between two objects in different physical contexts. The mathematical understanding of the concept of distance is concentrated in the notions of metric and metric space. These notions were introduced by M. Fréchet (1906) and F. Hausdorff (1914), generating special cases of topological spaces. The works of K. Menger (1928) and L.M. Blumenthal (1953) opened the prospect of deep research of the geometry of a metric space, reviewing at this level concepts, relationships and configurations of the Euclidean geometry.

The symmetries of geometric configurations, crystals and other microscopic physical objects had been observed and studied for a long time. In a modern expression, the symmetries of an object form a group, an algebraic notion which first appears at the beginning of the nineteenth century in the works of E. Galois and N. Abel. Thanks to the work of S. Lie, G.Frobenius, W. Killing and E. Cartan, I. Schur, H. Weyl, and many others, group theory has grown enormously, both in itself and in its applications. Applications in quantum mechanics and particle physics were investigated in the Twentieth century.

H. Weyl said that in order to understand a mathematical structure, it is necessary to study its group of symmetries. In the case of metric spaces, this idea naturally leads us to the study of their associated isometry groups. The study of isometries is a major subject in geometry, in connection with transformations that preserve angles, distances or different simple configurations. If the origin of the theory of Banach spaces is to be identified with the year of S. Banach's monograph (1932), then we can say that the study of the isometries of a Banach space, a particular metric space, starts at this time. The description of the isometry group of a given metric space is a problem that has attracted the attention of many mathematicians.

This work fits into this research direction and is divided into three chapters, which provide content unity and research topic relevance. The paper is based on a bibliography of 65 references. We will briefly present each chapter, focusing on the personal contributions of the author.

Chapter 1, entitled *Elements of metric space theory* is structured into five paragraphs and is mainly monographic. The main objective of this chapter is to present, in a succinct form, basic notions and results that will be used in later chapters. In paragraph 1.1, notions of metric space, metric and distance are defined, and examples of metric spaces are given. Definition 1.1.1 fixes notations for the Euclidean metric denoted d_2, the taxicab metric denoted d_1, the l^∞ metric, denoted d_∞, the spherical metric denoted d_{S^2}, and the intrinsic metric denoted $S \subset \mathbb{R}^3$. The notion of metric space is defined in Paragraph 1.2 and examples of subspaces are given. The notion of product space is defined and an example of a metric space using the distance function is given. In paragraph 1.3, the notions of convergent sequence, Cauchy sequence, and complete space are defined (Definitions 1.3.1, 1.3.2). Two theorems about convergent sequences are presented (Theorems 1.3.1, 1.3.2). Paragraph 1.4 introduces the notions of continuous function, uniformly continuous function, Lipschitz function, bi-Lipschitz function, and isometry, presented in Definition 1.4.1. Theorem 1.4.1 shows that a Lipschitz function is uniformly continuous. In paragraph 1.5, the notions of homeomorphism, homeomorphic spaces, bi-Lipschitz eqiuvalent maps, and isometric spaces (Definition 1.5.1) are defined. Theorem 1.5.1 shows the relationship between such spaces. This chapter is based on the monograph by D. Burago, Y. Burago, S. Ivanov [19].

Chapter 2, entitled *The isometry group of a metric space* is divided into eight paragraphs and contains original results of the author. The monographic part of the chapter is based on the works of D.J. Schattschneider [59], E.F. Krause [35], G. Chen [20], R. Kaya [32], M. Ozcan, R. Kaya [43], S. Mazur, S. Ulam [39], A. Vogt [63], M. Albertson, D. Boutin [1], M.M. Patnaik [46], M. Willar Jr. [64], H. Coxeter [22], D. Asimov [6], A. Papadopoulos [44]. The original part of the chapter is based on the works of D. Andrica, V. Bulgărean [3], [4], V. Bulgărean [14], [16], [17], [18],

[15]. In the introduction to this chapter, the set of all isometries of a metric space, (X, d), and the stabilizer of x or isotropy group of x (Definition 2.0.1) are defined. Theorem 2.0.1 establishes that $Iso(X, d)$ is a group with respect to composition and $Iso^{(x)}(X, d)$ is a subgroup of $Iso(X, d)$. Section 2.1 contains Theorem 2.1.1, which states that if the (X, d_X) and (U, d_Y) metric spaces are isometric then their isometry groups, $Iso(X, d_X)$ and $Iso(Y, d_Y)$, are isomorphic. Corollary 2.1.1 states that if the $Iso(X, d_X)$ and $Iso(Y, d_Y)$ groups are not isomorphic then the two spaces are not isometric. Corollary 2.1.3 builds an isomorphism between $Iso(X, d_X)$ and $Iso(X, d_X)$, where d_X is the metric defined on V by the scalar product and d_2 is the Euclidean metric on $\mathbb{R}^n$. In subparagraph 2.1.1 we define the notions of displacement of f, minimal displacement of f, minimal set of f, parabolic function, elliptical function, and hyperbolic function. Paragraph 2.2 presents Theorem 2.2.1, which states that the isometry group of $(\mathbb{R}, d_1)$ is isomorphic with the semi-direct product of the groups $\mathbb{Z}_2$ and $(\mathbb{R}, +)$. Remark 2.2.1 contains three examples of such isomorphisms. Section 2.3 contains the description of the Euclidean plane isometries. In subparagraph 2.3.1, we define the notions of linear transformation and affine linear transformation (Definition 2.3.1) for the Euclidean plane. As an example, a linear affine transformation is the composition of a translation and a linear transformation. A linear affine transformation transform a line into a line, a plane into a plane, etc. Subparagraph 2.3.2 defines translation, rotation and reflection (Definition 2.3.2). Subparagraph 2.3.3 contains Theorem 2.3.1, which states that an isometry $f : (\mathbb{R}^2, d) \to (\mathbb{R}^2, d)$ is a linear affine transformation, i.e. there exists a vector $b \in \mathbb{R}^2$ and a square matrix, such that $f(x) = Ax + b$, for any $x \in \mathbb{R}^2$. The proof of this theorem is done in two ways, using Lemmas 2.3.1 and 2.3.2. Paragraph 2.4 studies the isometries of the n-dimensional Euclidean space. Subparagraph 2.4.1 presents definitions for orthogonal matrices, proper isometries, improper isometries, the linear part of mapping f (Definitions 2.4.1 and 2.4.2). This paragraph also presents Theorem 2.4.1, about the $E(n)$, $SE(n)$, $O(n)$ and $SO(n)$ groups, which are: the isometry group on $\mathbb{R}^n$, the isometry group on $\mathbb{R}^n$ with $\det(A) = 1$, the group of orthogonal matrices, and the group of orthogonal matrices with determinant 1 respectively. In subparagraph 2.4.2 we define the notions of fixed point, axis of symmetry, axis of reflection, invariant

line, glide axial symmetry. The main result of this section is given in Theorem 2.4.2. As a conclusion, isometries other than $1_{\mathbb{R}^2}$ can be classified as follows: rotations and central symmetries are fixed point isometries, and translations and glide symmetries are isometries without a fixed point. Symmetries are the basic isometries of $\mathbb{R}^2$, meaning that all isometries can be obtained by compositions of symmetries. Corollary 2.4.1 states that every isometry of $\mathbb{R}^2$ can be obtained by the composition of at most three isometries. In particular, the $E(2)$ group is generated by symmetries. In subparagraph 2.4.3 we present the symmetries of the $O(n)$ group and Cartan's theorem. Cartan's Theorem 2.4.4 states that the $O(n)$ group is generated by its symmetries and in its proof, Theorem 2.4.3 is also used. Corollary 2.4.2 states that any isometry of the $\mathbb{R}^n$ Euclidean space is a composition of at most $n + 1$ symmetries. An isometry that fixes at least one point, is a composition of at most n symmetries. In paragraph 2.5 we study the isometry group of the $\mathbb{R}^2$ plane endowed with the Chinese checkers metric d_c . In Subparagraph 2.5.1, Proposition 2.5.1 is proved, which states that any translation of the Euclidean plane is an isometry of the $\mathbb{R}^2_c$ plane. Lemma 2.5.1 is useful for the determination of the axial symmetries in $\mathbb{R}^2_c$ which are isometries. Corollaries 2.5.1 and 2.5.2 show that the middle of a segment relative to the two metrics, d_E and d_c, is the same, and that the ratio defined by the distance d_c coincides with the ratio defined by d_E. Proposition 2.5.2 states that an axial symmetry with respect to the axis $y = mx$ is an isometry in $\mathbb{R}^2_c$ if and only if $m \in \{0, \pm 1, \pm(\sqrt{2} - 1), \pm(\sqrt{2} + 1), \infty\}$. Proposition 2.5.3 shows that there are only 8 Euclidean rotations that preserve d_c-distances, i.e., the set of isometric rotations in $\mathbb{R}^2_c$, is $R_c = \left\{ r_\theta : \theta = k\frac{\pi}{4},\ k = 0, 1, 2, \ldots, 7 \right\}$. Theorem 2.5.1 states that if $f : \mathbb{R}^2_c \to \mathbb{R}^2_c$ is an isometry then there exist $T_A \in T(2)$ and $g \in O_c(2)$ such that $f = T_A \circ g$, and these transformations are unique. This is proved by using Propositions 2.5.4, 2.5.5 and Corollary 2.5.3 respectively. Finally we obtain an important result, presented in Corollary 2.5.4, showing that $Iso(\mathbb{R}^2_c)$ is the semi-direct product of the $\mathbb{R}^2$ and D_8 gropus. In Subparagraph 2.5.2, Theorem 2.5.2 is proved, the theorem for the calculation of the area of a triangle in the $\mathbb{R}^2_c$ plane. In paragraph 2.6 we describe the $Iso_{d_p}(\mathbb{R}^n)$, $p \neq 2$ group. In Subparagraph 2.6.1 we present and prove the Mazur-Ulam theorem (Theorem 2.6.1), which states that any isometry $f : E \to F$ between real normal spaces is affine. In Subparagraph 2.6.2 we present original results of the author concerning the determination of the $Iso_{d_p}(\mathbb{R}^n)$ group. It is proven that if $p \neq 2$, all these groups are isomorphic and therefore they do not depend on the number p. These results appear in the papers D. Andrica, V. Bulgărean [4] and V. Bulgărean [15]. Theorem 2.6.2 states that for $p \neq 2$, $p \geq 1$ and $f_A : \mathbb{R}^n \to \mathbb{R}^n$, a linear function defined by the matrix $A \in M_n(\mathbb{R})$, $f_A \in Iso_{d_p}(\mathbb{R}^n)$ if and only if A is a permutation matrix i.e. each row and each column of A has exactly one nonzero element and this element is equal to ± 1. This result is shown in proven in the papers D. Andrica, V. Bulgărean [4] and V. Bulgărean [15]. In Subparagraph 2.6.3, the $Iso_{d_\infty}(\mathbb{R}^n)$ group is determined. Although the result is formulated in the same manner, we prefer to present it separately for the case $p = \infty$, because the proof is completely different from the one given in Theorem 2.6.2. Theorem 2.6.3 shows that for a linear function $f_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by the matrix $f_A \in Iso_{d_\infty}(\mathbb{R}^n)$, $A \in M_n(\mathbb{R})$ if and only if A is a permutation matrix, i.e. each row and each column of A has exactly one nonzero element and this element is equal to ± 1. Subparagraph 2.6.4 draws common conclusions for the $Iso_{d_p}(\mathbb{R}^n)$ and $Iso_\infty(\mathbb{R}^n)$ groups. These conclusions, together with the results of subparagraphs 2.6.2 and 2.6.3, lead us to the following common result for the $Iso_{d_p}(\mathbb{R}^n)$ and $Iso_{d_\infty}(\mathbb{R}^n)$ groups (Corollary 2.6.1): for $p \geq 1$, $p \neq 2$, a real number or $p = \infty$, the $Iso_{d_p}(\mathbb{R}^n)$ group is isomorphic to the semi-direct product of groups $(\mathbb{R}^n, +)$ and $S_p \times \mathbb{Z}_2^n$, where S_n is the group of permutations of the set $\{1, 2, \dots, n\}$. The subgroup of linear isometries of $Iso_{d_\infty}(\mathbb{R}^n)$ consists of $2^n \cdot n!$ linear mappings defined by the permutation matrices in Theorem 2.6.3. Subparagraph 2.6.5 introduces the d-isometric dimension of a finite group. The notion of d-isometric dimension of a group is defined. Theorem 2.6.4 shows that for a finite group G, the d_2-isometric dimension $\delta_{d_2}(G)$ equals the minimum dimension of the real representation of G. As a consequence of this theorem, Corollary 2.6.2 states that for finite groups $G_1, \dots, G_s$ the following inequality holds:

$$\delta_{d_2}(G_1 \oplus \ldots \oplus G_s) \leq \delta_{d_2}(G_1) + \ldots + \delta_{d_2}(G_s).$$

Theorem 2.6.5 shows that for any $p \geq 1$, $p \neq 2$, a real number or $p = \infty$ the inequality $\delta_{d_p}(S_n \times \mathbb{Z}_2^n) \leq n$ holds. Theorem 2.6.6 shows that the equality $\delta_c(D_8) = 2$ holds, where δ_c is the isometric dimension relative to the d_c metric of the $\mathbb{R}_c^2$ plane and

D_8 is the dihedral group. Two open problems are presented: The determination of $\delta_{d_p}(\mathbb{Z}_2^n)$; is it true that $\delta_{d_p}(S_n \times \mathbb{Z}_2^n) = n$ holds?. In paragraph 2.7 we study the problem of the geometric realization of finite groups. Theorem 2.7.1 shows that there exists a Riemann metric on the sphere S^{k-1} such that the associated isometry group is isomorphic with G. This theorem is due to D. Asimov and is proven using the following results: Proposition 2.7.1, which states that with the metric induced from $\mathbb{R}^k$ on the sphere S^{k-1}, the metric space X has its isometry group isomorphic with G, and Proposition 2.7.2 which shows the relationship $Iso(M) \simeq G$. Corollary 2.7.1 shows that any finite group G is isomorphic to the isometry group of a finite subset X_G of a Euclidean space. If $\operatorname{card}(G) = k$, then X_G can be chosen with $\operatorname{card}(X_G) = k^2 - k$, in a Euclidean space of dimension $k - 1$. This result is followed by some examples that illustrate the geometric realization of some finite groups, through isometries. All these examples are original contributions and are presented in the paper V. Bulgărean [16]. In paragraph 2.8 we present original results related to the isometry group of the French railway metric. These results are presented in paper V. Bulgărean [14]. Subparagraph 2.8.1 presents two theorems: Theorem 2.8.1, which shows that $Iso^{(p)}(X, d)$ is a subgroup of $Iso(X, d_{F,p})$, and in particular the inclusion $Iso^{(p)}(X, d) \subseteq Iso(X, d_{F,p})$ holds; and Theorem 2.8.2, which states that for any isometry $f \in Iso(X, d_{F,p})$, p is a fixed point, i.e. the relationship $f(p) = p$ holds. Corollary 2.8.1 shows that for any metric space (X, d) and for any point $p \in X$, the metric space $(X, d_{F,p})$ is elliptical, i.e. all its isometries are elliptical. In subparagraph 2.8.2 we are present comments regarding Theorem 2.8.2 in the case $X = \mathbb{R}^n$ and $d = d_2$.

Chapter 3, entitled *Special problems related to isometries* consists of four paragraphs. Paragraph 3.1 presents the notion of frieze, strips in the plane in which geometric patterns are repeated infinitely. In subparagraph 3.1.1 we present the notions of generators, words, the length of a word, reduced word, relationships, and group presentation. Lemma 3.1.1 shows that for TG a subgroup of G, t a generator for T, and $r \in G$, there exists $r^{-1}tr$, a generator for the group $r^{-1}Tr = \{r^{-1}xr : x \in T\}$. In Subparagraph 3.1.3 we present the classification of frieze groups. The notions of discrete subgroup of $E(2)$ are defined (Definitions 3.1.1, 3.1.2). We introduce two lemmas that illustrate some properties of translation (Lemma 3.1.2, 3.1.3). There are exactly seven frieze groups, given in Theorem 3.1.1. We also show some realizations of the 7 frieze groups, taking note that the horizontal axis is the axis of translation and the image is repeaded infinitely. Paragraph 3.2 studies mappings that preserve certain geometric properties. In Subparagraph 3.2.1,the general Aleksandrov-Rassias problem is presented, which states that if (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ is a continuous mapping, surjective, and preserving distance 1, it follows that f is an isometry. Even if we impose additional conditions on f the answer to this general problem can be negative. A good example in this regard (Example 3.2.1) is shown next. We also present an example in which the answer to the problem could be negative in the infinite dimensional case, even for Hilbert spaces. In subparagraph 3.2.2 we study mappings that transform cubes into cubes. Lemma 3.2.1 is presented, which states that for an injective map $f : \mathbb{R}^3 \to \mathbb{R}^3$ that transforms any cube into a cube, for any cubes of side length 1, A and B, if $\mathrm{Int}(A) \cap \mathrm{Int}(B) = \emptyset$, then $\mathrm{Int}\{f(A)\} \cap \mathrm{Int}\{f(B)\} = \emptyset$. Theorem 3.2.1 states that if the injective map $f : \mathbb{R}^3 \to \mathbb{R}^3$ transforms any cube into a cube, then f is a linear isometry up to a translation. Paragraph 3.3 shows results on the isometry group of the sphere. Theorem 3.3.1 is presented, which states that any isometry $f : S^2 \to S^2$ is a rotation or an axial symmetry. Theorem 3.3.2 shows that every isometry $f : S^2 \to S^2$ is a planar symmetry, rotation or rotosymmetry (composition of a rotation and a symmetry). Theorem 3.3.3 states that any isometry $f : S^n \to S^n$ is a composition of rotations and possibly a symmetry. In Theorem 3.3.4 shows that $Iso(S^n) \simeq O(n + 1)$. Theorem 3.3.5 states that any isometry $f \in Iso(S^n)$ is a composition of at most $\left[\dfrac{n+1}{2}\right]$ proper rotations of S^n and possibly a symmetry with respect to a hyperplane passing through the origin. Remark 3.3.1 shows that the $Iso(S^n)$ group is generated by rotations and symmetries. Theorem 3.3.6 states that for $f : S^n \to S^2$, $n \geq 2$, a function which preserves angles θ, $m\theta$, where $m\theta < \pi$ and m is a positive integer greater than 1, then f is an isometry, i.e. f preserves all angles. Theorem 3.3.7 shows that if $f : S^n \to S^p$, $p \geq n > 1$, is a continuous map which preserves angles θ, $m\theta$, where $m > 1$ and $m\theta < \pi$, then f is an isometry. Theorem 3.3.8 states that for $f : S^n \to S^n$ a map that preserves angle θ, or $\arccos\left(\dfrac{1}{m + \sec\theta}\right)$ is irrational for

$0 \le m \le n-1$, then f is an isometry. In paragraph 3.4 we study the isometry groups of locally compact metric spaces. Theorem 3.4.1 shows the general properties of the isometry group of a locally compact metric space (X,d). Example 3.4.1 shows such a space which is not a locally compact space. Example 3.4.2 shows a locally compact space. In subparagraph 3.4.1 we give properties related to the local compactness of the $Iso(X,d)$ group. Lemma 3.4.1 states that if (X,d) is a locally compact space, $F \subseteq Iso(X,d)$ and $K(F) = \{x \in X : F(x) = \{f(x) : f \in F\}$ is relatively compact$\}$, then $K(F)$ is an open and closed subset of X. Lemma 3.4.2 states that for (X,d) a locally compact metric space with quasi-compact connected component space $\Sigma(X)$, then condition (a) of Theorem 3.4.1 is satisfied. Example 3.4.3 shows an example of a limit of a sequence of isometries which is not surjective. Lemma 3.4.3 states that if $\Sigma(X)$ is quasi-compact and (f_n), $f_n \in Iso(X,d)$ is a sequence such that $f_n \to f$ with respect to punctual convergence topology, then $f(X)$ is open and closed in X. Proposition 3.4.1 shows the following result: if (X,d) is a locally compact metric space and $\Sigma(X)$ is quasi-compact, then $Iso(X,d)$ is closed in $C(X,X)$. Proposition 3.4.2 asserts that there exists a subsequence $\{S_{n_k}\}_{k\in\mathbb{N}}$ of $\{S_n\}$ such there exists $x_k \in S_k$ with $x_k \to x_0$, where $x_0 \in X$. Theorem 3.4.2 states that if $\Sigma(X)$ is quasi compact, then $Iso(X,d)$ is locally compact. Subparagraph 3.4.2 studies the proper action of the $Iso(X,d)$ group on the space X. Proposition 3.4.3 states that if (X,d) is locally compact and connected, then the $Iso(X,d)$ group is locally compact and its action on X is proper, so the quasi-compactness of $\Sigma(X)$ is not required for local compactness of $Iso(X,d)$. The chapter is based on the works of A.D. Aleksandrov [2], F.S. Beckman, D.A. Quarles [9], Th. M. Rassias [50], [51], [53], [54], B. Mielnik, Th. M. Rassias [40], [55], S.M. Jung [26], [27], S.M. Jung, Ki-Sik Lee [30], Th. M. Rassias, P. Semrl [56], D. van Dantzig, B.L. van der Waerden [23].

I would not wish to end this introduction without thanking Prof. Dorin Andrica, my scientific advisor, for his remarks, suggestions, substantial support and the amiability with which he has always answered my requests during the preparation of this work. I would also like to thank all the members of the Department of Geometry of the Babes-Bolyai University in Cluj-Napoca, for their trust and support given during the elaboration of this thesis.

Chapter 1

Elements of metric space theory

In this chapter we introduce the basic notions on metric spaces. We present specific examples and constructions that are useful in the development of the following chapters.

1.1 Metric spaces. Examples

A metric space is a pair (X, d) formed by a nonempty set X and a function $d : X \times X \to \mathbb{R}$ that satisfies the properties:

(1) (Positivity and nondegeneracy). For any $x, y \in X$, $d(x, y) \geq 0$. In addition, we have $d(x, y) = 0 \Leftrightarrow x = y$.

(2) (Symmetry) For any $x, y \in X$, $d(x, y) = d(y, x)$.

(3) For any $x, y, z \in X$ the inequality holds: $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality).

The function d is called the metric. It is also called the distance function.

Below we give some examples of metric spaces. In most of the examples conditions (1) and (2) of the above definition are easy to check. We mention these conditions only if there are problems in their establishment. It is usually more difficult to prove the triangle inequality and this is done in detail in some examples.

Example 1.1.1 Let $X = \mathbb{R}$ and the distance function $d(x, y) = |x - y|$.

Example 1.1.2 Let $X = \mathbb{R}^2$ and the usual Euclidean distance function

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$.

Example 1.1.3 Let $X = \mathbb{R}^n$ and the usual Euclidean distance

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2},$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

Example 1.1.4 Let $X = \mathbb{R}^n$ and $d_1(x, y) = |x_1 - y_1| + \ldots + |x_n - y_n|$, the taxicab metric. For $n = 2$, d is the usual distance that we use when we drive the car in a city where the street network is parallel to two perpendicular directions.

If we have x and z, the set of points y for which $d_1(x, z) = d_1(x, y) + d_1(y, z)$ is called the metric segment in Menger sense.

Example 1.1.5 Let $X = \mathbb{R}^n$ and $d_\infty(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

In order to prove the triangle inequality $d_\infty(x, z) \leq d_\infty(x, y) + d_\infty(y, z)$, we suppose that $d_\infty(x, z) = \max\limits_{1 \leq i \leq n}\{|x_i - z_i|\} = |x_k - z_k|$, for k fixed, $1 \leq k \leq n$. Then we have the relations:

$$|x_k - z_k| \leq |x_k - y_k| + |y_k - z_k|, \quad |x_k - y_k| \leq d_\infty(x, y), \quad |y_k - z_k| \leq d_\infty(y, z).$$

So we have: $d_\infty(x, z) \leq d_\infty(x, y) + d_\infty(y, z)$.

We will not discuss the equality case at this point, as we will study in detail in Chapter 2.

Example 1.1.6 Let $X = S^2 = \{x \in \mathbb{R}^3 : \ \|x\| = 1\}$, be the unit sphere in the Euclidean space $\mathbb{R}^3$. Let $d(x, y)$ be the lenght of the small arc that connects points x and y. This is how distances on Earth's surface are measured. An explicit formula for $d(x, y)$ is easily obtained. Let θ be the angle between the unit vectors x and y. The arc joining x to y belongs to the intersection of S^2 with the plane generated by x and y and the length of this arc is θ (see Figure 1.2). Therefore we have $\cos\theta = x \cdot y$ (Euclidean scalar product in $\mathbb{R}^3$), so $d(x, y) = \arccos(x \cdot y)$. It immediately follows that d is a metric on S^2.

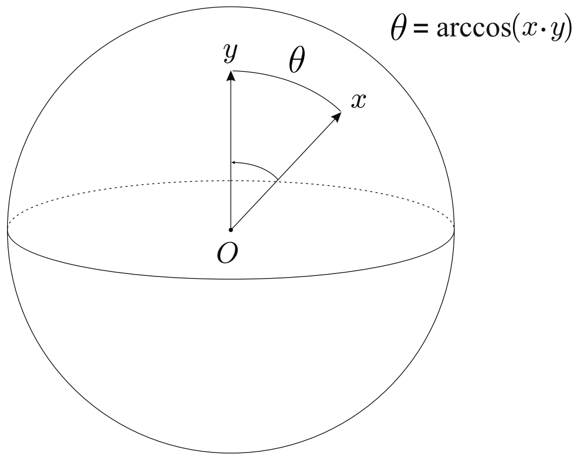

Figure 1.2. The distance on the sphere

Let $x_1, \ldots, x_m$ vectors in $\mathbb{R}^n$, where $m \leq n$. The Gram matrix defined by these vectors $x_1, \ldots, x_m$ is the square matrix of order m, A with elements $x_i \cdot x_j$. We remark that A is a symmetric matrix since we have $x_i \cdot x_j = x_j \cdot x_i$.

Theorem 1.1.1 *If A is the Gram matrix of vectors $x_1, \ldots, x_m$, then*

$$\det(A) \geq 0.$$

Additionally, we have $\det(A) = 0$ *if and only if the set* $\{x_1, \ldots, x_m\}$ *is linearly dependent.*

Remark 1.1.1 Note that if $m = 2$, meaning that we have two vectors $x, y \in \mathbb{R}^m$, then Theorem 1.1.1 reduces to

$$\det(A) = (x \cdot x)(y \cdot y) - (x \cdot y)^2 \geq 0,$$

which is the Cauchy-Schwarz inequality. In Examples 1.1.2 and 1.1.3 we have used this inequality to prove the triangle inequality for the Euclidean metric. We see that in the case $m = 3$ Theorem 1.1.1 is useful to prove the triangle inequality for the metric on the sphere in Example 1.1.6.

Example 1.1.7 Let X be a nonempty set and d defined as follows:

$$d(x,y) = \begin{cases} 0, & \text{if} \quad x = y \\ 1, & \text{if} \quad x \neq y. \end{cases}$$

This distance is called the discrete metric and (X, d) is called the discrete metric space.

Example 1.1.8 Let (X, d) be a metric space and $p \in X$ a fixed point. We define a metric on X, called the French railway metric, denoted $d_{F,p}$, where

$$d_{F,p}(x, y) = \begin{cases} 0 \text{ if and only if } x = y \\ d(x, p) + d(p, y) \text{ if } x \neq y \end{cases}$$

We obtain a new metric space $(X, d_{F,p})$. This metric is studied in paper V. Bulgărean [14] and in section 2.8.

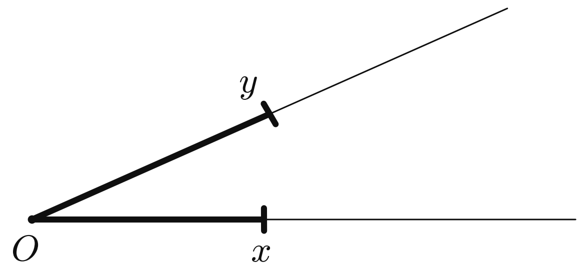

Figure 1.3. French railways metric in the Euclidean plane $\mathbb{R}^2$

The name of this metric comes from the following hypothetical situation. We are in a country (called France) in which there are railway lines passing through every town. We can travel between any two cities only if we pass through Paris.

Definition 1.1.1 Next we fix the notations that we have used so far the in metric spaces introduced. These will be used extensively in the following chapters.

(1) The metric in Example 1.1.3 is called the Euclidean metric and is denoted d_2. Then we have the relation:

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

(2) The metric in Example 1.1.4 is called the taxicab metric or l^1 metric and is denoted d_1. The formula for d_1 is:

$$d_1(x, y) = |x_1 - y_1| + \ldots + |x_n - y_n|.$$

(3) The metric in Example 1.1.5 is called the l^∞ metric and is denoted d_∞. The formula is:

$$d_\infty(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

(4) The metric in Example 1.1.6 is called the spherical metric and is denoted d_{S^2}.

1.2 Construction of metric spaces

There are some standard constructions of new metric spaces using the ones given so far. The most common constructions for spaces are the subspaces.

1.2.1 Subspaces

Let (X, d) be a metric space and let $Y \subset X$. We consider $d' = d|_{Y\times Y} : Y \times Y \to \mathbb{R}$, the restriction of d to $Y \times Y$. Then (Y, d') is the metric space called the subspace of (X, d). Usually the restriction d' is simply denoted d.

Examples of subspaces

(1) $\mathbb{Q}$ is a subspace of $\mathbb{R}$.

(2) Any subset of $\mathbb{R}$ is a subspace of $\mathbb{R}$. For example $(0, +\infty)$ is a subspace of $\mathbb{R}$.

(3) S^2 is a subspace of $\mathbb{R}^3$. But the metric of this subspace is not the same as the spherical metric from Example 1.1.6. If d' is the restriction to $S^2 \times S^2$ of the Euclidean metric d_2 on $\mathbb{R}^3$ and d_{S^2} is the spherical metric on S^2, then we have the inequality $d'(x, y) \leq d_{S^2}(x, y)$, for all $x, y \in S^2$, with equality if and only if $x = y$.

1.2.2 Product spaces

If (X_1, d_1) and (X_2, d_2) are metric spaces, their product is the space $(X_1 \times X_2, d)$, where:

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\},$$

for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$.

Remark the analogy with the d_∞ metric from Definition 1.1.1. Other metrics are possible on the product space, but this is a convenient choice.

1.2.3 Distance functions

Suppose that (X, d) is a metric space and the function $f : [0, +\infty) \to \mathbb{R}$ is strictly increasing with the property $f(0) = 0$ and is subadditive, satisfying the relation

$$f(a + b) \leq f(a) + f(b), \text{ for all } a, b \in [0, +\infty).$$

It is not difficult to see that $f \circ d : X \times X \to \mathbb{R}$ is a metric on X, so $(X, f \circ d)$ is a metric space.

1.3 Limits

The notion of metric space allows in this context the reformulation of many concepts and results of real analysis. We give some useful examples in the development of the following chapters. By sequence in a metric space (X, d) we understand, as usual, a function $\mathbb{N} \to X$ and use the notation $\{x_n\}$.

Definition 1.3.1 Let $\{x_n\}$ be a sequence in metric space (X, d).

(1) Let $x \in X$. We say that $\lim\limits_{n\to\infty} x_n = x$ if and only if for all $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d(x, x_n) < \varepsilon$, for all $n \geq N$.

(2) We say that $\{x_n\}$ converges if and only if there exists $x \in X$ such that we have $\lim\limits_{n\to\infty} x_n = x$.

(3) We say that $\{x_n\}$ is a Cauchy sequence if and only if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$, for all $m, n \geq N$.

Theorem 1.3.1 *Every convergent sequence is a Cauchy sequence.*

Theorem 1.3.2 *If the sequence $\{x_n\}$ is convergent, then its limit is unique.*

Definition 1.3.2 A metric space (X, d) is called complete if every Cauchy sequence is convergent.

The space $\mathbb{R}^n$, $n \geq 1$, is complete, while $\mathbb{Q}$ is not complete, with the usual Euclidean metric.

1.4 Mappings between metric spaces

Let (X, d) and (Y, d') be metric space and let the function $f : X \to Y$.

Definition 1.4.1 (1) Let $x \in X$. The mapping f is continuous at x if and only if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for all $y \in X$, if we have $d(x, y) < \delta$, then $d'(f(x), f(y)) < \varepsilon$.

(2) The mapping f is continuous on X if and only if it is continuous at any point $x \in X$. Explicitly, f is continuous if and only if for any $x \in X$ and $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon)$ such that $d'(f(x), f(y)) < \varepsilon$ for all $y \in X$ with $d(x, y) < \delta$.

(3) The mapping f is uniformly continuous if and only if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $d'(f(x), f(y)) < \varepsilon$ for all $x, y \in X$ with $d(x, y) < \delta$.

(4) The mapping f is a Lipschitz map if and only if there exists a constant $C > 0$ such that $d'(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$. The constant C is a Lipschitz constant for f.

(5) The mapping f if bi-Lipschitz if and only if there exist the constants $C_1, C_2 > 0$ such that

$$C_1 d(x, y) \leq d'(f(x), f(y)) \leq C_2 d(x, y)$$

for all $x, y \in X$.

(6) The mapping f is isometric if and only if $d'(f(x), f(y)) = d(x, y)$, for all $x, y \in X$.

In the following chapters we will thoroughly study these mappings.

Theorem 1.4.1 *If $f : (X, d) \to (Y, d')$ is a Lipschitz map, then f is uniformly continuous.*

1.5 Equivalence between metric spaces

We will define several types of equivalence between metric spaces, with the additional assumption that the maps defined in the previous section are bijective, and with any further assumptions when appropriately needed.

Definition 1.5.1 Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$ be a mapping. We say that:

(1) The map f is a homeomorphism if f is continuous, bijective and f^{-1} is continuous. If there exists such a map, we say that the spaces (X, d) and (Y, d') are homeomorphic.

(2) The mapping f is bi-Lipschitz equivalent if and only if f is surjective and bi-Lipschitz. If there exists a bi-Lipschitz equivalence, we say that the spaces (X, d) and (Y, d') are bi-Lipschitz equivalent.

(3) The spaces (X, d) and (Y, d') are isometric if and only if there exists a surjective isometry $f : (X, d) \to (Y, d')$.

Theorem 1.5.1 *Let (X, d) and (Y, d') be metric spaces.*

(1) If (X, d) and (Y, d') are isometric, then they are bi-Lipschitz equivalent.

(2) If (X, d) and (Y, d') are bi-Lipschitz equivalent, then they are homeomorphic.

Chapter 2

The isometry group of a metric space

Let (X, d) be a metric space and let f, g be two isometries of (X, d). Then the composition $f \circ g$ preserves distances, as for all $x, y \in X$ we have:

$$d(f \circ g(x), f \circ g(y)) = d(f(g(x)), f(g(y))) = d(g(x), g(y)) = d(x, y).$$

We also have the property that the inverse f^{-1} preserves distances, because

$$d(f^{-1}(x), f^{-1}(y)) = d(f(f^{-1}(x)), f(f^{-1}(y))) = d(x, y).$$

This means that the set of all isometries is a group with respect to the usual operation of function composition.

Definition 2.0.1. Let

$$Iso(X, d) = \{f : X \to X : \ f \text{ is an isometry of space } (X, d)\}$$

the set of all isometries of (X, d). If $x \in X$, we denote by

$$Iso^{(x)}(X, d) = \{f \in Iso(X, d) : \ f(x) = x\},$$

the set of isometries of X for which x is fixed. $Iso^{(x)}(X, d)$ is a subgroup of $Iso(X, d)$ called the stabilizer of x, or isotropy group of x.

Theorem 2.0.1. *The set $Iso(X, d)$ is a group with respect to the usual composition operation. The subset $Iso^{(x)}(X, d)$ is a subgroup of $Iso(X, d)$.*

2.1 General properties of the $Iso(X, d)$ group

Let (X, d_X), (Y, d_Y) be two metric spaces. The map $\alpha : X \to Y$ preserves distances if for all $x, x' \in X$ the relation $d_Y(\alpha(x), \alpha(x')) = d_X(x, x')$ holds. It is obvious that any mapping that preserves distances is injective.

The application $\alpha : X \to Y$ is called an isometry if it satisfies the following two properties:

1) α is surjective;

2) α preserves distances.

Obviously, an isometry $\alpha : X \to Y$ is a bijection.

The following result shows that the group of isometries of a metric space is invariant with isometric transformations.

Theorem 2.1.1 *If the metric spaces (X, d_X) and (Y, d_Y) are isometric, then their isometriy groups $Iso(X, d_X)$ and $Iso(Y, d_Y)$ are isomorphic.*

Corollary 2.1.1 *Consider the metric spaces (X, d_X) and (Y, d_Y). If the groups $Iso(X, d_X)$ and $Iso(Y, d_Y)$ are not isomorphic, then the two spaces are not isometric.*

Let $u : X \to X$ be a bijective map.

We define on the set X the metric $d_u : X \times X \to \mathbb{R}$, by $d_u(x, y) = d(u(x), u(y))$.

Corollary 2.1.2 *The relation $Iso(X, d_u) \simeq Iso(X, d)$ holds.*

Corollary 2.1.3 *Let V be a real n-dimensional linear space endowed with the scalar product $\langle \cdot, \cdot \rangle$. Then $Iso(V, d_V) \simeq Iso(\mathbb{R}^n, d_2)$, where d_V is the metric defined on V by the inner product and d_2 is the Euclidean metric on $\mathbb{R}^n$.*

2.1.1 General classification of the elements of $Iso(X, d)$

Let (X, d) be a metric space and $f : X \to X$. The function $x \mapsto d(x, f(x))$ is called the displacement of f. The number $\lambda(f)$, defined by

$$\lambda(f) = \inf_{x \in X} d(x, f(x))$$

is called the minimal displacement of f. The minimal set of f, denoted $\mathrm{Min}(f)$, is the subset of X defined by

$$\mathrm{Min}(f) = \{x \in X : \ d(x, f(x)) = \lambda(f)\}.$$

The following general classification of isometries on a metric space (X, d) with respect to invariants $\lambda(f)$ and $\mathrm{Min}(f)$ is given in the monograph by A. Papadopoulos [44]. Let $f \in Iso(X, d)$. Then

1. f is parabolic if $\mathrm{Min}(f) = \emptyset$;

2. f is elliptical if $\mathrm{Min}(f) \neq \emptyset$ and $\lambda(f) = 0$. Then f is elliptical if and only if $\mathrm{Fix}(f) \neq \emptyset$, where $\mathrm{Fix}(f)$ denotes the set of fixed points of f.

3. f is hyperbolic if $\mathrm{Min}(f) \neq \emptyset$ and $\lambda(f) > 0$.

2.2 The isometry group of the line

Consider the Euclidean line $\mathbb{R}$ endowed with the usual d_1 metric, where $d_1(x, y) = |x - y|$. Then the following result holds.

Theorem 2.2.1 *The isometry group* $(\mathbb{R}, d_1)$ *is isomorphic to the semidirect product of the groups* $\mathbb{Z}_2$ *and* $(\mathbb{R}, +)$*, i.e. we have*

$$Iso(\mathbb{R}, d_1) \simeq \mathbb{R} \rtimes \mathbb{Z}_2.$$

Remark 2.2.1 1) The set of matrices of the form

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & k \\ 0 & 1 \end{pmatrix}, \ k \in \mathbb{R}$$

form a non-commutative group with respect to multiplication. It is shown immediately that $Iso(\mathbb{R}, d_1)$ is isomorphic to this group.

2) From Theorem 2.2.1 it follows that $Iso(\mathbb{R}, d_1)$ is a nonconnected Lie group with two connected components. The connected component of the unit is the normal subgroup $N = \{f_k : \ k \in \mathbb{R}\}$, where $f_k(x) = x + k$. This is the subgroup of translations of group $Iso(\mathbb{R}, d_1)$.

3) Considering the metric space $X = (0, \infty)$ with the Euclidean metric d_1, the isometry group $Iso(X, d_1)$ reduces to the trivial group $\{1_X\}$. It is obvious that

all isometries determined in Theorem 2.2.1 preserve distances, but only the map $1_X : (0, \infty) \to (0, \infty)$ is surjective.

2.3 Euclidean plane isometries

The isometry group of a metric space can be very small, in fact it can be trivial, only containing the identical mapping. Next we study the case when the group is large.

We start with the study the group of isometries of $\mathbb{R}^2$ with Euclidean metric d_2. In this section we simply denote d instead of $d_{(2)}$, because it is the only metric considered. The goal is to determine all the isometries of the space $(\mathbb{R}^2, d)$ and to describe the group $Iso(\mathbb{R}^2, d)$.

2.3.1 Affine transformations of the Euclidean plane

We first recall some notions and results from linear algebra.

A transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ is called a linear transformation if and only if for every $r \in \mathbb{R}$, and for every $x, y \in \mathbb{R}^2$ we have the relations:

$$L(rx) = xL(x) \text{ and } L(x + y) = L(x) + L(y).$$

The previous definition is equivalent with the fact that for all $r, s \in \mathbb{R}$ and for all $x, y \in \mathbb{R}^2$ we have the relation:

$$L(rx + sy) = rL(x) + sL(y).$$

Definition 2.3.1 A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an affine linear transformation if there exists a matrix A of size $n \times n$ and a vector $b \in \mathbb{R}^n$ such that

$$f(x) = Ax + b, \ x \in \mathbb{R}^n.$$

An affine linear transformation is the composition of a linear transformation and a translation. An affine linear transformation transforms lines into lines, planes into planes, etc..

2.3.2 Classes of isometries of the Euclidean plane

The usual isometries of $\mathbb{R}^2$ are translations, rotations and symmetries. These are affine linear transformations on $\mathbb{R}^2$ of the form $f(x) = Ax + b$, where b is a vector and A is a 2×2 matrix. We use the following terminology:

Definition 2.3.2 An affine linear transformation $f(x) = Ax + b$ on $\mathbb{R}^2$ is called a:

(1) translation by b, denoted t_b if $A = I_2$, where I_2 is the identity matrix. Then we have

$$f(x) = t_b(x) = x + b.$$

(2)counterclockwise rotation by an angle θ about the origin , denoted R_θ, if $b = 0$ and

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.3.1}$$

(3) reflection about a line defined parametrically by

$$\left\{ t\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) :\ t \in \mathbb{R} \right\},$$

denoted by S_θ, if $b = 0$ and

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{2.3.2}$$

2.3.3 Euclidean plane isometry group determination

We want to prove that the examples of the form $f(x) = Ax + b$ discussed above give all isometries of the plane $\mathbb{R}^2$. The only difficulty is proving that an isometry of the plane $\mathbb{R}^2$ is an affine linear transformation.

Theorem 2.3.1 *Let $f : (\mathbb{R}^2, d) \to (\mathbb{R}^2, d)$ be an isometry. Then f is affine linear transformation, i.e. there is a vector $b \in \mathbb{R}^2$ and a square matrix such that $f(x) = Ax + b$, for all $x \in \mathbb{R}^2$.*

Lemma 2.3.1 *Let a, b be positive real numbers. We define the set $E(a, b)$ of triplets of points in $\mathbb{R}^2$*

$$E(a,b) = \{(x,y,z) :\ x,y,z \in \mathbb{R}^2,\ d(x,y) = a,\ d(y,z) = b \text{ and } d(x,z) = a + b\}.$$

Suppose that $(x_1, y_1, z_1), (x_2, y_2, z_2) \in E(a, b)$ *and assume that two of the three equalities* $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$ *are true. Then the third equality also holds.*

Lemma 2.3.2 *Suppose* $f : \mathbb{R} \to \mathbb{R}^2$ *is an isometry, with* $f((0,0)) = (0,0)$, $f((1,0)) = (1,0)$ *and* $f((0,1)) = (0,1)$. *Then* $f = 1_{\mathbb{R}^2}$.

2.4 The isometries of the n-dimensional Euclidean space

2.4.1 The $E(n)$, $SE(n)$, $O(n)$, $SO(n)$ groups

Definition 2.4.1 We denote by $O(n)$ the set of orthogonal matrices, by $SO(n)$ the set of orthogonal matrices with determinant 1, by $E(n)$ the set of isometries on $\mathbb{R}^n$ and by $SE(n)$ the set of isometries of $\mathbb{R}^n$, $f(x) = Ax + b$ with $\det(A) = 1$. The elements of $SE(n)$ are called proper isometries (or isometries that preserve orientation) of $\mathbb{R}^n$. The elements of $E(n)$ that are not in $SE(n)$ are called improper isometries (or isometries which do not preserve orientation) of $\mathbb{R}^n$.

The notations $O(n)$, $SO(n)$ are standard.

We will use the notation $f_{A,b}$ for the isometry $f_{A,b}(x) = Ax + b$ on $\mathbb{R}^n$.

Definition 2.4.2 We define the map $l : E(n) \to O(n)$ by $l(f_{A,b}) = A$. The matrix $l(f)$ defines the linear part of f.

Theorem 2.4.1 *(1) The sets* $E(n)$, $SE(n)$, $O(n)$, $SO(n)$ *are groups (with respect to composition or matrix multiplication, depending on the case).*

(2) The map $l : E(n) \to O(n)$ *is a group morphism and* $\operatorname{Ker} l$ *is the group of translations of* $\mathbb{R}^n$, *which is isomorphic with the group* $(\mathbb{R}^n, +)$.

(3) The map $\det : O(n) \to \{1, -1\}$ *is a morphism of groups with kernel* $SO(n)$.

(4) The composition $E(n) \xrightarrow{l} O(n) \xrightarrow{det} \{-1, 1\}$ *is a morphism of groups with kernel* $SE(n)$.

2.4.2 The classification of Euclidean plane isometries

In the following section we classify the isometries of the plane $\mathbb{R}^2$, dividing them into four classes with respect to their fixed points.

Let $f \in SE(2)$ be a proper isometry of $\mathbb{R}^2$, and suppose $f \neq 1_{\mathbb{R}^2}$. A point $x \in \mathbb{R}^2$ is called a fixed point of f if the relation $f(x) = x$ holds. To find the fixed points, it is more convenient to use the identification of $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and the general form of isometries in this context given by the equation

$$R_\theta(z) = e^{i\theta} z.$$

Theorem 2.4.2 *The composition of two reflections of $\mathbb{R}^2$ is:*

(1) A translation if the axes of the two symetries are parallel. More precisely, if b is a vector perpendicular to both axes and of length the distance between them, then their composition is the translation $t_{\pm b}$ (the sign depends on the order of composition).

(2) A rotation of angle $\pm 2\alpha$ and centered at the intersection of the two axes, if they meet at an angle α (the sign depends on the order of composition).

The composition of three reflections is either a reflection or a glide-reflection. Every glide-reflection can be obtained by composing three reflections, two of the axes being parallel and the third perpendicular to both.

Corollary 2.4.1 *Every isometry of $\mathbb{R}^2$ can be obtained by the composition of at most three reflections. In particular, the Euclidean group $E(2)$ is generated by symmetries.*

2.4.3 The symmetries of the $O(n)$ group. Cartan's theorem

Symmetries with respect to hyperplanes of the $\mathbb{R}^n$ Euclidean space play an essential role in the generation of the orthogonal group $O(n)$. Consider a hyperplane H passing through the origin of $\mathbb{R}^n$, with respect to which we define the symmetry.

Let $L = H^\perp$ be the 1-dimensional space complementary to H. We have the decomposition $\mathbb{R}^n = H \oplus L$, so any vector $v \in \mathbb{R}^n$ can be uniquely written in the form $v = w + u$, where $w \in H$ şi $u \in L$. We define the symmetry with respect to H of the space $\mathbb{R}^n$ as the map

$$s_H : \mathbb{R}^n \to \mathbb{R}^n,\ s_H(v) = s_H(w + u) = w - u.$$

It is clear that s_H fixes the points of hyperplane H, and the point $u \in L$ transforms into $-u$, its symmetric with respect to the origin. Also, s_H is a linear map and because $w \perp u$, we have $\|s_H(v)\|^2 = \|w\|^2 + \|u\|^2 = \|v\|^2$, that is $\|s_H(v)\| = \|v\|$, $v \in \mathbb{R}^n$. Therefore $\|s_H(v) - s_H(v')\| = \|s_H(v - v')\| = \|v - v'\|$, for all $v, v' \in \mathbb{R}^n$, so $s_H \in Iso(\mathbb{R}^n)$.

Theorem 2.4.3 *Let w and w' be two distinct points in $\mathbb{R}^n$. There exists a unique symmetry s'_H of $\mathbb{R}^n$ such that $s'_H(w) = w'$. Additionally, we have $s'_H \in O(n)$ if and only if $\|w\| = \|w'\|$.*

Theorem 2.4.4 (Cartan) *The group $O(n)$ is generated by its symmetries.*

Corollary 2.4.2 *Any isometry of Euclidean space $\mathbb{R}^n$ is a composition of at most $n+1$ symmetries. An isometry that fixes at least one point, is a composition of at most n symmetries.*

The group S_n is found as a subgroup of $O(n)$, by identifying $\sigma \mapsto X_\sigma$, where X_σ is the matrix that has one element 1 on each row and on each column, and the others equal to 0. Moreover, considering this morphism of groups as being $u : S_n \to O(n)$, we have

$$\begin{aligned} detu(\sigma) = \det X_\sigma &= \sum_{\tau \in S_n} (-1)^{\operatorname{sgn}(\tau)} a_{1\tau(1)} \dots a_{n\tau(n)} \\ &= (-1)^{\operatorname{sgn}(\tau)} a_{1\sigma(1)} \dots a_{n\sigma(n)} = (-1)^{\operatorname{sgn}(\sigma)}, \end{aligned}$$

where $X_\sigma = (a_{ij})_{1 \le i,j \le n}$. This calculation shows that the following diagram is commutative

$$\begin{tikzcd} S_n \arrow[rr, "u"] \arrow[dr, "\operatorname{sgn}"'] & & O(n) \arrow[dl, "\det"] \\ & \{-1, 1\} & \end{tikzcd}$$

Therefore, if $\sigma \in A_n$, then $\det u(\sigma) = 1$, so we have $u|_{A_n} \to SO(n)$, which shows that the alternate group A_n identifies with a subgroup of $SO(n)$.

2.5 The isometries of the CC plane

2.5.1 The isometry group of the CC plane

One of the main problems in the geometric investigations of the metric space X endowed with the metric d, is describing the $Iso(X, d)$ group of isometries. If X is the Euclidean plane with the usual Euclidean metric, we saw that $Iso(X, d)$ consists of all translations, rotations, central symmetries and axial symmetries. Moreover, a consequence of Theorem 2.4.1 is that for the Euclidean plane, the isometry group $E(2)$ is the semi-direct product of its two subgroups, $O(2)$ (the orthogonal group) and $T(2)$ (the translation group). The taxicab plane isometry group was determined by D.J. Schattschneider the paper [59], a result which we will reobtain in a general context. The taxicab metric provides an important first example of metric that does not come from a scalar product, which decisively affects the structure of its group of isometries.

In this section we study the general problem of the above isometry group, for the $\mathbb{R}^2$ plane endowed with the Chinese checkers game metric d_c defined by:

$$d_c(X, Y) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sqrt{2} - 1)\min\{|x_1 - x_2|, |y_1 - y_2|\},$$

where $X = (x_1, y_1)$ şi $Y = (x_2, y_2)$.

Proposition 2.5.1 *Any translation of the Euclidean plane is an isometry of* $\mathbb{R}^2_c$.

Definition 2.5.1 Let P be a point and l be a Euclidean line in $\mathbb{R}^2_c$. Let Q be a point on l such that $PQ \perp l$. If P' is a point in the opposite half-plane defined by the line l, such that we have $d_c(P, Q) = d_c(P', Q)$, then P' is called the reflection on P with respect to l. The line l is called the axis of symmetry.

Lemma 2.5.1 *Let l be the line determined by the points $A(x_1, y_1)$, $B(x_2, y_2)$ in the Euclidean plane and d_E be the usual Euclidean metric. If we denote by m the slope of l, then we have the relation:*

$$d_c(A, B) = \frac{M}{\sqrt{m^2 + 1}} d_E(A, B),$$

where

$$M = \begin{cases} 1 + (\sqrt{2} - 1)|m| & \text{if} \quad |m| \leq 1 \\ |m| + \sqrt{2} - 1 & \text{if} \quad |m| \geq 1. \end{cases}$$

Corollary 2.5.1 *If A, B and X are three collinear points in $\mathbb{R}^2$, then*

$$d_E(X, A) = d_E(X, B)$$

if and only if $d_c(X, A) = d_c(X, B)$, i.e. the middle of a segment is the same, relative to the two metrics considered.

Corollary 2.5.2 *If A, B and X are three distinct collinear points in the Euclidean plane, then we have $d_c(X, A)/d_c(X, B) = d_E(X, A)/d_E(X, B)$, i.e. the ratio defined by the distance d_c coincides with the ratio defined by distance d_E.*

Remark 2.5.1 The last corollary shows the validity of the well-known theorems of Menelaus and Ceva in the $\mathbb{R}_c^2$ plane.

The next result determines the axial symmetries which are isometries of the $\mathbb{R}_c^2$ plane.

Proposition 2.5.2 *An axial symmetry with axis of equation $y = mx$ is an isometry in $\mathbb{R}_c^2$ if and only if*

$$m \in \{0, \pm 1, \pm(\sqrt{2} - 1), \pm(\sqrt{2} + 1), \infty\}.$$

Proposition 2.5.3 *There are only 8 Euclidean rotations that preserve d_c-distances. In other words, the set of isometric rotations in $\mathbb{R}_c^2$, is*

$$R_c = \left\{ r_\theta : \ \theta = k\frac{\pi}{4}, \ k = 0, 1, \ldots, 7 \right\}.$$

Thus we have determined the "orthogonal group" of plane $\mathbb{R}_c^2$, consisting of 8 axial symmetries and 8 rotations, i.e. we have $O_c(2) = R_c \cup S_c$. This is the dihedral group D_8, the Euclidean symmetry group of a regular octagon. Now we show that the group $Iso(\mathbb{R}_c^2)$ is isomorphic to $T(2) \rtimes O_c(2)$, the semi-direct product of these groups.

Definition 2.5.2 Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two fixed points in $\mathbb{R}^2_c$. The d_c-segment determined by the points A and B is the set

$$\widehat{AB} = \{X : d_c(A, X) + d_c(B, X) = d_c(A, B)\}.$$

Proposition 2.5.4 *Let $\phi : \mathbb{R}^2_c \to \mathbb{R}^2_c$ be an isometry and let $\widehat{AB}$ be the standard parallelogram of the points A and B. The relation holds:*

$$\phi(\widehat{AB}) = \widehat{\phi(A)\phi(B)}.$$

Corollary 2.5.3 *Let $\phi : \mathbb{R}^2_c \to \mathbb{R}^2_c$ be an isometry and let $\widehat{AB}$ be the standard parallelogram of the points A and B. The mapping ϕ transforms its vertices into vertices and preserves the lengths of the sides of $\widehat{AB}$.*

Proposition 2.5.5 *Let $f : \mathbb{R}^2_c \to \mathbb{R}^2_c$ be an isometry which fixes the origin, i.e. satisfies $f(O) = O$. Then $f \in R_c$ or $f \in S_c$.*

Theorem 2.5.1 *Let $f : \mathbb{R}^2_c \to \mathbb{R}^2_c$ be an isometry. Then there exists $T_A \in T(2)$ and $g \in O_c(2)$ such that $f = T_A \circ g$, and these transformations are unique.*

Corollary 2.5.4 *The relation holds:*

$$Iso(\mathbb{R}^2_c) \simeq \mathbb{R}^2 \rtimes D_8.$$

2.5.2 The area formula for CC triangles

The area of a triangle in the Euclidean plane can be calculated by using the well-known formula

$$A = \frac{b \cdot h}{2},$$

which in general is not true in the $\mathbb{R}^2_c$ plane. Formulas for calculating the area of a triangle in the taxicab metric are given by R. Kaya in [32] and M. Ozcan, R. Kaya in [43]. If we know the d_c-lengths b_c and h_c of the base, and of the corresponding height of a triangle in plane $\mathbb{R}^2_c$, we are interested in the calculation of its area. The following theorem answers this question and provides the formula for the Euclidean surface area of a triangle in terms of d_c-distances.

Theorem 2.5.2 *Let b_c and h_c, be the d_c-lengths of a base, and the height of a triangle in the plane $\mathbb{R}_c^2$ respectively. If we denote by m the slope of the base, then the area of the triangle is given by the formula*

$$A = \frac{1+m^2}{2M^2} b_c h_c,$$

where

$$M = \begin{cases} 1 + (\sqrt{2}-1)|m| & \text{if } \ |m| \leq 1 \\ |m| + \sqrt{2} - 1 & \text{if } \ |m| \geq 1. \end{cases}$$

2.6 The $Iso_{d_p}(\mathbb{R}^n)$, $p \neq 2$ group

2.6.1 The Mazur-Ulam theorem: a powerful tool for the investigation of isometry groups

In this subsection, E and F denote two real normal spaces. We consider the metrics d_E and d_F induced on E and F by the norms that define the two spaces. We have $d_E(x,y) = \|x-y\|_E$ but since there is no danger of confusion, we simplify by using the same notation for the two norms. A function $f : E \to F$ is an isometry, if it is surjective and preserves distances, i.e. we have

$$\|f(x) - f(y)\| = \|x - y\|, \ \forall \ x, y \in E.$$

The function f is affine if it satisfies the relation

$$f((1-t)a + tb) = (1-t)f(a) + tf(b), \tag{2.6.1}$$

for all $a, b \in E$ and $0 \leq t \leq 1$. Obviously, f is affine if and only if the function $T : E \to F$, $T(x) = f(x) - f(0)$ is linear.

Theorem 2.6.1 (Mazur-Ulam) *Any isometry $f : E \to F$, between real normed spaces is affine.*

2.6.2 Determination of the $Iso_{d_p}(\mathbb{R}^n)$ group

In this subsection we consider $X = \mathbb{R}^n$ and for all real numbers $p \geq 1$ we define the d_p metric by:

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x^i - y^i|^p \right)^{1/p}, \tag{2.6.4}$$

where $x = (x^1, \ldots, x^n), y = (y^1, \ldots, y^n) \in \mathbb{R}^n$. If $p = \infty$, then the d_∞ metric is defined by

$$d_\infty(x,y) = \max\{|x^1 - y^1|, \ldots, |x^n - y^n|\}. \tag{2.6.5}$$

In the case $p = 2$, we obtain the well-known Euclidean metric on $\mathbb{R}^n$. In this case the Ulam theorem says that $Iso_{d_2}(\mathbb{R}^n)$ is isomorphic to the semidirect product of the orthogonal group $O(n)$ and $T(n)$, where $T(n)$ is the group of translations of $\mathbb{R}^n$. The case $p \neq 2$ is very interesting. The main purpose of this chapter is to describe the groups $Iso_{d_p}(\mathbb{R}^n)$ for $p \geq 1$ and $p = \infty$. We prove that if $p \neq 2$, all these groups are isomorphic and hence they do not depend on the number p. These results appear in the paper D. Andrica, V. Bulgărean [4].

Theorem 2.6.2 *Let $p \neq 2$ be a real number, $p \geq 1$, and let $f_A : \mathbb{R}^n \to \mathbb{R}^n$ a linear function defined by the matrix $A \in M_n(\mathbb{R})$. Then $f_A \in Iso_{d_p}(\mathbb{R}^n)$ if and only if A is a matrix of permutations, i.e. each row and each column of A has exactly one nonzero element and this element is equal with ± 1.*

2.6.3 Determination of the $Iso_{d_\infty}(\mathbb{R}^n)$ group

For $p = \infty$ the unit sphere is

$$S^{n-1}_{d_\infty} = \{x \in \mathbb{R}^n : \max\{|x^1|, \ldots, |x^n|\} = 1\}. \tag{2.6.10}$$

Theorem 2.6.3 *Let $f_A : \mathbb{R}^n \to \mathbb{R}^n$ a linear function defined by the matrix $A \in M_n(\mathbb{R})$. Then $f_A \in Iso_{d_\infty}(\mathbb{R}^n)$ if and only if A is a matrix of permutation, i.e. each row and each column of A has exactly on nonzero element and this element is equal to ± 1.*

2.6.4 Common conclusion for the $Iso_{d_p}(\mathbb{R}^n)$ and $Iso_{d_\infty}(\mathbb{R}^n)$ groups

Summing up the results of subparagraphs 2.6.2 and 2.6.3 we obtain the following common result for the $Iso_{d_p}(\mathbb{R}^n)$ and $Iso_{d_\infty}(\mathbb{R}^n)$ groups:

Corollary 2.6.1 *Let $p \geq 1$, $p \neq 2$, a real number or $p = \infty$. Then the group $Iso_{d_p}(\mathbb{R}^n)$ is isomorphic with the semidirect product of groups $(\mathbb{R}^n, +)$ and $S_n \times \mathbb{Z}_2^n$, where S_n is the group of permutations of the set $\{1, 2, \ldots, n\}$.*

Remark 2.6.1 One can show directly that the $Iso(\mathbb{R}^2, d_1)$ and $Iso(\mathbb{R}^2, d_\infty)$ groups are isomorphic, considering the isometry $\alpha : (\mathbb{R}^2, d_1) \rightarrow (\mathbb{R}^2, d_\infty)$, defined by $\alpha(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$ and the applying Theorem 2.1.1.

The subgroup of linear isometries of $Iso_{d_p}(\mathbb{R}^3)$ consists of 48 linear maps defined by the corresponding matrices described in Theorem 2.6.3. Also, these linear functions define all symmetries of the sphere $S^2_{d_p}$. For $p = 1$, the sphere $S^2_{d_1}$ is the border of an octahedron with vertices at points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$.

The subgroup of linear isometries of $Iso_{d_\infty}(\mathbb{R}^n)$ consists of $2^n n!$ linear maps defined by the matrices of permutations in Theorem 2.6.3. These linear maps define all the symmetries of the sphere $S^{n-1}_{d_\infty}$, which is the border of the n-cube with vertices at points $(\pm 1, \ldots, \pm 1)$, for all choices of signs $+$ and $-$.

2.6.5 The d-isometric dimension of a finite group

For a finite group G we define the d-isometric dimension of G as the smallest integer n with the property that the group can be realized as an isometry group of a subset of $\mathbb{R}^n$, where d is a given metric on $\mathbb{R}^n$.

Theorem 2.6.4 *Let G be a finite group. Then the d_2-isometric dimension $\delta_{d_2}(G)$ equals the minimum dimension of the real representation of G.*

As a consequence of Theorem 2.6.4, the following result is proved in [46]:

Corollary 2.6.2 *If $G_1, \ldots, G_s$ are finite groups, then the inequality holds*

$$\delta_{d_2}(G_1 \oplus \ldots \oplus G_s) \leq \delta_{d_2}(G_1) + \ldots + \delta_{d_2}(G_s).$$

Theorem 2.6.5 *Let $p \geq 1$, $p \neq 2$, a real number, or $p = \infty$. The inequality holds*

$$\delta_{d_p}(S_n \times \mathbb{Z}_2^n) \leq n,$$

where S_n is the group of permutations of the set $\{1, 2, \ldots, n\}$.

Theorem 2.6.6 *The relation holds*

$$\delta_C(D_8) = 2,$$

where δ_C is the isometric dimension relative to the metric d_c of the plane $\mathbb{R}_c^2$, and D_8 is the dihedral group.

2.7 The geometric realization of a finite group. Asimov's Theorem

The problem of the realization a group as an isometry group of a metric space is important. In this section we present the results of D. Asimov [6], which contain affirmative solution of this problem in the case of finite groups.

Let G be a finite group with $k+1$ elements $\{1, g_1, \ldots, g_k\}$.

Theorem 2.7.1 *There exists a Riemann metric on the sphere S^{k-1} such that the associated isometry group is isomorphic to G.*

Proposition 2.7.1 *With the metric induced from $\mathbb{R}^k$ on the sphere S^{k-1}, the metric space X has its isometry group isomorphic to G.*

Proposition 2.7.2 *The relation holds $Iso(M) \simeq G$.*

Corollary 2.7.1 *Any finite group G is isomorphic with the group of isometries of a finite subset X_G of a Euclidean space. If* $\operatorname{card}(G) = k$, *then X_G can be chosen with* $\operatorname{card}(X_G) = k^2 - k$, *in a Euclidean space of dimension $k-1$.*

Example 2.7.1 Consider the triangle ABC in the Euclidean plane, having unequal sides and let $X = \{A, B, C\}$ with the induced Euclidean metric. It is clear that the only isometry $f : X \to X$ is $f = 1_X$, so we have $Iso(X) \simeq 0$.

Example 2.7.2 If ABC is an isosceles triangle, $AB = AC \neq BC$, then the space $X = \{A, B, C\}$ has two isometries, namely 1_X and $g : X \to X$ defined by $g(A) = A$, $g(B) = C$, $g(C) = B$. In this case, we obtain

$$Iso(X) \simeq \mathbb{Z}_2.$$

Example 2.7.3 If ABC is an equilateral triangle, then the space $X = \{A, B, C\}$ has 6 isometries, namely 1_X, three axial symmetries with axes the heights of the triangle, 2 rotations of angle $\frac{2\pi}{3}$ and center in the center of the triangle. In this case we have

$$Iso(X) \simeq D_3,$$

the dihedral group of order 6. In this case the geometric realization of the group D_3 is optimal. Indeed, from Corollary 2.7.1 we have $k = 6$, so the set X_{D_3} may contain $36 - 6 = 30$ points, and can be considered in the space $\mathbb{R}^5$, which is far from the case of the example above.

2.8 Remarks on the isometry group of the French railway metric

France is a centralized country in terms of railways, almost any train traveling between two cities having to pass through Paris. This motivates the name of the French railway metric for the next construction. Let (X, d) be a metric space and $p \in X$, a fixed point. We define a new metric on X, denoted $d_{F,p}$, by

$$d_{F,p}(x, y) = \begin{cases} 0 \text{ if and only if } x = y \\ d(x, p) + d(p, y) \text{ if } x \neq y \end{cases}$$

and called the French railway metric.

Following the work of V. Bulgărean [14], in this section we study the properties of isometries with respect to the $d_{F,p}$ metric. Let us begin by noticing that the geometry generated by the metric $d_{F,p}$ is very poor, all geometrical properties being concentrated at point p.

2.8.1 The isometry group of the $d_{F,p}$ metric

Let (X,d) be a metric space and $p \in X$, a fixed point. Consider $Iso^{(p)}(X,d)$ the subgroup of $Iso(X,d)$ defined by all isometries of the space (X,d) that fix the point p, i.e.

$$Iso^{(p)}(X,d) = \{f \in Iso(X,d) : \ f(p) = p\}.$$

Theorem 2.8.1 *$Iso^{(p)}(X,d)$ is a subgroup of $Iso(X,d_{F,p})$. In particular, the inclusion $Iso^{(p)}(X,d) \subseteq Iso(X,d_{F,p})$ holds.*

Theorem 2.8.2 *For any isometry $f \in Iso(X,d_{F,p})$, the point p is fixed, i.e. the relation $f(p) = p$ holds.*

Taking into account the classification of isometries presented in subsection 2.1.1, we obtain the following result:

Corollary 2.8.1 *For any metric space (X,d) and any point $p \in X$, the metric space $(X,d_{F,p})$ is elliptical, i.e. all its isometries are elliptical.*

Chapter 3

Special problems related to isometries

3.1 Frieze groups in the Euclidean plane

The notion of frieze is defined in DEX as "an ornament in the form of a horizontal strip with paintings or reliefs, around a bowl, a room, a coffin, and so on". We will consider such strips, in the plane, in which some simple geometric pattern repeats indefinitely. In this section we describe, in the language of group theory, the possible configurations, a central role being played by symmetries and translations in the Euclidean plane.

3.1.1 Generators and relations in a group

We leave aside for now the Euclidean plane isometries to discuss the concepts that will be used to describe and classify the Frieze groups.

We denote $G = \langle X|R\rangle$, where the symbols X, R, G have the following significance:

The set X of generators consists by symbols, usually finite, $x_1, \ldots, x_n$, $n \in \mathbb{N} \cup \{0\}$. We think of the symbols $x_i^{\pm 1}$, $1 \leq i \leq n$, as being the letters in an alphabet $X^{\pm}$, used to form words. The length of a word is the number of letters used. This number is finite, and zero if the word is "empty", denoted e. A word is reduced if it doesn't contain the letters $x_i^{\pm 1}$ on neighboring places, for all $i \in \{1, \dots, n\}$. We denote by $F(X)$ the set of reduced words.

The set R consists of relations, meaning equations between words, usually finite in number, of the form $u_i = v_i$, where $u_i, v_i \in F(X)$, $i \in \{1, \dots, m\}$, $m \in \mathbb{N} \cup \{0\}$. We say that $\langle X | R \rangle$ is a presentation of the group G, or equivalently, $G = \langle X | R \rangle$, if the following three conditions hold:

(i) every element of G can be written as a word in $X^{\pm}$;

(ii) the equations of R occur between elements of G;

(iii) any equation between words in $X^{\pm}$ of G is obtained from the relations contained in R.

Lemma 3.1.1 *Let T be a cyclical subgroup of G, t a generator for T and $r \in G$. Then $r^{-1}Tr = \{r^{-1}xr : \ x \in T\}$ is cyclical subgroup of G and $r^{-1}tr$ is its generator.*

3.1.2 Composition of reflections across different axes

Back to isometries in Euclidean plane, we analyze in detail the composition of symmetries of different axes.

3.1.3 Classification of frieze groups

Definition 3.1.1 A subgroup G of $E(2)$ is discrete if for any point $O \in \mathbb{R}^2$, any disk of center O contains a finite number of points from the set $\{gO : \ g \in G\}$.

Definition 3.1.2 A frieze group is a discrete subgroup of $E(2)$ which has infinite subgroups of cyclic translations, i.e. a subgroup of $E(2)$ that has subgroups of translations generated by a single translation.

There are exactly seven frieze groups. Before we begin to classify them, we give two lemmas that illustrate some properties of translations.

Lemma 3.1.2 *Let r be a glide symmetry and t be a translation such that the axes of r and t are parallel. Then r and t commute.*

Lemma 3.1.3 *Let $T \leq G \leq E(2)$, where T is a subgroup of translations of G. Then T is normal subgroup of G.*

Theorem 3.1.1 *If F is a frieze group, then F is one of the following 7 possible groups:*

$F_1 = \langle t | \rangle$
$F_1^2 = \langle t, r | r^2 = 1, r^{-1}tr = t \rangle$
$F_1^2 = \langle t, r | r^2 = 1, r^{-1}tr = t^{-1} \rangle$
$F_1^3 = \langle t, r | r^2 = 1, r^{-1}tr = t \rangle$
$F_2 = \langle t, s | t^s = t^{-1}, s^2 = 1 \rangle$
$F_2^1 = \langle t, s, r | s^2 = 1, t^s = t^{-1}, r^2 = 1, t^r = t, (sr)^2 = 1 \rangle$
$F_2^2 = \langle t, s, t | s^2 = 1, t^s = t^{-1}, r^2 = t, t^r = t, (sr)^2 = 1 \rangle.$

3.2 Mappings that preserve some geometric properties

3.2.1 The Aleksandrov-Rassias problem

Let (X, d_X), (Y, d_Y) be two metric spaces and the function $f : X \to Y$. We say that f preserves distances $r > 0$, if we have $d_Y(f(x), f(y)) = r$, for all $x, y \in X$ with $d_X(x, y) = r$. Evidently, f is an isometry if and only if f is surjective and preserves any distance.

In 1970, A.D. Aleksandrov [2] raised the question that if f preserves a single distance, does it follow that it is an isometry? The answer was given in the case of Euclidean spaces, by F.S. Beckman and D.A. Quarles [9]. They considered the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} x + 1 & \text{if} \quad x \in \mathbb{Z} \\ x & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$$

The Aleksandrov-Rassias problem. *If (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ is a continuous applications, surjective, and preserves distance 1, does it follow that f is an isometry?*

Example 3.2.1 Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = x + \frac{1}{7} \sin 2\pi x.$$

This is a diffeomorphism, preserving distance 1, but it is not an isometry.

B. Mielnik and Th. M. Rassias [40] proved the following result: Any homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 3)$ which preserves a distance r, is an isometry of $\mathbb{R}^n$. Regarding the isometries between different spaces, Th. M. Rassias [50] proved that the following property holds: For any natural number $n \geq 1$, there exists a natural number m_n such that for $N \geq m_n$ there exist maps $f : \mathbb{R}^n \to \mathbb{R}^N$ which are not isometries but preserve distance 1.

3.2.2 Mappings in $\mathbb{R}^3$ that transform cubes into cubes

It is very interesting to investigate whether the distance conservation property, $r > 0$, can be replaced by conservation properties of simple geometric configurations, in order to have an affirmative answer to the appropriate Aleksandrov-Rassias problem.

S.M. Jung has shown that if we have an injective map $f : \mathbb{R}^n \to \mathbb{R}^n$ $(n \geq 2)$ which transforms any equilateral triangle (quadrilateral or hexagon) with side length $a > 0$ in a figure of the same type but with side length $b > 0$ then, up to a translation, there exists a linear isometry $g : \mathbb{R}^n \to \mathbb{R}^n$ such that we have

$$f(x) = \left(\frac{b}{a}\right) g(x).$$

Furthermore, the authors of paper [29] proved that if an injective map $f : \mathbb{R}^n \to \mathbb{R}^n$ transforms any circle of radius 1 into a circle of radius 1, then f is a linear isometry up to a translation. We extend the results of [28] to the 3-dimensional case and we prove the following result: if we have an injective map $f : \mathbb{R}^3 \to \mathbb{R}^3$ that transforms any cube into a cube, then f is a linear isometry up to a translation.

Lemma 3.2.1 *Let the injective map $f : \mathbb{R}^3 \to \mathbb{R}^3$ that transforms any cube into a cube. For any cubes with side length 1, A and B, if $\mathrm{Int}(A) \cap \mathrm{Int}(B) = \emptyset$, then $\mathrm{Int}\{f(A)\} \cap \mathrm{Int}\{f(B)\} = \emptyset$.*

Next we show that if an injective function transforms cubes into cubes, then it is an isometry. More precisely we have the theorem:

Theorem 3.2.1 *If the injective map* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *transforms any cube into a cube, then* f *is a linear isometry up to a translation.*

3.3 The isometry group of the sphere. Results on isometries between spheres

Given two metric spaces X and Y, an important problem is finding minimal conditions for an map $f : X \to Y$ to be an isometry. There is a rich literature in this direction, in the case when the domain and codomain of the map have the same size and the map preserves only one distance.

Let S^n be the unit n-dimensional sphere in $\mathbb{R}^{n+1}$. We determine the group of isometries $Iso(S^n)$, with the induced metric of $\mathbb{R}^{n+1}$. For the cases $n = 1$ and $n = 2$, we give geometric proofs for the classification of isometries. Also, we show that a map $f : S^n \to S^p$, $p \geq n > 1$, which preserves two distances and keeps an angle invariant, is an isometry. This problem was proposed by Th. M. Rassias. The general proof for $\mathbb{R}^n$ does not work in this context because it uses the properties of equilateral triangles and the rhombus, geometric properties of the Euclidean plane. In this section we present a proof for the problem mentioned above. Assuming continuity of f, we show that, assuming that it preserves an irrational angular distance, then f is an isometry. For simplicity we use the notations $A, B, C, \ldots$ for the points from the domain and $A', B', C', \ldots$ for their corresponding images through f.

Theorem 3.3.1 *Any isometry* $f : S^1 \to S^1$ *is a rotation or an axial symmetry.*

Theorem 3.3.2 *Any isometry* $f : S^2 \to S^2$ *is a planar symmetry, a rotation or a roto-symmetry (composition of a rotation and a symmetry).*

Theorem 3.3.3 *Any isometry* $f : S^n \to S^n$ *is a composition of rotations and possibly a symmetry.*

Theorem 3.3.4 *The relation holds* $Iso(S^n) \simeq O(n + 1)$.

Theorem 3.3.5 *Any isometry* $f \in Iso(S^n)$ *is a composition of at most* $\left[\dfrac{n+1}{2}\right]$ *proper rotations of* S^n *and possibly a symmetry with respect to a hyperplane passing through the origin.*

Remark 3.3.1 1. The above results show that the $Iso(S^n)$ group is generated by rotations and symmetries.

2. Theorem 3.3.2 implies that the isometry group of a regular polyhedron may contain only proper rotations of the S^2 sphere, planar symmetries and roto-symmetries (compositions of proper rotations with planar symmetries). In fact, according to Hessel's theorem, there are only 14 possible types of such groups.

Theorem 3.3.6 *There not exist functions $f : S^n \to S^2, n \geq 3$, that preserves angles $\theta, m\theta$, where $m\theta < \pi$ and m is a natural number ≥ 2.*

Remark 3.3.2 1) By the above proof it results that every application $f : S^2 \to S^2$, that preserves angles θ, $m\theta$, where $m\theta < \pi$ and m is a natural number ≥ 2, is an isometry of the sphere S^2.

2) The above proof is valid, with appropriate modifications, if we replace codomain S^2 with S^p, $p \geq 2$. Suppose that f preserves angles θ and 2θ. If we fix the images of A, B, in the plane X_1X_p with A as the north pole and $B = (\sin\theta, 0, \ldots, 0, \cos\theta)$, where $\widehat{AOB} = \widehat{BOC} = \theta$ and $\widehat{AOC} = 2\theta$ as above, then a possible position for C' would be the intersection of $(p-1)$-spheres of equations

$$x_1^2 + x_2^2 + \ldots + x_{p-1}^2 = \sin^2 2\theta, \quad x_p = \cos 2\theta$$

and

$$(x_1 - \sin\theta\cos\theta)^2 + x_2^2 + \ldots + (x_p - \cos^2\theta)^2 = \sin^2\theta.$$

Theorem 3.3.7 *Let $f : S^n \to S^p$, $p \geq n > 1$, be a continuous map that preserves angles $\theta, m\theta$, where $m > 1$ and $m\theta < \pi$. Then f is an isometry.*

Theorem 3.3.8 *Let $f : S^n \to S^n$ be a map that preserves angle θ. Suppose $\arccos\left(\frac{1}{m+\sec\theta}\right)$ is irrational for $0 \leq m \leq n-1$. Then f is an isometry.*

Remark 3.3.3 If there exists an angle θ such that $\arccos\left(\dfrac{1}{n+\sec\theta}\right)$ is irrational for all $n \geq 0$, then any continuous map $f : S^n \to S^n$ that preserves angle θ is an isometry.

3.4 The isometry group of a locally compact metric space

It is well known from the classical work of D. van Dantzig and B.L. van de Waerden [23] that if (X, d) is a locally compact and connected metric space, then its $Iso(X, d)$ group of isometries equipped with punctual convergence topology is locally compact and acts properly on X. Recently it was shown that punctual closure of $Iso(X, d)$ is locally compact if the space $\Sigma(X)$ of connected components of X is quasi-compact (is compact but not necessarily Hausdorff) with respect to the quotient topology. The problem of whether $Iso(X, d)$ is closed in $C(X, X)$ (the space of all continuous mappings from X to X, endowed with punctual convergence topology) is still unresolved. The purpose of this section is to show that if $\Sigma(X)$ is quasi-compact then $Iso(X, d)$ coincides with his Ellis semigroup. More precisely, we prove the following result.

Theorem 3.4.1 *Let (X, d) be a locally compact metric space. We denote by $Iso(X, d)$ the isometry group endowed with punctual convergence topology and by $\Sigma(X)$ the space of connected components of X, endowed with the quotient topology. Then:*

1. If $\Sigma(X)$ is not quasi-compact, then it is not necessary that $Iso(X, d)$ be locally compact, or to act properly on X.

2. If $\Sigma(X)$ is quasi-compact, then the following properties hold:

(a) $Iso(X, d)$ is locally compact;

(b) the action of $Iso(X, d)$ on X is not always proper;

(c) the action of $Iso(X, d)$ on X is proper under the assumption that the space X is connected.

Our approach is based on the sets $(x, V_x) = \{g \in Iso(X, d) : g(x) \in V_x\}$, where V_x is a neighborhood of $x \in X$. These sets form a subbase of neighborhoods of the identity (unit group) with respect to punctual convergence topology, which is the natural topology of $Iso(X, d)$.

The following two simple examples establish claims 1 and 2 (b) of the theorem stated above.

Example 3.4.1 Let $X = \mathbb{Z}$ endowed with the discrete metric. Obviously $\Sigma(X)$ is not a quasi-compact space. It can be easily seen that $Iso(X, d)$ is the group of all bijections of $\mathbb{Z}$. This is not locally compact with respect to the punctual convergence topology, so it cannot act on its own on a locally compact space.

Example 3.4.2 Let $X = Y \cup \{(1, 0)\} \subset \mathbb{R}^2$ where $Y = \{(0, y) : y \in \mathbb{R}\}$ şi $d = \min\{1, \delta\}$, where δ is the Euclidean metric. As we shall see, the group $Iso(X, d)$ is locally compact. However, the action of $Iso(X, d)$ on X is not proper, because the isotropy group of point $(1, 0)$ is not compact, since it contains the translations of Y. Thus, the action of $Iso(X, d)$ on X is not proper, even if X has two connected components.

3.4.1 Local compactness of the $Iso(X, d)$ group

The next result is essential for the investigation of the conditions (a) and (b) above:

Lemma 3.4.1 *Let (X, d) be a locally compact metric space, $F \subseteq Iso(X, d)$ and*

$$K(F) = \{x \in X : F(x) = \{f(x) : f \in F\} \text{ is relatively compact}\}.$$

Then $K(F)$ is an open and closed subset of X.

Lemma 3.4.2 *Let (X, d) be a locally compact metric space with quasi-compact space of connected components $\Sigma(X)$. Then condition (a) is satisfied.*

Example 3.4.3 Let $X = \mathbb{Z}$ be endowed with the discrete metric. If $f_n(z) = z$ for $-n < z < 0$, $f_n(-z) = 0$, and $f_n(z) = z + 1$ in other cases, then $f_n \to f$, where $f(z) = z$ for $z < 0$, and $f(z) = z + 1$ for $z \geq 0$. It follows that f_n is an isometry for all n, but f is not surjective because $0 \notin f(\mathbb{Z})$.

Lemma 3.4.3 *If $\Sigma(X)$ is quasi-compact and (f_n), $f_n \in Iso(X, d)$ is a sequence such that $f_n \to f$ with respect to the punctual convergence topology, then $f(X)$ is open and closed in X.*

Proposition 3.4.1 *If (X, d) is a locally compact metric space and $\Sigma(X)$ is quasi-compact, then $Iso(X, d)$ is closed in $C(X, X)$.*

Proposition 3.4.2 *There exists a subsequence $\{S_{n_k}\}_{k \in \mathbb{N}}$ of $\{S_n\}$ such that there exists $x_k \in S_k$ with $x_k \to x_0$, where $x_0 \in X$.*

Theorem 3.4.2 *If $\Sigma(X)$ is quasi-compact, then $Iso(X, d)$ is locally compact.*

3.4.2 The proper action of the $Iso(X, d)$ group on the space X

In this section, applying the methods previously used, we present a complete proof of the following result:

Proposition 3.4.3 *If (X, d) is locally compact and connected, then the $Iso(X, d)$ group is locally compact and its action on X is proper.*

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