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# Contributions to the Theory of Equilibrium Problems and Variational Inequalities

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# Contents

# Introduction

One of the most important topics in nonlinear analysis and several applied fields is the so called *equilibrium problem*. The equilibrium theory, which is a part of the nonlinear analysis, provides a general, unified and natural framework for the study of a large variety of problems, such as: *optimization problems, variational inequalities problems, saddle point problems, complementarity problems, Nash equilibria problems and fixed point problems.* These problems often occur in economics, finance, network analysis, mechanics, physics, etc.
According to the bifunctions involved, one can distinguish two forms: the *scalar equilibrium problem* and the *vector equilibrium problem*. In this thesis we deal with both forms.

The first equilibrium problem studied in the literature, was the scalar equilibrium problem which can be stated as:

$$(SEP) \qquad \text{find } \bar{a} \in A \text{ such that } f(\bar{a}, b) \geq 0 \text{ for all } b \in B,$$

where $A$ and $B$ are two nonempty sets, and $f : A \times B \to \mathbb{R}$ is a given bifunction.

As far as we know the term "equilibrium problem" was attributed in E. Blum and W. Oettli [22], but the problem itself has been investigated more than twenty years before in a paper of Ky Fan [44] in connection with the so called "intersection theorems" (i.e., results stating the nonemptiness of the intersection of a certain family of sets). Ky Fan considered $(SEP)$ in the special case $A = B$ a compact convex subset of a Hausdorff topological vector space and termed it "minimax inequality". In the same year, H. Brézis, G. Nirenberg and G. Stampacchia [23] improved Ky Fan's result, extending it to a not necessarily compact set, but assuming instead a so-called "coercivity condition", which is automatically satisfied when the set is compact.

$(SEP)$ has been extensively studied in recent years, see, for instance, M. Bianchi and R. Pini [18], M. Bianchi, G. Kassay and R. Pini [20], M. Bianchi and S. Schaible [16], G. Bigi, M. Castellani and G. Kassay [21], W. Oettli [104], A.N. Iusem, G. Kassay and W. Sosa [67], A.N. Iusem and W. Sosa [64].

New necessary (and in some cases also sufficient) conditions for existence of solutions in infinite dimensional spaces were proposed, among others, in A.N. Iusem and W. Sosa [64], A.N. Iusem, G. Kassay and W. Sosa [67] and G. Kassay and M. Miholca [78]. Note that A.N. Iusem, G. Kassay and W. Sosa [66] obtained existence results concerning $(SEP)$ in finite dimensional cases, where, obviously the assumptions are less demanding.

The extension of the scalar equilibrium problem to vector equilibrium problems can be achieved in different ways. Given a real topological vector space $Y$, a convex cone $K \subseteq Y$ with $\operatorname{int} K \neq \emptyset$ (where $\operatorname{int} K$ denotes the interior of $K$), two nonempty sets $A$ and $B$, and a bifunction $f : A \times B \to Y$, it can be formulated the following vector equilibrium problem, known in the literature as the weak vector equilibrium problem:

$$(VEP) \qquad \text{find } \bar{a} \in A \text{ such that } f(\bar{a}, b) \notin -\operatorname{int} K \text{ for all } b \in B.$$

$(VEP)$ has also been extensively studied in recent years, see, for instance, Q.H. Ansari, S. Schaible and J.C. Yao [8], M. Bianchi, N. Hadjisavvas and S. Schaible [17], A. Capătă and G. Kassay [30], D.T. Luc [93], and the references therein.

The vector equilibrium problems contain as particular cases *vector optimization problems, vector cone saddle point problems, vector variational inequality problems*, which arise in economics, physics, mechanics, etc. The latter will be described in what follows.

Suppose that $X$ is also a real topological vector space and denote by $L(X,Y)$ the set of all linear and continuous maps from $X$ to $Y$.
Let $T : X \to L(X,Y)$, set $A = B := \operatorname{dom} T$.

(i) The Stampacchia vector variational inequality problem can be formulated as follows:

$$(SVVI) \qquad \text{find } \bar{a} \in A \text{ such that } \langle T(\bar{a}), b - \bar{a}\rangle \notin -\operatorname{int} K \text{ for all } b \in A.$$

If we consider $f : A \times A \to Y$ given by

$$f(a,b) = \langle T(a), b - a\rangle,$$

then $(VEP)$ is equivalent with $(SVVI)$.

(ii) The Minty vector variational inequality problem can be formulated as follows:

$$(MVVI) \qquad \text{find } \bar{a} \in A \text{ such that } \langle T(b), b - \bar{a}\rangle \notin -\operatorname{int} K \text{ for all } b \in A.$$

If we consider $g : A \times A \to Y$ given by

$$g(a,b) = f(b,a) = \langle T(b), a - b\rangle,$$

with $f$ given at item (i), then one may see that $(VEP)$ with respect to $g$ is equivalent with $(MVVI)$.

Vector variational inequalities have shown to be important mathematical models in the study of many real problems, in particular in network equilibrium models ranging from spatial price equilibrium problems and imperfect competitive oligopolistic market equilibrium problems to general financial or traffic equilibrium problems.

In the setting of vector equilibrium problem, if we take $Y := \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$ a pointed convex cone, $A = B$ nonempty set, and $f(a,b) = F(b) - F(a)$, where $F : A \to \mathbb{R}^n$, then the $(VEP)$ becomes the (finite dimensional) vector optimization problem, denoted $(VOP)$. For a presentation and results concerning $(VOP)$ the reader can consult, for instance, the monograph of N. Popovici [109] and the references therein.

In 1998, Giannessi [54] first used the so called *Minty-type vector variational inequality* to establish the necessary and sufficient conditions for a point to be an efficient solution of a vector optimization problem for differentiable and convex functions. Since then, several researchers have studied $(VOP)$ by using several kinds of Minty-type vector variational inequality under different assumptions, see S. Al-Homidan and Q.H. Ansari [1], Q.H. Ansari and G.M. Lee [3], Q.H. Ansari, M. Rezaie and J. Zafarani [5], Q.H. Ansari and J.-K. Yao [6], G.R. Garzon, R.O. Gomez and A.R. Lizana [53], S.K. Mishra and S.Y. Wang [101] and the references therein. Consequently, vector variational inequalities have been generalized in various directions, in particular, vector variational-like inequality problems, see S. Al-Homidan and Q.H. Ansari [1], F. Giannessi, A. Maugeri and P.M. Pardalos [56], T. Jabarootian and J. Zafarani [69], S.K. Mishra and S.Y. Wang [101], M. Rezaie and J. Zafarani [110], X.M. Yang and X.Q. Yang [122] and the references therein.

The aim of this thesis is to extend some of the known existence results and to present new results for $(SEP)$, $(VEP)$, and for different kinds of variational inequalities.

The thesis consists on three chapters.

The mathematical notions and results needed for the study of scalar and vector equilibrium problems are recalled in Chapter 1. Section 1.1 contains properties concerning cones, convex sets, separation and intersection theorems in infinite dimensional spaces, different generalizations of the upper semicontinuity from the scalar case and classical monotonicity for operators. Then, in Section 1.2, there are presented weakened convexity notions for vector-valued functions and their characterizations.

Chapter 2 is devoted to some existence results for $(SEP)$ and $(VEP)$. More specifically, Section 2.1 deals with $(SEP)$ in the case of compact or noncompact sets. In Subsection 2.1.1, using pseudo-upper semicontinuity instead of upper semicontinuity, we extend the existence result of G. Kassay and J. Kolumban [77]. In Subsection 2.1.2, in the case of noncompact sets, we are able to obtain existence results for $(SEP)$ in infinite dimensional case, without monotonicity by strengthening the continuity assumption on $f$ with respect to the first variable. The absence of compactness of the set $A$ can be overcome by considering different types of so-called *coercivity conditions*.

In Section 2.2 we deal with $(VEP)$ and under $K$-pseudo upper semicontinuity we extend a result of A. Capătă and G. Kassay [30].

In the last section of this chapter, Section 2.3, we obtain existence results for a special case of $(VEP)$ when $f(x, y) = g(x, y) + h(x, y)$. This problem, with $g, h : A \times A \to \mathbb{R}$, captured less attention, although it was investigated already in E. Blum and W. Oettli [22], where the authors obtained existence results by imposing their assumptions separately on $g$ and $h$. As stressed in [22], if $g = 0$, the result becomes a variant of Ky Fan's theorem [46], whereas for $h = 0$ it becomes a variant of the Browder-Minty theorem for variational inequalities (see F.E. Browder [25], [26], G.J. Minty [100]). Also, in this section the special case of reflexive Banach spaces endowed with the weak topology is separately treated; in that case mild sufficient conditions for guaranteeing coercivity are presented.

In Chapter 3 we study different kinds of variational inequalities. Section 3.1 is devoted to the study of *generalized monotone operators*. The original concept of monotonicity has been extended in various directions. Apart from their theoretical interest, *generalized monotone operators* are often more suitable to describe problems than the original concept of monotonicity, in disciplines such as economics, management science, probability theory and other applied sciences. Results concerning surjectivity are therefore especially important, as they guarantee, in particular, the existence of zeros of these operators.

In Section 3.2 sufficient conditions for the solvability of variational inequalities in infinite dimensional case, given by properly quasimonotone and quasicoercive operators, are provided. In the special case when the domain of the operator is the whole space, it shows the existence of zeroes for such kind of operators. The surjectivity results from Section 3.3 are obtained without monotonicity but with a stronger continuity condition. In Section 3.4 we deal with $\mathcal{C}$-pseudomonotone operators and we establish a result concerning variational inequalities given by $\mathcal{C}$-pseudomonotone operators, which can be seen in the framework of a reflexive Banach space a generalization of a similar result by D. Inoan and J. Kolumbán [63].

Section 3.5 is devoted to the study of (Minty and Stampacchia) generalized variational-like inequalities with set-valued mappings in topological spaces, which include as a special case the strong vector variational-like inequalities. Motivated by the works of T.Q. Bao, B.S. Mordukhovich [13], Y.P. Fang, N.J. Huang [47], Y.P. Fang, N.J. Huang [48], A.P. Farajzadeh, A.A. Harandi, K.R. Kazmi [49] and J. Zeng, S.J. Li [129], in this section we introduce several kinds of generalized invexity for set-valued mappings and study the relationships among them. Then, we establish solution relationships between several kind of vector variational-like inequality problems and a set-valued optimization problem by means of weak contingent generalized subdifferential defined for set-valued mappings.

In Section 3.6 we study the Minty vector variational-like inequality and (weak) Stampacchia vector variational-like inequality (which are closely related to the concepts of invex and preinvex functions which generalize the notion of convexity of functions), defined by means of Mordukhovich limiting subdifferentials in Asplund spaces. The concept of invexity was first introduced by M.A. Hanson [60]. More recently, the characterization and applications for generalized invexity were studied by many authors, see Q.H. Ansari and J.-K. Yao [6], A. Cambini and L. Martein [29], A. Chinchuluun and P.M. Pardalos [37], T. Jabarootian and J. Zafarani [69], G.R. Garzon, R.O. Gomez and A.R. Lizana [52], M. Soleimani-damaneh [116], M. Soleimani-damaneh [117], X.M. Yang, X.Q. Yang and K.L. Teo [123], X.M. Yang, X.Q. Yang and K.L. Teo [126] and the references therein.

We obtain some relationships between a solution of $(VOP)$ and these vector variational-inequalities using the concept of generalized invexity for vector valued functions. The main results in S. Al-Homidan and Q.H. Ansari [1], Q.H. Ansari, M. Rezaie and J. Zafarani [5], M. Rezaie and J. Zafarani [110] were obtained in the setting of Clarke subdifferential. Since the class of Clarke subdifferential is larger than the class of Mordukhovich subdifferential (see B. Mordukhovich [102]), some authors studied the vector variational-like inequalities and vector optimization problems in the setting of Mordukhovich subdifferential in order to obtain stronger results. By means of several examples we show that our results are stronger than the similar statements in the literature.

The author's original contributions are the following:

*Chapter 2*: Definition 2.1.1, Lemma 2.1.2, Example 2.1.4, Theorem 2.1.5, Theorem 2.1.11, Theorem 2.1.17, Theorem 2.1.18, Lemma 2.2.3, Definition 2.2.4, Definition 2.2.5, Definition 2.2.6, Theorem 2.2.7, Theorem 2.2.10, Theorem 2.2.11, Theorem 2.2.14, Lemma 2.3.16, Definition 2.3.20, Proposition 2.3.21, Example 2.3.22, Lemma 2.3.23, Lemma 2.3.24, Lemma 2.3.25, Example 3.6.19, Theorem 2.3.28, Example 2.3.31, Theorem 2.3.33, Proposition 2.3.34, Remark 2.3.35, Lemma 2.3.36, Lemma 2.3.38, Proposition 2.3.39, Lemma 2.3.40, Theorem 2.3.41 and Corollary 2.3.42.

*Chapter 3*: Definition 3.1.5, Remark 3.1.6, Lemma 3.1.14, Theorem 3.2.1, Example 3.2.4, Definition 3.3.1, Remark 3.3.2, Example 3.3.3, Theorem 3.3.4, Corollary 3.3.5, Example 3.3.6, Theorem 3.4.1, Theorem 3.4.2, Definition 3.5.7, Definition 3.5.13, Proposition 3.5.14, Example 3.5.15, Example 3.5.16, Definition 3.5.20, Definition 3.5.21, Theorem 3.5.22, Example 3.5.23, Theorem 3.5.31, Theorem 3.5.32, Corollary 3.5.33, Corollary 3.5.34, Corollary 3.5.35, Theorem 3.5.36, Corollary 3.5.37, Theorem 3.5.38, Corollary 3.5.39, Corollary 3.5.40, Corollary 3.5.41, Theorem 3.5.42, Corollary 3.5.43, Corollary 3.5.44, Corollary 3.5.45, Theorem 3.5.46, Corollary 3.5.47, Theorem 3.5.48, Corollary 3.5.49, Corollary 3.5.50, Corollary 3.5.51, Theorem 3.5.52, Corollary 3.5.53, Example 3.6.9, Definition 3.6.10, Definition 3.6.13, Theorem 3.6.16, Theorem 3.6.17, Example 3.6.19, Theorem 3.6.21, Corollary 3.6.22, Example 3.6.24, Theorem 3.6.25, Corollary 3.6.26, Theorem 3.6.27 and Example 3.6.29.

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1. **M. Miholca**, *Vector variational-like inequalities and set-valued optimization problems*, Ann. of Tiberiu Popoviciu Seminar **10** (2012), 75-85.

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3. **M. Miholca**, *On set-valued optimization problems and vector variational-inequalities*, Optim. Lett. **8**(**2**) (2014), 463-476.

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5. G. Kassay, **M. Miholca**, *On vector equilibrium problems given by a sum of two functions*, submitted.

6. **M. Miholca**, *Lipschitz B-preinvex functions in Asplund spaces*, submitted.

7. **M. Miholca**, *On vector variational-like inequalities and vector optimization problems in Asplund spaces*,

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while some of them are completely new in the sense that they appear for the first time here.

A significant part of the original results proved in this thesis were also presented at the following scientific conferences:

• $10^{th}$ International Symposium on Generalized Convexity and Monotonicity (GCM10), August 22-27, 2011, Cluj-Napoca, Romania.

• $11^{th}$ EUROPT Workshop on Advances in Continuous Optimization, June 26-28, 2013, Firenze, Italy.

• The Fourth International Conference on Continuous Optimization (ICCOPT), July 27-August 1, 2013, Lisbon, Portugal.

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# Chapter 1

# Preliminary notions and results

In this thesis, we use the well-known notions of vector space, topological space, and topological vector space. All vector spaces are real. Basic definitions and properties concerning these and other spaces can be found, for instance, in the books by I. Muntean [103], H.L. Royden [112], W. Rudin [114], J. Schauder [115], J. Von Neumann [119] and K. Yosida [127].

If $X$ and $Y$ are topological vector spaces, we denote by

$$L(X,Y) := \{\xi : X \to Y \mid \xi \ \text{ linear and continuous}\},$$

and by $\langle \cdot, \cdot \rangle : L(X,Y) \times X \to Y$ we understand

$$\langle \xi, x \rangle = \xi(x), \quad \forall \xi \in L(X,Y), \quad \forall x \in X.$$

In the particular case when $Y := \mathbb{R}$, $L(X,Y) := X^*$ (the dual of $X$) and $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ is the *duality pairing* given by

$$\langle x^*, x \rangle = x^*(x), \quad \forall x^* \in X^*, \quad \forall x \in X.$$

Let $\{x_\alpha\}_\alpha \subseteq X$ be a net and $x \in X$. Recall that $x_\alpha$ *converges weakly* to $x$ (denoted $x_\alpha \rightharpoonup x$) iff for each $x^* \in X^*$ one has $\langle x^*, x_\alpha \rangle$ converges (in $\mathbb{R}$) to $\langle x^*, x \rangle$. Recall also that in a reflexive Banach space, every closed, convex and bounded subset is weakly compact.

## 1.1 Convex sets, convex cones and intersection theorems

**Definition 1.1.1.** A subset $A$ of a vector space $X$ is called *convex* iff

$$\lambda A + (1-\lambda)A \subseteq A \ \text{ for all } \ \lambda \in [0,1].$$

For a subset $A$ of a topological vector space $X$, we denote by intA, clA, coA, riA, *the interior*, *the closure*, *the convex hull*, and *the relative interior* of $A$, respectively.
We recall here that a cluster point of a set $A$ in a topological space is a point $a$ whose neighborhoods all contain at least one point of the set $A$ other than $a$. By $A^{'}$ we denote the set of *cluster points* of $A$.

**Definition 1.1.2.** Let $X$ be a vector space. A subset $K \subseteq X$ is said to be a *cone* iff it is nonempty and $\lambda K \subseteq K$ for each $\lambda \geq 0$. A cone $K \subseteq X$ is said to be *convex* iff it is a convex set.

A cone $K$ in a vector space is convex if and only if $K + K \subseteq K$. For more details, see R. Hartley [61].

**Definition 1.1.3.** A cone in a vector space is said to be:

*(i)* *nontrivial* iff $K \neq \{0\}$;

*(ii)* *proper* iff $K \neq X$;

*(iii)* *pointed* iff $K \cap (-K) = \{0\}$;

*(iv)* *solid* iff $\text{int}K \neq \emptyset$.

Given a convex cone $K$ of a topological vector space $X$, *the dual cone* of $K$ is defined by

$$K^* := \{x^* \in X^* \mid \quad x^*(k) \geq 0 \ \text{ for all } \ k \in K\}.$$

Often, as in the case of real numbers, we need an order relation on vector spaces. This relation will be defined below. Given a pointed convex cone $K$ in a vector space $X$, the relation $\leq_K$ is defined as follows

$$y \leq_K x \ \text{ iff } \ x - y \in K.$$

For arbitrary $x, y, z, t$, the following properties are satisfied:

(i) $x \leq_K x$;

(ii) $x \leq_K y$ and $z \leq_K t$ implies $x + z \leq_K y + t$;

(iii) $x \leq_K y$ and $y \leq_K x$ implies $x = y$;

(iv) $x \leq_K y$ and $\lambda \in \mathbb{R}_+$ implies $\lambda x \leq_K \lambda y$.

The relation above is in particular a *partial order*.

**Lemma 1.1.4.** (see, for instance, A. Capătă and G. Kassay [30]) *Let $K$ be a nontrivial solid convex cone of a topological vector space $X$. If $k^* \in K^*$ is a nonzero functional, then $k^*(k) > 0$ for all $k \in \text{int}\, K$.*

If $X$ denotes a normed space or a Banach space and $K$ is a closed convex set in $X$, we denote by $P_K(y)$ the projection of $y$ onto $K$, that is $P_K(y) \in K$ and

$$\|y - P_K(y)\| = d(y, K) := \inf_{z \in K} \|z - y\|.$$

The next result is a variant of the celebrated Hahn-Banach theorem. Known as the separation of convex sets by closed hyperplanes, it is a cornerstone property of nonlinear and convex analysis.

**Theorem 1.1.5.** [41] *Let $X$ be a topological vector space, and let $A$ and $B$ be nonempty convex subsets of $X$ satisfying the following conditions:*

(i) *$B$ has a nonempty interior;*

(ii) $A \cap (\text{int}\, B) = \emptyset$.

*Then there exist $x^* \in X^*$, $x^* \neq 0$, and $r \in \mathbb{R}$ such that $x^*(a) \leq r \leq x^*(b)$ whenever $a \in A$ and $b \in B$.*

If $X$ is a topological vector space, $A$ and $B$ are nonempty subsets of $X$, we say that $A$ and $B$ can be *strongly separated* iff there exists $x^* \in X^*$, $x^* \neq 0$ such that

$$\inf_{a\in A}\langle x^*, a\rangle > \sup_{b\in B}\langle x^*, b\rangle.$$

**Remark 1.1.6.** Assumption $(i)$ is essential in Theorem 1.1.5 even in case $A \cap B = \emptyset$, provided $X$ is infinite dimensional, as the example below shows.

**Example 1.1.7.** Let $X$ be an infinite dimensional topological vector space and let $x^* : X \to \mathbb{R}$ be a discontinuous linear functional. Set

$$C = \{x \in X \mid x^*(x) = 0\} \quad \text{and} \quad D = \{x \in X \mid x^*(x) = 1\}.$$

We can observe that these sets are convex (in fact affine) and dense in $X$, $\operatorname{int} C = \operatorname{int} D = \emptyset$, $C \cap D = \emptyset$ and cannot be separated by a closed hyperplane.

**Remark 1.1.8.** In finite dimensional spaces, any nonempty convex set has nonempty relative interior (see [111]) and Theorem 1.1.5 holds without (i) and a weaker condition (ii).

**Theorem 1.1.9.** [111] *Let $A$ and $B$ be nonempty convex sets in $\mathbb{R}^n$ such that* $\mathrm{riA} \cap \mathrm{riB} = \emptyset$. *Then there exists a hyperplane separating $A$ and $B$ properly, that is there exists some $y \in \mathbb{R}^n$ satisfying:*

$$\inf_{a\in A}\langle y, a\rangle \geq \sup_{b\in B}\langle y, b\rangle$$

*and*

$$\langle y, a_1\rangle > \langle y, b_1\rangle,$$

*for some $a_1 \in A$ and $b_1 \in B$.*

Let $X$ be a vector space and $F : X \to [-\infty, +\infty]$ be a function. The effective domain, level sets, and epigraph of $F$ are defined by

$$\begin{aligned}
\operatorname{dom} F &:= \{x \in X \mid F(x) < \infty\},\\
L(F, r) &:= \{x \in X \mid F(x) \leq r\}, r \in \mathbb{R},\\
\operatorname{epi} F &:= \{(x, r) \in X \times \mathbb{R} \mid F(x) \leq r\},
\end{aligned}$$

respectively.

**Definition 1.1.10.** (see, for instance, [15], [111]) Let $X$ be a topological space and $F : X \to [-\infty, +\infty]$ be a function.

(i) $F$ is said to be *lower semicontinuous* at a point $x_0 \in X$ iff for any $\epsilon > 0$ there exists a neighbourhood $U$ of $x_0$ such that $f(x_0) < f(x) + \epsilon$ for all $x \in U$.

(ii) $F$ is said to be *upper semicontinuous* at a point $x_0 \in X$ iff for any $\epsilon > 0$ there exists a neighbourhood $U$ of $x_0$ such that $f(x) - \epsilon < f(x_0)$ for all $x \in U$.

In the case of normed spaces, $F$ is *lower semicontinuous* at a point $x_0 \in X$ iff for any sequence $\{x_n\}_n \subseteq X$,

$$x_n \to x_0 \Rightarrow F(x_0) \leq \liminf_{n\to\infty} F(x_n)$$

and it is *upper semicontinuous* at a point $x_0 \in X$ iff for any sequence $\{x_n\}_n \subseteq X$,

$$x_n \to x_0 \Rightarrow \limsup_{n\to\infty} F(x_n) \leq F(x_0).$$

**Theorem 1.1.11.** (see, for instance, [15], [111]) *Let $X$ be a Hausdorff topological vector space and $F : X \to [-\infty, +\infty]$ be a function. Then the following assertions are equivalent:*

*(i) $F$ is lower semicontinuous, i.e., $F$ is lower semicontinuous at every point in $X$;*

*(ii)* epi$F$ *is closed in $X \times \mathbb{R}$;*

*(iii) for every $r \in \mathbb{R}$, $L(F, r)$ is closed in $X$.*

**Definition 1.1.12.** Let $X$ be a normed space and $F : X \to [-\infty, +\infty]$ be a function. $F$ is said to be *upper hemicontinuous* iff it is upper semicontinuous on any segment contained on $X$.

Let $T : X \to 2^Y$ be a set-valued map and $K \subseteq Y$ a pointed convex cone. The domain, graph and epigraph of $T$ are defined by

$$\operatorname{dom} T := \{x \in X \mid T(x) \neq \emptyset\},$$
$$\operatorname{gph} T := \{(x, y) \in \times Y \mid x \in \operatorname{dom} T,\ y \in T(x)\},$$
$$\operatorname{epi} T := \{(x, y) \in X \times Y \mid x \in \operatorname{dom} T,\ y \in T(x) + K\},$$

respectively.

The next concept turned to be very important in the last fifty years.

**Definition 1.1.13.** (see, for instance, [25]) Let $T : X \to 2^{X^*}$ be a set-valued map. We say that $T$ is *monotone* iff for every $x_1, x_2 \in \operatorname{dom} T$, $y_1 \in T(x_1)$,
$y_2 \in T(x_2)$ one has

$$\langle y_2 - y_1, x_2 - x_1 \rangle \geq 0.$$

The classical concept of monotonicity has been extended in several ways. Among them we mention the following two.

**Definition 1.1.14.** (see, for instance, [58]) Let $T : X \to 2^{X^*}$ be a set-valued map. We say that $T$ is *pseudomonotone* iff for every $x_1, x_2 \in \operatorname{dom} T$,
$y_1 \in T(x_1), y_2 \in T(x_2)$ one has

$$\langle y_1, x_2 - x_1 \rangle \geq 0 \Rightarrow \langle y_2, x_2 - x_1 \rangle \geq 0.$$

**Definition 1.1.15.** (see, for instance, [58]) Let $T : X \to 2^{X^*}$ be a set-valued map. We say that $T$ is *quasimonotone* iff for every $x_1, x_2 \in \operatorname{dom} T$,
$y_1 \in T(x_1), y_2 \in T(x_2)$ one has

$$\langle y_1, x_2 - x_1 \rangle > 0 \Rightarrow \langle y_2, x_2 - x_1 \rangle \geq 0.$$

In what follows, in this section, we suppose that $A$ is a nonempty subset of a topological vector space $X$ and $T : X \to 2^X$ is a set-valued map with $\operatorname{dom} T = A$.

**Definition 1.1.16.** $T$ is said to be a *KKM-mapping* iff, for every finite subset $\{a_1, a_2, ..., a_n\}$ of $A$, the following inclusion holds:

$$\operatorname{co}\{a_1, a_2, ..., a_n\} \subseteq \bigcup_{i=1}^{n} T(a_i).$$

The next celebrated lemma was given by B. Knaster, C. Kuratowski and S. Mazurkiewicz [83] in finite dimensional spaces, while in infinite dimensional spaces it was established by Ky Fan [44].

**Lemma 1.1.17.** [44] *Let $T : A \to 2^X$ be a KKM-mapping satisfying the following conditions:*

*(i) $T(a)$ is closed for all $a \in A$;*

*(ii) there exists $\bar{a} \in A$ such that $T(\bar{a})$ is a compact set.*

*Then* $\bigcap_{a \in A} T(a) \neq \emptyset$.

To establish our results in Section 3.4 below we use the following important intersection theorem due to H. Brézis, G. Nirenberg and G. Stampacchia [23] which extended the above intersection theorem of Ky Fan [44].

**Theorem 1.1.18.** [23]. *Let $T : A \to 2^X$ be KKM mapping such that:*

*(i)* $\operatorname{cl} T(y_0)$ *is compact for some $y_0 \in A$;*

*(ii) For each $y \in A$ and for each finite dimensional subspace $Z$ of $X$, $T(y) \cap Z$ is closed;*

*(iii) For each line segment $D$ of $X$ :*

$$\operatorname{cl}(\bigcap_{y \in A \cap D} T(y)) \cap D = (\bigcap_{y \in A \cap D} T(y)) \cap D.$$

*Then* $\bigcap_{y \in A} T(y) \neq \emptyset$.

**Remark 1.1.19.** *If $A$ is convex, closed and $T(y) \subseteq A$ for every $y \in A$, then the hypothesis $(iii)$ can be replaced by $(iii\prime)$ for every line segment $D$ of $X$*

$$\operatorname{cl}(\bigcap_{y \in D} T(y)) \cap D = (\bigcap_{y \in D} T(y)) \cap D.$$

## 1.2 Functions satisfying certain weakened convexity

Let $X$ be a vector space and $A$ a nonempty subset of it.

**Definition 1.2.1.** (see, for instance, [58]) If $A \subseteq X$ is convex, a function $F : A \to \mathbb{R}$ is called *quasiconvex* iff for all $x, y \in A$ and for each $\lambda \in [0, 1]$ the following inequality holds:

$$F(\lambda x + (1 - \lambda)y) \leq \max\{F(x), F(y)\}.$$

**Definition 1.2.2.** (see, for instance, [58]) If $A \subseteq X$ is convex, a function $F : A \to \mathbb{R}$ is called *semistrictly quasiconvex* iff for all $x, y \in A$ the following implication holds:

$$F(x) < F(y) \Rightarrow \ \forall \lambda \in (0, 1), \ F(\lambda x + (1 - \lambda)y) < F(y).$$

If $F$ is semistrictly quasiconvex and lower semicontinuous, then it is quasiconvex. For more details, see N. Hadjisavvas [58], I. Konnov [87] and I. Konnov, D.T. Luc and A.M. Rubinov [89].

**Definition 1.2.3.** (see, for instance, [58]) If $A \subseteq X$ is convex, a function $F : A \to \mathbb{R}$ is called *pseudo-convex* iff for all $x, y \in K$ and for each $\lambda \in (0, 1)$ the following implication holds:

$$F(\alpha x + (1 - \alpha)y) \geq F(x) \Rightarrow F(\alpha x + (1 - \alpha)y) \leq F(y).$$

The next concept was introduced by Ky Fan [43] in order to extend some minimax results.

**Definition 1.2.4.** [43] Let $B$ be a nonempty set. A bifunction $f : A \times B \to \mathbb{R}$ is said to be *Ky Fan concave-convex* iff for all $a_1, a_2 \in A$ and all $\lambda \in [0,1]$ there exists $a \in A$ such that

$$f(a,b) \geq \lambda f(a_1,b) + (1-\lambda) f(a_2,b) \quad \text{for all} \quad b \in B,$$

and for all $b_1, b_2 \in B$ and all $\mu \in [0,1]$ there exists $b \in B$ such that

$$f(a,b) \leq \mu f(a,b_1) + (1-\mu) f(a,b_2) \quad \text{for all} \quad a \in A.$$

Some years later, H. König [86] extended the above notion of Ky Fan in order to generalize his minimax result.

**Definition 1.2.5.** [86] Let $B$ be a nonempty set. A bifunction $f : A \times B \to \mathbb{R}$ is said to be *König concave-convex* iff for all $a_1, a_2 \in A$ there exists $a \in A$ such that

$$f(a,b) \geq \frac{1}{2}[f(a_1,b) + f(a_2,b)] \quad \text{for all} \quad b \in B,$$

and for all $b_1, b_2 \in B$ there exists $b \in B$ such that

$$f(a,b) \leq \frac{1}{2}[f(a,b_1) + f(a,b_2)] \quad \text{for all} \quad a \in A.$$

The convexity concept for scalar functions has been extended in a natural way for vector-valued functions, according to the partial order introduced by the cone $K$.

**Definition 1.2.6.** Let $X$ and $Y$ be vector spaces, let $A$ be a nonempty subset of $X$, and let $K \subseteq Y$ be a convex cone. A function $F : A \to Y$ is said to be:

(i) *$K$-convex* iff $A$ is convex and, for all $a_1, a_2 \in A$ and all $\lambda \in [0,1]$, the following inequality holds:

$$F(\lambda a_1 + (1-\lambda)a_2) \leq_K \lambda F(a_1) + (1-\lambda) F(a_2);$$

$F$ is said to be *$K$-concave* iff $-F$ is $K$-convex.

(ii) *$K$-convexlike* iff, for all $a_1, a_2 \in A$ and all $\lambda \in [0,1]$, there exists $a \in A$ such that

$$F(a) \leq_K \lambda F(a_1) + (1-\lambda) F(a_2);$$

(iii) *$K$-subconvexlike* iff there exists $k \in \operatorname{int} K$ such that, for all $a_1, a_2 \in A$, $\lambda \in [0,1]$ and all $\epsilon > 0$, there exists $a \in A$ satisfying the following inequality:

$$F(a) \leq_K \lambda F(a_1) + (1-\lambda) F(a_2) + \epsilon k.$$

Definition 1.2.6 (i) can be found in B. D. Craven [38], (ii) is due to Ky Fan [43], while (iii) originates from V. Jeyakumar [71]. Obviously, when $\operatorname{int} K \neq \emptyset$, then the $K$-convexlikeness of $F$ implies the $K$-subconvexlikeness of $F$.

Next we present some monotonicity conditions for scalar bifunctions considered within the literature in the recent years. Most of these notions were inspired by similar (generalized) monotonicity concepts defined for operators acting from a topological vector space to its dual space.

Let $A \subseteq X$ be a convex set.

**Definition 1.2.7.** (see, for instance, [18]) The bifunction $f : A \times A \to \mathbb{R}$ is said to be

(i) *monotone* iff $f(x, y) + f(y, x) \leq 0$ for all $x, y \in A$;

(ii) *quasimonotone* iff the following implication holds:

$$f(x, y) > 0 \Rightarrow f(y, x) \leq 0;$$

(iii) *properly quasimonotone* iff for all $x_1, \cdots, x_n \in A$ and all $\lambda_1, \cdots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ it holds that

$$\min_{1 \leq i \leq n} f(x_i, \sum_{j=1}^n \lambda_j x_j) \leq 0.$$

Proper quasimonotonicity was introduced by Zhou and Chen in [130] under the name of $0-$*diagonal quasiconcavity* (see also [18]).

In the case of variational inequalities, proper quasimonotonicity is stronger than quasimonotonicity, as defined, e.g., in S. Karamardian and S. Schaible [74]. In the general case it neither implies quasimonotonicity nor is implied by it, see M. Bianchi and R. Pini [18].

The result below, known as "Ky Fan's" minimax inequality theorem, plays a crucial role within nonlinear and convex anaysis and it is an easy consequence of Lemma 1.1.17.

**Theorem 1.2.8.** [46]. *Let $A$ be a compact convex set in a Hausdorff topological vector space. If $f : A \times A \to \mathbb{R}$ is such that for every $x \in A$, $f(x, \cdot)$ is quasiconvex and for every $y \in A$, $f(\cdot, y)$ is upper semicontinuous, then there exists $\overline{x} \in A$ such that*

$$f(\overline{x}, y) \geq \inf_{x \in A} f(x, x) \quad \textit{for all} \quad y \in A.$$

# Chapter 2

# On Equilibrium Problems

One of the most important topics in nonlinear analysis (see, for instance, P.M. Pardalos, T.M. Rassias and A.A. Khan [107]) and several applied fields is the so called equilibrium problem. According to the bifunctions involved, one can distinguish two forms: the scalar equilibrium problem and the vector equilibrium problem. In this chapter we deal with both forms.
The equilibrium problem has been extensively studied in recent years (see, for instance, M. Bianchi, G. Kassay and R. Pini [20], O. Chadli, Y. Chiang and S. Huang [31], X.H. Gong [57], A.N. Iussem and W. Sossa [65], P. Kas, G. Kassay and Z. Boratas-Sensoy [75], G. Kassay [76], I.V. Konnov and J.C. Yao [88]). One of the reasons is that it has among its particular cases, optimization problems, saddlepoint (minimax) problems (see, for instance, [42], F. Ferro [50], S. Paeck [106]), variational inequalities (monotone or otherwise) (see, for instance, L.B. Batista Dos Santos, G. Ruiz-Garzón and M.A. Rojas-Medar [14], S.S. Chang and Y. Zhang [32], G.-Y. Chen and Q.M. Cheng [33], B. Chen and N.J. Huang [34], C. Finet, L. Quarta and C. Troestler [51], F. Giannessi [54], K.L. Lin, D.P. Yang and J.C. Yao [91]), Nash equilibrium problems, and other problems of interest in many applications (see, J.P. Aubin [10], T. Basar and G. J. Olsder [12], A.J. Jones [72], H.W. Kuhn [90], N.N. Vorob'ev [120] or E. Blum and W. Oettli [22] for a survey).

## 2.1 Existence results for the scalar equilibrium problem

Throughout this section we suppose that $A$ is a nonempty subset of a topological space $X$ and $B$ is a nonempty set.

Let $f : A \times B \to \mathbb{R}$ be a given bifunction. The scalar equilibrium problem consists on:

$$(SEP) \qquad \text{finding an element } a \in A \text{ such that } f(a,b) \geq 0 \text{ for all } b \in B.$$

$(SEP)$ has been extensively studied in recent years, see, for instance, M. Bianchi and R. Pini [18], [19], M. Bianchi and S. Schaible [16], G. Bigi, M. Castellani and G. Kassay [21], E. Blum and W. Oettli [22], A.N. Iusem, G. Kassay and W. Sosa [67], A.N. Iusem and W. Sosa [64]. As we have already mentioned in the Introduction, the term "equilibrium problem" was attributed in E. Blum and W. Oettli [22], but the problem itself has been investigated more than twenty years before in a paper of Ky Fan [44] in connection with the so called "intersection theorems" (i.e., results stating the nonemptiness of a certain family of sets). Ky Fan considered $(SEP)$ in the special case $A = B$ a compact convex subset of a Hausdorff topological vector space and termed it "minimax inequality". In the same year, H. Brézis, G. Nirenberg and G. Stampacchia [23] improved Ky Fan's result, extending it to a not necessarily compact set, but assuming instead a so-called "coercivity condition", which is automatically satisfied when the set is compact.

Recent results on $(SEP)$ emphasizing existence of solutions can be found in M. Bianchi and R. Pini [18], [19], M. Bianchi and S. Schaible [16], [70], W. Oettli [104], and many other papers. New necessary (and in some cases also sufficient) conditions for existence of solutions in infinite dimensional spaces were proposed, among others, by A.N. Iusem and W. Sosa [64], A.N. Iusem, G. Kassay and W. Sosa [67] and G. Kassay and M. Miholca [78].

Looking on the proofs given for existence results, one may detect two fundamental methods: fixed point methods (intersection theorems mostly based on Brouwer's fixed point theorem), and separation methods (Hahn-Banach type theorems).

In what follows we divide our study on $(SEP)$ according to the nature of the set $A$, namely when this set is compact and when is not.

### 2.1.1 Existence results for equilibria on compact sets

In this subsection we introduce a definition for scalar functions which is weaker than that of upper semicontinuity.

**Definition 2.1.1.** A function $F : A \to \mathbb{R}$ is said to be *pseudo-upper semicontinuous* at $a \in A$ iff $F(a) < 0$ implies that there exist $c > 0$ and a neighbourhood $U$ of $a$ such that $F(x) + c < 0$ for all $x \in A \cap U$.

$F$ is said to be *pseudo-upper semicontinuous* on $A$ iff it is pseudo-upper semicontinuous at $a \in A$ for all $a \in A$.

The pseudo-upper semicontinuity of a function can be characterized as follows. Let $\mathcal{V}(a)$ be the set of all neighbourhoods of $a$.

We have the following characterization which, as far as we know, is new in the literature.

**Lemma 2.1.2.** *$F$ is pseudo-upper semicontinuous at $a \in A \cap A'$ if and only if*

$$F(a) < 0 \implies \limsup_{x \to a} F(x) < 0.$$

**Remark 2.1.3.** It is obvious that if a function $F$ is upper semicontinuous on $A$ then it is also pseudo-upper semicontinuous on $A$.

Next, we construct an example with a function $F$ which is pseudo-upper semicontinuous at $a \in A$ but not upper semicontinuous at $a \in A$.

**Example 2.1.4.** Let $F : [-2, 2] \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} -x, & x \in [-2, 1), \\ -2, & x = 1, \\ x - 2, & x \in (1, 2]. \end{cases}$$

It is easy to check that the function above is pseudo-upper semicontinuous at $a = 1$. On the other hand, it is not upper semicontinuous at $a = 1$.

Now we obtain a more general existence result for the problem $(SEP)$ than that which has been established by G. Kassay and J. Kolumbán in [77]. In this result, instead of the upper semicontinuity assumption upon the bifunction $f$, the pseudo-upper semicontinuity assumption is supposed.

**Theorem 2.1.5.** *Let $A$ be a compact set, let $B$ be a nonempty set, and let $f : A \times B \to \mathbb{R}$ be a given bifunction which satisfies the following conditions:*

*(i) for each $b \in B$, the function $f(\cdot, b) : A \to \mathbb{R}$ is pseudo-upper semicontinuous on $A$;*

*(ii) for each $a_1, ..., a_m \in A$, $b_1, ..., b_n \in B$, $\lambda_1, ..., \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$, the inequality*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i f(a_i, b_j) \leq \sup_{a \in A} \min_{1 \leq j \leq n} f(a, b_j)$$

*holds;*

*(iii) for each $b_1, ..., b_n \in B$, $\mu_1, ..., \mu_n \geq 0$ with $\sum_{j=1}^n \mu_j = 1$, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} \mu_j f(a, b_j) \geq 0.$$

*Then the scalar equilibrium problem $(SEP)$ admits a solution.*

In G. Kassay and J. Kolumban [77], the authors obtained sufficient conditions for the existence of solutions of $(SEP)$ using upper semicontinuity. Since pseudo-upper semicontinuity is weaker than upper semicontinuity, their result becomes a corollary of our Theorem 2.1.5.

**Corollary 2.1.6.** [77] *Let $A$ be a compact set, let $B$ be a nonempty set, and let $f : A \times B \to \mathbb{R}$ be a given bifunction which satisfies the following conditions:*

*(i) for each $b \in B$, the function $f(\cdot, b) : A \to \mathbb{R}$ is upper semicontinuous on $A$;*

*(ii) for each $a_1, ..., a_m \in A$, $b_1, ..., b_n \in B$, $\lambda_1, ..., \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$, the inequality*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i f(a_i, b_j) \leq \sup_{a \in A} \min_{1 \leq j \leq n} f(a, b_j)$$

*holds;*

*(iii) for each $b_1, ..., b_n \in B$, $\mu_1, ..., \mu_n \geq 0$ with $\sum_{j=1}^n \mu_j = 1$, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} \mu_j f(a, b_j) \geq 0.$$

*Then the scalar equilibrium problem $(SEP)$ admits a solution.*

While assumption $(i)$ of Theorem 2.1.5 is of topological nature, $(ii)$ and $(iii)$ of this result express some kinds of generalized concavity/convexity conditions. We conclude this subsection with this issue.

A. Capătă and G. Kassay [30] introduced a new convexity concept for scalar bifunctions.

**Definition 2.1.7.** A bifunction $f : A \times B \to \mathbb{R}$ is said to be:

(i) *subconcavelike* in its first variable iff, for all $k > 0$, $a_1, a_2 \in A$, and $\lambda \in [0, 1]$ there exists $a \in A$ such that

$$f(a, b) \geq \lambda f(a_1, b) + (1 - \lambda) f(a_2, b) - k,$$

for all $b \in B$;

*(ii)* *subconvexlike* in its second variable iff, for all $k > 0$, $b_1, b_2 \in A$, and $\lambda \in [0,1]$ there exists $b \in A$ such that

$$f(a,b) \leq \lambda f(a,b_1) + (1-\lambda) f(a,b_2) + k,$$

for all $a \in A$;

*(iii)* *subconcavelike-subconvexlike* iff it is subconcavelike in its first variable and subconvexlike in its second variable.

The subconcavelikeness of a bifunction can be characterized as follows, see [30], [77].

**Proposition 2.1.8.** *A bifunction $f : A \times B \to \mathbb{R}$ is subconcavelike in its first variable if and only if for all $k > 0$, $a_1, a_2, ..., a_m \in A$, $\lambda_1, \lambda_2, ..., \lambda_m \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ there exists $a \in A$ such that*

$$f(a,b) \geq \sum_{i=1}^{m} \lambda_i f(a_i, b) - k \quad \textit{for all} \quad b \in B.$$

A similarly property holds for subconvexlikeness.

**Proposition 2.1.9.** [30] *A bifunction $f : A \times B \to \mathbb{R}$ is subconvexlike in its second variable if and only if for all $k > 0$, $b_1, b_2, ..., b_m \in B$, $\lambda_1, \lambda_2, ..., \lambda_m \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ there exists $b \in B$ such that*

$$f(a,b) \leq \sum_{i=1}^{m} \lambda_i f(a, b_i) + k \quad \textit{for all} \quad a \in A.$$

**Remark 2.1.10.** Assumption *(iii)* of Theorem 2.1.5 is satisfied if the bifunction $f$ is subconvexlike in its second variable and the condition

$$\sup_{a \in A} f(a,b) \geq 0 \quad \text{for each} \quad b \in B, \tag{2.1}$$

is satisfied. In the case when $A = B$, this additional condition is satisfied in particular if $f(a,a) = 0$ for all $a \in A$.

Using this kind of generalized convexity and Proposition 2.1.8 we obtain the following result.

**Theorem 2.1.11.** *Let $A$ be a compact set and let the bifunction $f : A \times B \to \mathbb{R}$ satisfy the following conditions:*

*(i)* *for each $b \in B$, the function $f(\cdot, b) : A \to \mathbb{R}$ is pseudo-upper semicontinuous on $A$;*

*(ii)* *$f$ is subconcavelike-subconvexlike;*

*(iii)* $\sup_{a \in A} f(a,b) \geq 0$ *for each $b \in B$.*

*Then the scalar equilibrium problem $(SEP)$ admits a solution.*

The next corollary is a result obtained by A. Capătă and G. Kassay [30] using upper semicontinuity instead of pseudo-upper semicontinuity.

**Corollary 2.1.12.** [30] *Let $A$ be a compact set, and let the bifunction*
*$f : A \times B \to \mathbb{R}$ satisfies the following conditions:*

*(i)* *for each $b \in B$, function $f(\cdot, b) : A \to \mathbb{R}$ is upper semicontinuous on $A$;*

*(ii)* $f$ *is subconcavelike-subconvexlike;*

*(iii)* $\sup_{a \in A} f(a,b) \geq 0$ *for each* $b \in B$.

*Then the scalar equilibrium problem* $(SEP)$ *admits a solution.*

### 2.1.2 Equilibria on noncompact sets

Throughout this subsection, $X$ is a real reflexive Banach space with $X^*$ its dual.

A.N. Iusem, G. Kassay and W. Sosa [66], [67] obtained existence results concerning $(SEP)$ in both finite and infinite dimensions. As shown below, in finite dimensions, the existence result can be obtained without any monotonicity assumption on the bifunction, while in infinite dimensions apparently it is not possible. The finite dimensional result is the following.

**Theorem 2.1.13.** [66] *Let* $A \subseteq \mathbb{R}^n$ *be a closed convex set and* $f : A \times A \to \mathbb{R}$ *a given bifunction. Suppose that the following assumptions are satisfied:*

*(i)* $f(x,x) = 0$ *for all* $x \in A$;

*(ii)* $f(x,\cdot)$ *is pseudo-convex and lower semicontinuous for all* $x \in A$;

*(iii)* $f(\cdot,y)$ *is upper semicontinuous for all* $y \in A$;

*(iv) For any sequence* $\{x_n\} \subseteq A$ *satisfying* $\lim_{n\to\infty} \|x_n\| = +\infty$, *there exist* $u \in A$ *and* $n_0 \in \mathbb{N}$ *such that* $f(x_n,u) \leq 0$ *for all* $n \geq n_0$.

*Then* $(SEP)$ *admits a solution.*

The next result, in infinite dimensions, is a particular case of Theorem 4.2 in [67] and plays an important role for our purposes.

**Theorem 2.1.14.** [67]. *Let* $A \subseteq X$ *be a closed convex set and* $f : A \times A \to \mathbb{R}$ *a given bifunction. Suppose that the following assumptions are satisfied:*

*(i)* $f(x,x) = 0$ *for all* $x \in A$;

*(ii)* $f(x,\cdot)$ *is convex and lower semicontinuous for all* $x \in A$;

*(iii)* $f(\cdot,y)$ *is upper hemicontinuous for all* $y \in A$;

*(iv)* $f$ *is properly quasimonotone;*

*(v) For any sequence* $\{x_n\} \subseteq A$ *satisfying* $\lim_{n\to\infty} \|x_n\| = +\infty$, *there exist* $u \in A$ *and* $n_0 \in \mathbb{N}$ *such that* $f(x_n,u) \leq 0$ *for all* $n \geq n_0$.

*Then* $(SEP)$ *admits a solution.*

The main difficulty in obtaining an existence result for $(SEP)$ in infinite dimensions without monotonicity assumptions originates from the fact that the closed balls are not compact (with respect to the strong topology). However, in a reflexive Banach space they are weakly compact and this allows us to obtain the following result for $(SEP)$ without monotonicity by strengthening the continuity assumption on $f$ with respect to its first variable. Given $A \subseteq X$ and $r > 0$, set $B_r := \{x \in A : \|x\| \leq r\}$.

**Remark 2.1.15.** In order to establish our results we need to consider a slightly weaker *coercivity property* of $f$ than the one in item (v) above. Namely, let us consider

(v') For any sequence $\{x_n\} \subseteq A$ satisfying $\lim_{n\to\infty} \|x_n\| = +\infty$, there exist a sequence $\{u_n\} \subseteq A$ with $\|u_n\| < \|x_n\|$ and $n_0 \in \mathbb{N}$ such that $f(x_n, u_n) \leq 0$ for all $n \geq n_0$.

It is easy to check that the same proof as in [67] shows that Theorem 2.1.14 remains valid when (v) is replaced by (v').

**Remark 2.1.16.** Note that similar existence results concerning $(SEP)$ have been established in M. Bianchi and R. Pini [19] (see for instance Theorem 4.2), but these results are not comparable with Theorem 2.1.14 (none of them can be deduced from the other).

The absence of compactness of the set $A$ can be overcome by considering different types of so-called *coercivity conditions*. In this subsection we deal with this issue.

**Theorem 2.1.17.** [78] *Suppose that $A \subseteq X$ is closed and convex.*
*Let $f : A \times A \to \mathbb{R}$ be a bifunction satisfying:*

*(i)* $f(x, x) = 0$ *for all* $x \in A$;

*(ii)* $f(x, \cdot)$ *is semistrictly quasiconvex and lower semicontinuous for all* $x \in A$;

*(iii)* $f(\cdot, y)$ *is weakly upper semicontinuous for all* $y \in A$;

*(iv) There exists* $r > 0$ *such that for each* $x \in A\backslash B_r$, *there exists* $y \in A$ *with* $\|y\| < \|x\|$ *and* $f(x, y) \leq 0$.

*Then scalar the equilibrium problem* $(SEP)$ *has a solution.*

The following theorem also gives sufficient conditions for the existence of solutions of $(SEP)$.

**Theorem 2.1.18.** [78] *Let $A$ be a closed, convex subset of $X$ and let*
$f : A \times A \to \mathbb{R}$ *be a bifunction satisfying:*
*(i)* $f(x, x) = 0$ *for all* $x \in A$;
*(ii)* $f(x, \cdot)$ *is semistrictly quasiconvex and lower semicontinuous for all* $x \in A$;
*(iii)* $f(\cdot, y)$ *is upper semicontinuous on the intersection of $A$ with any finite dimensional subspace of $X$, for any* $y \in A$;
*(iv) For any* $\{x_\alpha\}_{\alpha\in I} \subseteq A$, $x_\alpha \rightharpoonup \overline{x}$ *and for any line segment* $D \subseteq A$ *such that* $\overline{x} \in D$,

$$f(x_\alpha, y) \geq 0,\ \forall\ \alpha \in I,\ \forall\ y \in D \Rightarrow f(\overline{x}, y) \geq 0,\ \forall\ y \in D;$$

*(v) There exists* $r > 0$ *such that for each* $x \in A\backslash B_r$, *there exists* $y \in A$ *with* $\|y\| < \|x\|$ *and* $f(x, y) \leq 0$.
*Then the scalar equilibrium problem* $(SEP)$ *has a solution.*

## 2.2 Existence results for the vector equilibrium problem

Throughout this section we suppose that $A$ is a nonempty subset of a topological space $X$, $B$ is a nonempty set, $Y$ is a topological vector space, and $K \subseteq Y$ is a convex cone, which is also solid, if not otherwise specified.

Let $f : A \times B \to Y$ be a given bifunction. The vector equilibrium problem consists on:

$(VEP)$ finding an element $a \in A$ such that $f(a,b) \notin -\mathrm{int}K$ for all $b \in B$.

$(VEP)$ has been extensively studied in recent years, see, for instance, Q.H. Ansari, S. Schaible and J.C. Yao [8], M. Bianchi, N. Hadjisavvas and S. Schaible [17], A. Capătă and G. Kassay [30], D.T. Luc [93], and the references within.

In the literature several concepts for extending the classical notions of *infimum* and *supremum* for a certain set in a topological vector space ordered by a cone have been considered. In what follows we deal with two of such concepts. The first type of infima and suprema was introduced by Q.H. Ansari, X.C. Yang and J.C. Yao [7].

For a subset $C$ of $Y$, the infimum of $C$ with respect to $K$ is defined by

$$\mathrm{Inf}(C,K) = \{y \in \mathrm{cl}\, C : (y - \mathrm{int}K) \cap C = \emptyset\}$$

and the supremum of $C$ with respect to $K$ is defined by

$$\mathrm{Sup}(C,K) = \{y \in \mathrm{cl}\, C : (y + \mathrm{int}K) \cap C = \emptyset\}.$$

It follows from the definitions above that $\mathrm{Inf}(C,K)$ and $\mathrm{Sup}(C,K)$ can be empty even if $C$ is nonempty. Sufficient conditions for the nonemtiness of $\mathrm{Inf}(C,K)$ and $\mathrm{Sup}(C,K)$ can be found, for instance, in Y. Chiang [36].

In the rest of this section, we shall consider the superior and inferior of subsets of $Y$ with respect to a fixed ordering cone. Therefore, we will simply write $\mathrm{Inf}\, C$ and $\mathrm{Sup}\, C$; for more details, see Y. Chiang [36].

Another type, different from that of the above is the following (see A. Löhne [92]). For a subset $C$ of $Y$, $\inf C$ is an element $u \in Y$ such that $u \leq_K y$ for all $y \in C$ and for each $v \in C$ such that $v \leq_K y$ for all $y \in C$, we have $u \leq_K v$. Also, $\sup C$ is an element $u \in Y$ such that $y \leq_K u$ for all $y \in C$ and for each $v \in C$ such that $y \leq_K v$ for all $y \in C$, we have $u \leq_K v$.

By the above definitions it follows that $\sup C$ and $\inf C$ cannot have more than one element whenever we assume (additionally) that the cone $K$ is pointed.

The following two properties have been stated in [36].

**Proposition 2.2.1.** *If* $y \in \mathrm{Inf}\, C$ *and* $k \in K$, *then* $(y + k - \mathrm{int}\, K) \cap C \neq \emptyset$.

**Proposition 2.2.2.** *If* $y \in C$, *then* $\mathrm{Inf}(\mathrm{cl}\, C \cap (y - K)) \subseteq \mathrm{Inf}\, C$.

In order to prove the following results, we need the next lemma.

**Lemma 2.2.3.** *Suppose there exists* $c \in C \cap (-\mathrm{int}K)$ *such that*

$$\mathrm{Inf}(\mathrm{cl}\, C \cap (c - K)) \neq \emptyset.$$

*Then*

$$(\mathrm{Inf}\, C) \cap (-\,\mathrm{int}\, K) \neq \emptyset.$$

In the next definition, we introduce two useful notions $\lim_{x\to a} \operatorname{Sup} F(x)$ and $\lim_{x\to a} \operatorname{Inf} F(x)$.

**Definition 2.2.4.** For an $a \in A^{'}$ and a function $F : A \to Y$ we define

(i) $\lim_{x\to a} \operatorname{Sup} F(x) =$
$\operatorname{Inf}\{\sup\{F(x) : x \in A \cap U \setminus \{a\}\} : U \in \mathcal{V}(a), A \cap U \setminus \{a\} \neq \emptyset\}$.

(ii) $\lim_{x\to a} \operatorname{Inf} F(x) =$
$\operatorname{Sup}\{\inf\{F(x) : x \in A \cap U \setminus \{a\}\} : U \in \mathcal{V}(a), A \cap U \setminus \{a\} \neq \emptyset\}$.

Now likewise for the scalar case, we introduce the concept of $K$-pseudo-upper semicontinuity for vector-valued functions.

**Definition 2.2.5.** A function $F : A \to Y$ is said to be *$K$-pseudo-upper semicontinuous* at $a \in A$ iff $F(a) \in -\operatorname{int} K$ implies there exist $k \in \operatorname{int} K$ and a neighbourhood $U$ of $a$ such that $F(x) + k \in -\operatorname{int} K$ for all $x \in U \cap A$.
$F$ is said to be *$K$-pseudo-upper semicontinuous on $A$* iff it is K-pseudo-upper semicontinuous at $a \in A$ for all $a \in A$.

**Definition 2.2.6.** A function $F : A \to Y$ is said to satisfy *property* $(LS)$ at $a \in A \cap A^{'}$ iff

$$F(a) \in -\operatorname{int} K \quad \Longrightarrow \quad \lim_{x\to a} \operatorname{Sup} F(x) \cap (-\operatorname{int} K) \neq \emptyset.$$

Next we give a sufficient condition in terms of lim Sup for pseudo-upper semicontinuity.

**Theorem 2.2.7.** *If a function $F : A \to Y$ satisfies property $(LS)$ at $a \in A \cap A^{'}$ then $F$ is K-pseudo-upper semicontinuous at $a \in A$.*

**Definition 2.2.8.** [93] Let $X$ be a topological space, let $A \subseteq X$ be a nonempty set, let $Y$ be a topological vector space, and let $K \subseteq Y$ be a convex cone (not necessarily solid). A function $F : A \to Y$ is said to be:

*(i)* $K$-upper semicontinuous at a point $a \in A$ iff, for any neighbourhood $V$ of $F(a)$, there exists a neighbourhood $U$ of $a$ such that $F(x) \in V - K$ for all $x \in U \cap A$.

*(ii)* $K$-upper semicontinuous on $A$ iff it is $K$-upper semicontinuous at each $a \in A$.

*(iii)* $K$-lower semicontinuous at a point $a \in A$ (respectively $K$-lower semicontinuous on $A$) iff $-F$ is $K$-upper semicontinuous at $a$ (respectively $K$-upper semicontinuous on $A$).

**Remark 2.2.9.** Under the hypotheses that $K$ is a solid convex cone, T. Tanaka [118] showed that item (i) of the above definition is equivalent to: for any $k \in \operatorname{int} K$, there exists a neighborhood $U$ of $a$ such that
$F(x) \in F(a) + k - \operatorname{int} K$ for all $x \in A \cap U$.

**Theorem 2.2.10.** *If $F : A \to Y$ is $K$-upper semicontinuous at $a \in A$ then $F$ is $K$-pseudo-upper semicontinuous at $a \in A$.*

The next theorem states the existence of solutions of the vector equilibrium problem under $K$-pseudo-upper semicontinuity.

**Theorem 2.2.11.** *Let $A$ be a compact set, let $B$ be a nonempty set, and let $f : A \times B \to Y$ be a given bifunction which satisfies the following conditions:*

*(i) for each $b \in B$, the function $f(\cdot, b) : A \to Y$ is $K$-pseudo-upper semicontinuous on $A$;*

*(ii) for each $a_1, ..., a_m \in A$, $b_1, ..., b_n \in B$, $\lambda_1, ..., \lambda_m \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$, there exists $k^* \in K^* \setminus \{0\}$ such that*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i k^*(f(a_i, b_j)) \leq \sup_{a \in A} \min_{1 \leq j \leq n} k^*(f(a, b_j));$$

*(iii) for each $b_1, ..., b_n \in B$ and $k_1^*, ..., k_n^* \in K^*$ not all zero, one has*

$$\sup_{a \in A} \sum_{j=1}^{n} k_j^* f(a, b_j) \geq 0.$$

*Then the vector equilibrium problem $(VEP)$ admits a solution.*

Assumption (ii) of Theorem 2.2.11 is a kind of generalized concavity of the bifunction $f$ in the first variable with respect to the cone $K$. In a similar way as in the scalar case (Definition 2.1.7), A. Capătă and G. Kassay introduced a new concept for vector-valued bifunctions. A related notion has been introduced by Jeyakumar [71] for vector-valued functions, see Definition 1.2.6.

**Definition 2.2.12.** [30] A bifunction $f : A \times B \to Y$ is said to be:

*(i)* *$K$-subconcavelike* in its first variable iff, for all $k \in \mathrm{int}K$, $a_1, a_2 \in A$ and $\lambda \in [0, 1]$ there exists $a \in A$ such that

$$f(a, b) \geq_K \lambda f(a_1, b) + (1 - \lambda) f(a_2, b) - k,$$

for all $b \in B$;

*(ii)* *$K$-subconvexlike* in its second variable iff, for all $k \in \mathrm{int}K$, $b_1, b_2 \in A$ and $\lambda \in [0, 1]$ there exists $b \in A$ such that

$$f(a, b) \leq_K \lambda f(a, b_1) + (1 - \lambda) f(a, b_2) + k,$$

for all $a \in A$;

*(iii)* *$K$-subconcavelike-subconvexlike* iff it is $K$-subconcavelike in its first variable and $K$-subconvexlike in its second variable.

The $K$-subconcavelikeness of a vector-valued bifunction can be characterized as follows.

**Proposition 2.2.13.** [30] *A bifunction $f : A \times B \to Y$ is $K$-subconcavelike in its first variable if and only if for all $k \in \mathrm{int}\, K$, $a_1, a_2, ..., a_m \in A$, $\lambda_1, \lambda_2, ..., \lambda_m \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ there exists $a \in A$ such that*

$$f(a, b) \geq_K \sum_{i=1}^{m} \lambda_i f(a_i, b) - k \quad \text{for all} \quad b \in B. \tag{2.2}$$

By Proposition 2.2.13 it easy to observe that (2.2) it is a sufficient condition to obtain condition (*iii*) from Theorem 2.2.11. Therefore, by Theorem 2.2.11 and the observation above we obtain the following result.

**Theorem 2.2.14.** *Let $A$ be a compact set and let the bifunction $f : A \times B \to Y$ satisfies the following conditions:*

*(i) for each $b \in B$, the function $f(\cdot, b) : A \to Y$ is $K$-pseudo-upper semicontinuous on $A$;*

*(ii) $f$ is $K$-subconcavelike in its first variable;*

*(iii) for all $b_1, b_2, ..., b_n \in B$, $k_1^*, k_2^*, ..., k_n^* \in K^*$, not all zero, one has*

$$\sup_{a\in A}\sum_{i=1}^{m} k_j^*(f(a,b)) \geq 0 \quad \textit{for each} \quad b \in B.$$

*Then the vector equilibrium problem $(VEP)$ admits a solution.*

The next corollary is a result obtained by A. Capătă and G. Kassay [30] for vector-valued bifunctions using upper semicontinuity instead of $K$-pseudo-upper semicontinuity.

**Corollary 2.2.15.** *Let $A$ be a compact set and let the bifunction $f : A \times B \to Y$ satisfies the following conditions:*

*(i) for each $b \in B$, the function $f(\cdot, b) : A \to Y$ is upper semicontinuous on $A$;*

*(ii) $f$ is $K$-subconcavelike in its first variable;*

*(iii) for all $b_1, b_2, ..., b_n \in B$, $k_1^*, k_2^*, ..., k_n^* \in K^*$, not all zero, one has*

$$\sup_{a\in A}\sum_{i=1}^{m} k_j^*(f(a,b)) \geq 0 \quad \textit{for each} \quad b \in B.$$

*Then the vector equilibrium problem $(VEP)$ admits a solution.*

## 2.3 On vector equilibrium problems given by a sum of two functions

As we know by now, $(SEP)$ has been extensively studied in recent years. However, a seemingly interesting special case, where $f(x, y) = g(x, y) + h(x, y)$ with $g, h : A \times A \to \mathbb{R}$ captured less attention, although it was investigated already in E. Blum and W. Oettli [22], where the authors obtained existence results by imposing their assumptions separately on $g$ and $h$. As stressed in [22], if $g = 0$, the result becomes a variant of Ky Fan's theorem [46], whereas for $h = 0$ it becomes a variant of the Browder-Minty theorem for variational inequalities (see F.E. Browder [25], [26], G.J. Minty [100]).

Throughout this section, if not otherwise stated, $X$ and $Y$ denote (real) topological vector spaces, $A, B \subseteq X$ nonempty convex sets ($B$ being typically a compact subset of $A$, but not always), and $K \subseteq Y$ a proper convex cone with nonempty interior such that $0 \in K \cap (-K)$.

If $\mathbb{R}$ is replaced by $Y$, i.e., $f$ becomes a vector-valued function of the form $f : A \times A \to Y$, one may consider the so-called *vector equilibrium problem* as follows:

$$\textit{(VEP)} \text{ find an element } \overline{x} \in A \text{ such that } f(\overline{x}, y) \notin -\mathrm{intK} \quad \text{for all} \quad y \in A.$$

This problem has also attracted much attention in the recent years especially due to its applications within the fields of vector optimization and vector variational inequalities (see, for instance A. Capătă and G. Kassay [30], Y.P. Fang and N.J. Huang [47], N. Hadjisavvas and S. Schaible [59] and the references therein).

Following the idea and the steps of the proofs given by Blum and Oettli, K.R. Kazmi [80] obtained an existence result for $(VEP)$ in the case when $f(x, y) = g(x, y) + h(x, y)$, with $g, h : A \times A \to Y$. However, his assumptions on the vector-valued functions turn to be too strong in order to recover the result of Blum and Oettli when $Y := \mathbb{R}$ and $K := [0, \infty)$.

The results of this section have been stated in G. Kassay and M. Miholca [79] aiming to weaken the assumptions of Kazmi in such a way to be able to recover Blum-Oettli's results on one hand, and by assuming alternative conditions on the vector functions, to deduce new existence theorems, on the other hand. The special case of reflexive Banach spaces endowed with the weak topology is separately treated; in that case mild sufficient conditions for guaranteing coercivity are presented.

Let us first recall the following concept. If $B \subseteq A$, then $\text{core}_A B$, the core of $B$ relative to $A$, is defined through

$$a \in \text{core}_A B \Longleftrightarrow (a \in B \ \text{ and } \ B \cap (a, y] \neq \emptyset \ \text{ for all } \ y \in A \backslash B),$$

where $(a, y] = \{\lambda a + (1-\lambda) y : \lambda \in [0, 1)\}$. Note that $\text{core}_A A = A$.

The following simple property will be useful in the sequel.

**Lemma 2.3.16.** *For all $x, y \in Y$ we have:*

$$x \in K, \ y \notin -\text{int} K \Rightarrow x + y \notin -\text{int} K.$$

In vector optimization various relaxations and modifications of the classical *lower/upper semicontinuity* for scalar functions have been investigated for vector-valued functions to explore and characterize efficient solutions.

For our next investigations we use the $K$-upper (lower) semicontinuity, already presented in Definition 2.2.8.

The next characterizations of $K$-upper semicontinuity has been given by T. Tanaka [118] (see also Remark 2.2.9).

**Lemma 2.3.17.** *The following three statements are equivalent:*

*(i) $F$ is $K$-upper semicontinuous on $X$;*

*(ii) for any $x \in X$, for any $k \in \text{intK}$, there exists a neighborhood $U \subseteq X$ of $x$ such that $F(u) \in F(x) + k - \text{intK}$ for all $u \in U$;*

*(iii) for any $a \in Y$, the set $\{x \in X : F(x) - a \in -\text{intK}\}$ is open.*

Aiming to obtain existence results for $(SEP)$, A. N. Iusem, G. Kassay and W. Sosa [67] introduced the following (slightly stronger) variant of proper quasimonotonicity (called by themselves *property P4"*):

**Definition 2.3.18.** A bifunction $f : A \times A \to \mathbb{R}$ is said to be *essentially quasimonotone* iff for every $x_1, \cdots, x_n \in A$ and $\lambda_1, \cdots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$, it holds that

$$\sum_{i=1}^n \lambda_i f(x_i, \sum_{j=1}^n \lambda_j x_j) \leq 0.$$

Now let us turn to define the corresponding concepts of monotonicity, proper quasimonotonicity (see Definition 1.2.7), and essential quasimonotonicity for vector-valued bifunctions.

**Definition 2.3.19.** (see, for instance, [80]) A bifunction $f : A \times A \to Y$ is said to be *$K$-monotone* iff

$$f(x, y) + f(y, x) \in -K \ \text{ for all } \ x, y \in A.$$

**Definition 2.3.20.** [79] A bifunction $f : A \times A \to Y$ is said to be

(i) $K$-*properly quasimonotone* iff for all $x_1, ...x_n \in A$ and all $\lambda_1, ...\lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ there exists an $i_0$ such that
$$f(x_{i_0}, \sum_{j=1}^n \lambda_j x_j) \notin \text{int}K;$$

(ii) $K$-*essentially quasimonotone* iff for all $x_1, ...x_n \in A$ and all $\lambda_1, ...\lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ it holds that
$$\sum_{i=1}^n \lambda_i f(x_i, \sum_{j=1}^n \lambda_j x_j) \notin \text{int}K.$$

It easy to observe that if $f$ is $K$-essentially quasimonotone then $f$ is $K$-properly quasimonotone. The relationship between $K$-monotonicity and $K-$essentially quasimonotonicity is stressed in the following statement.

**Proposition 2.3.21.** [79] *Suppose that $f : A \times A \to Y$ is $K$-monotone and $K$-convex in the second argument. Then $f$ is $K$-essentially quasimonotone.*

The next example shows that a $K$-essentially quasimonotone bifunction is not necessarily $K$-monotone, even if it is $K$-convex in the second argument.

**Example 2.3.22.** [79] Let $f : [0,1] \times [0,1] \to \mathbb{R}^2$ given by $f = (f_1, f_2)$, where $f_1(x,y) = |x-y|$, $f_2(x,y) = 0$ for every $x, y \in [0,1]$. It is easy to see that $f$ is $\mathbb{R}^2_+$-essentially quasimonotone, $\mathbb{R}^2_+$-convex in the second argument, but not $\mathbb{R}^2_+$-monotone, since $f(1,0) + f(0,1) = (2,0) \notin -\mathbb{R}^2_+$.

### 2.3.1 Existence results of $(VEP)$

In what follows we are interested to obtain existence results for $(VEP)$ when the bifunction $f : A \times A \to Y$ is given by $f(x,y) = g(x,y) + h(x,y)$ where $g, h : A \times A \to Y$. Such a situation was already explored by Blum and Oettli [22] in the particular case when $Y := \mathbb{R}$ and $K := [0, \infty)$, i.e., the scalar equilibrium problem $(SEP)$. Their result was obtained with $g$ monotone satisfying a mild upper semicontinuity in the first argument, whereas $h$ is not necessarily monotone, but has to satisfy a much stronger upper semicontinuity condition in the first argument. Later, Kazmi [80], made an attempt to extend these results to the general case of $(VEP)$ by following the ideas of [22], but many of his assumptions are too strong, therefore he couldn't recover the results of Blum and Oettli. We only mention here that while the latter assume semicontinuity for scalar functions, Kazmi requires continuity of the corresponding vector functions. The aim of this section is to improve Kazmi's results by weakening their conditions (both topological and algebraical) in such a way to recover Blum and Oettli's results (the three Lemmas and Theorem 2.3.28 below), and to obtain a variant without monotonicity requirement upon $g$; this is substituted by assuming $K$-concavity of $g$ in its first variable (Theorem 3.6.21 below).

To start, let us first prove the following three lemmas. The first one can also be seen as an existence result for a special vector equilibrium problem.

**Lemma 2.3.23.** [79] *Let $B$ be a compact subset of $X$, let $g : B \times B \to Y$ and $h : B \times B \to Y$ be given bifunctions satisfying:*

*(i) $g$ is $K$-essentially quasimonotone and $K$-lower semicontinuous in the second argument;*

*(ii) $h$ is $K$-upper semicontinuous in the first argument and $K$-convex in the second argument; $h(x,x) \in K$ for all $x \in B$.*

*Then there exists $\overline{x} \in B$ such that*

$$h(\overline{x}, y) - g(y, \overline{x}) \notin -\mathrm{int}K,$$

*for all $y \in B$.*

**Lemma 2.3.24.** [79] *Let $g : B \times B \to Y$ and $h : B \times B \to Y$ be given bifunctions satisfying:*

*(i) $g$ is $K$-convex in the second argument, $g(x, x) \in K$ for all $x \in B$, and for all $x, y \in B$ the function $t \in [0, 1] \to g(ty + (1-t)x, y)$ is $K$-upper semicontinuous at 0;*

*(ii) $h$ is $K$-convex in the second argument; $h(x, x) = 0$ for all $x \in B$.*

*If there exists $\overline{x} \in B$ such that $h(\overline{x}, y) - g(y, \overline{x}) \notin -\mathrm{int}K$ for all $y \in B$ then $h(\overline{x}, y) + g(\overline{x}, y) \notin -\mathrm{int}K$ for all $y \in B$.*

**Lemma 2.3.25.** [79] *Let $B \subseteq A$. Assume that $F : A \to Y$ is $K$-convex, $x_0 \in \mathrm{core}_A B$, $F(x_0) \in -K$, and $F(y) \notin -\mathrm{int}K$ for all $y \in B$. Then $F(y) \notin -\mathrm{int}K$ for all $y \in A$.*

**Remark 2.3.26.** As the next example shows, the assumption $F(x_0) \in -K$ within Lemma 2.3.25 cannot be weakened to $F(x_0) \notin \mathrm{int}K$, and in this way, Lemma 10 in Kazmi [80] is false.

**Example 2.3.27.** [79] Let $X = A := \mathbb{R}$, $B := [-1, 1]$, $Y := \mathbb{R}^2$, $K := \mathbb{R}^2_+$, $F(x) = (x+1, x-1)$, and $x_0 = 0$. Then $F$ is obviously $\mathbb{R}^2_+$-convex (furthermore, both components are affine functions), $x_0 \in \mathrm{core}_A B$, $F(x_0) = (1, -1) \notin \mathrm{int}K$ and $F(y) \notin -\mathrm{int}K$ for each $y \in B$. However, for instance, for $y = -2$ we get $F(-2) = (-1, -3) \in -\mathrm{int}K$.

Now we are in a position to state our main results of this section.

**Theorem 2.3.28.** [79] *Suppose that $g : A \times A \to Y$ and $h : A \times A \to Y$ satisfy:*

*(i) $g$ is $K$-essentially quasimonotone, $K$-convex and $K$-lower semicontinuous in the second argument; $g(x, x) \in K \cap (-K)$ for all $x \in A$, and for all $x, y \in A$ the function $t \in [0, 1] \to g(ty + (1-t)x, y)$ is $K$-upper semicontinuous at 0;*

*(ii) $h$ is $K$-upper semicontinuous in the first argument and $K$-convex in the second argument; $h(x, x) = 0$ for all $x \in A$;*

*(iii) There exists a nonempty compact convex subset $C$ of $A$ such that for every $x \in C \setminus \mathrm{core}_A C$ there exists an $a \in \mathrm{core}_A C$ such that*

$$g(x, a) + h(x, a) \in -K.$$

*Then there exists $\overline{x} \in C$ such that*

$$g(\overline{x}, y) + h(\overline{x}, y) \notin -\mathrm{int}K,$$

*for all $y \in A$.*

The next result due to K.R. Kazmi [80] is a particular case of Theorem 2.3.28. $X$ and $Y$ are the same as before, $A \subseteq X$ is a nonempty closed convex set and $K \subseteq Y$ is a proper pointed closed convex cone with nonempty interior.

**Corollary 2.3.29.** *Suppose that $g : A \times A \to Y$ and $h : A \times A \to Y$ satisfy:*

*(i) $g$ is $K$-monotone, $K$-convex and continuous in the second argument; $g(x, x) = 0$ for all $x \in A$, and for all $x, y \in A$ the function $t \in [0, 1] \to g(ty + (1-t)x, y)$ is continuous at 0;*

*(ii) $h$ is continuous in the first argument and $K$-convex in the second argument; $h(x, x) = 0$ for all $x \in A$;*

*(iii) There exists a nonempty compact convex subset $C$ of $A$ such that for every $x \in C \setminus \mathrm{core}_A C$ there exists an $a \in \mathrm{core}_A C$ such that*

$$g(x,a) + h(x,a) \in -K.$$

*Then there exists $\overline{x} \in C$ such that*

$$g(\overline{x}, y) + h(\overline{x}, y) \notin -\mathrm{int}K,$$

*for all $y \in A$.*

**Remark 2.3.30.** Theorem 2.3.28 improves from several points of view the above result of Kazmi. With respect to the monotonicity, notice that the bifunction $f$ defined in Example 2.3.22 satisfies all requirements demanded upon $g$ in item (i) of Theorem 2.3.28, but since it is not $\mathbb{R}^2_+$-monotone, it doesn't satisfy item (i) of Corollary 2.3.29. Another improvement, related to the continuity assumptions, can be identified within the next example, obtained by a slight modification of Example 9.27 in H.H. Bauschke and P.L. Combettes [15].

**Example 2.3.31.** [79] Let $A := \{(x,y) \in \mathbb{R}^2 : x > 0\} \cup \{(0,0)\}$ and consider the function $f : A \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{y^2}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

It is obvious that $f$ is convex and lower semicontinuous, but not continuous at $(0,0)$. Indeed, take any sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n > 0$ and $x_n \to 0$ whenever $n \to \infty$. Then $f(x_n^2, x_n) = 1$ for each $n$, but $f(0,0) = 0$, i.e., $f$ is not continuous at $(0,0)$. Now let $Y := \mathbb{R}$ and $K := [0,\infty)$, and consider the bifunction $g : A \times A \to \mathbb{R}$ given by $g(a,b) := f(b) - f(a)$, where $a = (x,y), b = (u,v) \in A$. Then $g$ satisfies all assumptions of Theorem 2.3.28 (i), but not item (i) of Corollary 2.3.29 due to the lack of continuity.

Thanks to the improvements made for Theorem 7 of K.R. Kazmi [80], the next result of E. Blum and W. Oettli [22] becomes a particular case of Theorem 2.3.28.

**Corollary 2.3.32.** *Let $X$ be a real topological vector space, $A \subseteq X$ a nonempty closed convex set, $g : A \times A \to \mathbb{R}$ and $h : A \times A \to \mathbb{R}$ satisfying:*

*(i) $g$ is monotone, convex and lower semicontinuous in the second argument; $g(x,x) = 0$ for all $x \in A$, and for all $x, y \in A$ the function $t \in [0,1] \to g(ty + (1-t)x, y)$ is upper semicontinuous at $0$;*

*(ii) $h$ is upper semicontinuous in the first argument and convex in the second argument; $h(x,x) = 0$ for all $x \in A$;*

*(iii) There exists a nonempty compact convex subset $C$ of $A$ such that for every $x \in C \setminus \mathrm{core}_A C$ there exists an $a \in \mathrm{core}_A C$ such that*

$$g(x,a) + h(x,a) \leq 0.$$

*Then there exists $\overline{x} \in C$ such that*

$$g(\overline{x}, y) + h(\overline{x}, y) \geq 0,$$

*for all $y \in A$.*

We conclude this section with a variant of Theorem 2.3.28 in which no monotonicity assumptions are made on the bifunction $g$; the lack of this requirement is substituted by assuming $K$-concavity of $g$ in its first argument. In this way the algebraic conditions upon $g$ become symmetric. Apparently this provides us a new result even in the particular case of scalar functions, i.e., where $Y := \mathbb{R}$ and $K := [0,\infty)$.

**Theorem 2.3.33.** [79] *Suppose that $g : A \times A \to Y$ and $h : A \times A \to Y$ satisfy:*

*(i) $g$ is $K$-concave in the first argument, $K$-convex and $K$-lower semicontinuous in the second argument; $g(x,x) \in K \cap (-K)$ for all $x \in A$, and for all $x, y \in A$ the function $t \in [0,1] \to g(ty + (1-t)x, y)$ is $K$-upper semicontinuous at $0$;*

*(ii) $h$ is $K$-upper semicontinuous in the first argument and $K$-convex in the second argument; $h(x,x) = 0$ for all $x \in A$;*

*(iii) There exists a nonempty compact convex subset $C$ of $A$ such that for every $x \in C \setminus \mathrm{core}_A C$ there exists an $a \in \mathrm{core}_A C$ such that*

$$g(x,a) + h(x,a) \in -K.$$

*Then there exists $\overline{x} \in C$ such that*

$$g(\overline{x}, y) + h(\overline{x}, y) \notin -\mathrm{int}K,$$

*for all $y \in A$.*

### 2.3.2 The case of reflexive Banach spaces

The aim of this subsection is to provide several sufficient conditions for the coercivity required in assumption (iii) of Theorem 2.3.28 (or Theorem 2.3.33) when $X$ is a reflexive Banach space endowed with the weak topology. It is well-known that in this setting every closed, convex and bounded set (in particular closed balls) are (weakly) compact. Hence, all conditions which we formulate below are vacuously satisfied when the (closed and convex) set $A \subseteq X$ is bounded. Thus, in the sequel we shall suppose that $A$ is unbounded. Let $a \in A$ be a fixed element. Let us start with the following condition:

(C) there exists $\rho > 0$ such that for all $x \in A$ with $\|x - a\| = \rho$ one has $g(x,a) + h(x,a) \in -K$.

**Proposition 2.3.34.** [79] *Under condition (C), assumption (iii) in Theorem 2.3.28 is satisfied.*

Next we give mild sufficient conditions separately on $g$ and $h$ for (C). Consider the following assumptions.

(G) (upper boundedness of $g(\cdot, a)$ on a closed ball): there exist $M \in Y$ and $r > 0$ such that

$$M - g(x,a) \in K, \text{ whenever } x \in A,\ \|x - a\| \leq r,$$

and

(H) there exists an element $u \in -\mathrm{int}K$ such that for all $t > 0$ there is an $R > 0$ satisfying

$$\forall x \in A:\ \|x - a\| \geq R:\ tu\|x - a\| - h(x,a) \in K.$$

**Remark 2.3.35.** (H) is obviously fulfilled when $Y := \mathbb{R}$, $K := [0, \infty)$ and

$$\frac{h(x,a)}{\|x - a\|} \to -\infty \text{ whenever } \|x - a\| \to \infty, \quad x \in A.$$

(Condition (c) in [22].) Indeed, take $u = -1$ and arbitrary $t > 0$. Since

$$\frac{h(x,a)}{\|x - a\|} \to -\infty \quad \text{whenever} \quad \|x - a\| \to \infty,$$

we can find $R > 0$ such that for all $x \in A:$

$$\|x - a\| \geq R, \quad \frac{h(x,a)}{\|x - a\|} \leq -t,$$

proving the assertion.

In what follows we shall need the property below. We denote by $B(u,r)$ the open ball centered at $u \in X$ and radius $r > 0$.

**Lemma 2.3.36.** [79] *Let $v \in Y$ and $u \in -\mathrm{int}K$ be arbitrary. Then there exists $t > 0$ such that $v + tu \in -\mathrm{int}K$.*

**Remark 2.3.37.** The proof above shows that a similar statement holds when in Lemma 2.3.36 we take $\mathrm{int}K$ instead of $-\mathrm{int}K$. More precisely, the following property holds: for any $v \in Y$ and $u \in \mathrm{int}K$ there exists $t > 0$ such that $v + tu \in \mathrm{int}K$.

Next we need the following technical lemma concerning the function $g$.

**Lemma 2.3.38.** [79] *Suppose that $g$ satisfies (G) and $g(\cdot, a)$ is $K$-concave with $g(a,a) \in K$. Then*

$$\frac{M}{r} - \frac{g(x,a)}{\|x-a\|} \in K, \quad \forall x \in A,\ \|x-a\| \geq r,$$

*where the vector $M$ and the number $r$ are defined in (G).*

Now we are able to provide sufficient conditions, separately on $g$ and on $h$, for the coercivity assumption (iii) in Theorem 2.3.28.

**Proposition 2.3.39.** [79] *Assume that*

*(i) $g$ satisfies (G) and $g(\cdot, a)$ is $K$-concave with $g(a,a) \in K$;*

*(ii) $h$ satisfies (H).*

*Then condition (iii) of Theorem 2.3.28 is fulfilled.*

The next result, which might have some interest on its own, gives sufficient conditions for lower (upper) boundedness of a $K$-lower ($K$-upper) semicontinuous function in the sense of Definition 2.2.8.

**Lemma 2.3.40.** [79] *Let $C$ be a compact subset of $X$ and $F : C \to Y$.*

*(i) if $F$ is $K$-lower semicontinuous on $C$, then it is lower bounded, i.e. there exists a vector $m \in Y$ such that $F(x) - m \in \mathrm{int}K$ for all $x \in C$;*

*(ii) if $F$ is $K$-upper semicontinuous on $C$, then it is upper bounded, i.e. there exists a vector $M \in Y$ such that $M - F(x) \in \mathrm{int}K$ for all $x \in C$.*

Summarizing, we have the following existence result in reflexive Banach spaces. The basic assumptions upon the set $A$, the space $Y$, and the cone $K$ remain the same.

**Theorem 2.3.41.** [79] *Suppose that $X$ is a reflexive Banach space. Let $g : A \times A \to Y$ and $h : A \times A \to Y$ satisfying the following properties:*

*(i) $g$ is $K$-concave and weakly $K$-upper semicontinuous in the first argument; $K$-convex and weakly $K$-lower semicontinuous in the second argument; $g(x,x) \in K \cap (-K)$ for all $x \in A$;*

*(ii) $h$ is weakly $K$-upper semicontinuous in the first argument; $K$-convex in the second argument; $h(x,x) = 0$, for all $x \in A$ and (H) holds.*

*Then there exists $\overline{x} \in A$ such that*

$$g(\overline{x}, y) + h(\overline{x}, y) \notin -\mathrm{int}K, \quad \forall y \in A.$$

In the scalar case, i.e., when $Y := \mathbb{R}$ and $K := [0, \infty]$, we obtain a simplified form of Theorem 2.3.41.

**Corollary 2.3.42.** [79] *Let $g : A \times A \to \mathbb{R}$ and $h : A \times A \to \mathbb{R}$ satisfying the following properties:*

*(i) $g$ is concave and upper semicontinuous in the first argument, convex and lower semicontinuous in the second argument, and $g(x, x) = 0$ for all $x \in A$;*

*(ii) $h$ is weakly upper semicontinuous in the first argument, convex in the second argument, $h(x, x) = 0$ for all $x \in A$, and*

$$\frac{h(x, a)}{\|x - a\|} \to \infty \text{ whenever } \|x - a\| \to \infty, \quad x \in A.$$

*Then there exists $\overline{x} \in A$ such that*

$$g(\overline{x}, y) + h(\overline{x}, y) \geq 0, \quad \forall y \in A.$$

# Chapter 3

# Existence results for variational inequalities

Let $X$ be a topological vector space and $X^*$ its dual space. Given a set-valued operator $T : X \to 2^{X^*}$, $D(T)$ stands for its effective domain, i.e.,the set

$$D(T) = \{x \in X : T(x) \neq \emptyset\}.$$

The *variational inequality problem* can be regarded as a particular case of the scalar equilibrium problem $(SEP)$. Given a nonempty, closed, convex subset $A$ of $X$, the variational inequality problem we consider in this chapter can be formulated as follows:

$$(VI) \qquad \text{find an element } \overline{x} \in K \text{ such that } \sup_{x^* \in T(\bar{x})} \langle x^*, y - \overline{x} \rangle \geq 0 \;\text{ for all }\; y \in A.$$

The former has many applications in the field of partial differential equations as, for example, the Signorini problem, the obstacle problem and the elasto-plastic problem.

In this section we establish some similar results valid in infinite dimensional reflexive Banach spaces.

## 3.1 Generalized monotone and hemicontinuous operators

Section 3.1 is devoted to the study of *generalized monotone operators.*

The concept of *monotone operator,* see Definition 1.1.13, introduced some fifty years ago by the celebrated works of F.E. Browder and G.J. Minty (see, for example, F.E. Browder [25], F.E. Browder [26] and G.J. Minty [100]), is considered being one of the cornerstones for the development of nonlinear functional analysis, especially for its usefulness in the theory of partial differential equations and, as well, in modeling different phenomena arising from mechanics, engineering and economy.

The original concept of monotonicity has been extended in various directions. Apart from their theoretical interest, *generalized monotone operators* are often more suitable to describe problems than the original concept of monotonicity, in disciplines such as economics, management science, probability theory and other applied sciences. Results concerning surjectivity are therefore especially important, as they guarantee, in particular, the existence of zeros of these operators. Furthermore, due to the fact that the subdifferential operator of a (generalized) convex function is a (generalized) monotone operator, finding a zero of such an operator is of special interest. Indeed, zeros of the subdifferential operator of a function defined on the same space, are precisely the minimum points of this function. Hence, there is an important link between the theory of (generalized) monotone operators and optimization theory.

One of the earliest surjectivity results with respect to monotone operators is due to G.J. Minty [100] stating that the sum of a maximal monotone operator defined on a Hilbert space and the identity operator is surjective. This result has been extended for Banach spaces where the role of the identity was played by the duality mapping (see, e.g., Chapter 4 in R.S. Burachik and A.N. Iusem [28]). A somewhat more general result where the identity (or duality) mapping is not involved is due to F.E. Browder and G.J. Minty, saying that a monotone, *hemicontinuous* and *coercive* operator on a reflexive Banach space is surjective (see, for instance, E. Zeidler [128]). Here the absence of the duality mapping is substituted by the assumption of coercivity upon the operator, which, in a more relaxed form, plays a very important role in the present thesis as well.

In their attempt to extend the surjectivity results of Minty and Browder-Minty, A.N. Iusem, G. Kassay and W. Sosa [66] introduced the so-called *premonotone operators* which include, in particular $\epsilon$-monotone operators, that are related to the very useful $\epsilon$-subdifferentials. Their surjectivity results are limited to finite dimensional spaces since they are consequences of an existence theorem for finite dimensional equilibrium problem. As commented in the last section of A.N. Iusem, G. Kassay and W. Sosa [66], the limitation of finite dimensionality couldn't be overcome. This fact is due to the specific kind of *coercivity condition* they were somehow forced to use.

Existence results for variational inequalities, if their domains are unbounded, usually require a kind of coercivity condition upon the operator involved. A classical notion of coerciveness is the following.

**Definition 3.1.1.** (see, for instance, [2], [66]) An operator $T : X \to 2^{X^*}$ is called *coercive* iff

$$\lim_{\|x\|\to\infty} \frac{\inf\limits_{x^*\in T(x)} \langle x^*, x\rangle}{\|x\|} = \infty.$$

In this thesis, we use a more general coercivity condition introduced in M.H. Alizadeh, N. Hadjisavvas and M. Roohi [2].

**Definition 3.1.2.** An operator $T$ is said to be *quasicoercive* iff

$$\lim_{\|x\|\to\infty} \inf_{x^*\in T(x)} \|x^*\| = \infty$$

and

$$\liminf_{\|x\|\to\infty} \frac{\inf\limits_{x^*\in T(x)} \langle x^*, x\rangle}{\|x\|} > -\infty.$$

Clearly, each coercive operator is quasicoercive. The converse is not true as the following example (given in [2]) shows. Consider the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (-y, x)$. It is immediate to see that $T$ is quasicoercive without being coercive.

In order to investigate some continuity properties of monotone operators, J. Kolomý [84] introduced the following concept, useful for us in the next section.

**Definition 3.1.3.** Let $X$ be a normed space. The operator $T : X \to 2^{X^*}$ is called *hemiclosed* at $\overline{x} \in D(T)$ iff for every $z \in X$, for each sequence $\{t_n\}$ of positive numbers converging to zero such that $x_n = \overline{x} + t_n z \in D(T)$ for sufficiently large $n$ and each $x_n^* \in T(x_n)$ such that $\|x_n^*\| \le R$, for some constant $R > 0$, there exists a subsequence $\{x_{n_k}^*\}$ of $\{x_n^*\}$ having weak* limit $x^* \in T(\overline{x})$.

If we consider a bifunction $f : D(T) \times X \to \mathbb{R}$ given in the form

$$f(x, y) = \sup_{x^*\in T(x)} \langle x^*, y - x\rangle,$$

it is immediate to see that the continuity (with respect to the norm) of the operator $T$ does not guarantee the weak continuity of $f$ with respect to its first variable: moreover, it doesn't even guarantee the weak upper semicontinuity of $f(\cdot, y)$, in general. To avoid this difficulty, Ky Fan [46], [45], introduced the following concept for single-valued operators (called $F$-hemicontinuity in A. Maugeri and F. Raciti [94]).

**Definition 3.1.4.** A mapping $T : K \to X^*$ is called $F$-*hemicontinuous* iff for all $y \in A$, the function $x \mapsto \langle T(x), x - y\rangle$ is weakly lower semicontinuous on $A$.

The next notion is a corresponding notion for set-valued maps.

**Definition 3.1.5.** [78] An operator $T : X \to 2^{X^*}$ is said to be $F$-*hemicontinuous* at $\overline{x} \in D(T)$ iff for any net $\{x_\alpha\}_{\alpha\in I} \subseteq D(T)$, with $x_\alpha \neq \overline{x}$, converging weakly to $\overline{x}$, for any $x^*_\alpha \in T(x_\alpha)$ and for each $y \in D(T)$ there exists $x^*_y \in T(\overline{x})$ such that

$$\langle x^*_y, \overline{x} - y\rangle \leq \liminf_{\alpha} \langle x^*_\alpha, x_\alpha - y\rangle.$$

$T$ is said to be $F$-hemicontinuous iff it is $F$-hemicontinuous at each point of its domain.

**Remark 3.1.6.** Replacing liminf by limsup in Definition 3.1.5 we apparently obtain a weaker notion. But in reality not, as the following argument shows.
Suppose that for any $\overline{x} \in D(T)$, any net $\{x_\alpha\}_{\alpha\in I} \subseteq D(T)$, $x_\alpha \rightharpoonup \overline{x}$, any $x^*_\alpha \in T(x_\alpha)$ and for each $y \in D(T)$ there exists $x^*_y \in T(\overline{x})$ such that

$$\langle x^*_y, \overline{x} - y\rangle \leq \limsup_{\alpha} \langle x^*_\alpha, x_\alpha - y\rangle. \tag{3.1}$$

Suppose by contradiction that there exist $x_\alpha \rightharpoonup \overline{x}$, $x^*_\alpha \in T(x_\alpha)$, $y \in D(T)$ such that for all $x^* \in T(\overline{x})$

$$\langle x^*, \overline{x} - y\rangle > \liminf_{\alpha} \langle x^*_\alpha, x_\alpha - y\rangle. \tag{3.2}$$

Then there exists $\{x_{\alpha_\beta}\} \subseteq \{x_\alpha\}$ and $\{x^*_{\alpha_\beta}\} \subseteq \{x^*_\alpha\}$ such that

$$\liminf_{\alpha} \langle x^*_\alpha, x_\alpha - y\rangle = \lim_{\beta} \langle x^*_{\alpha_\beta}, x_{\alpha_\beta} - y\rangle.$$

Let $z_\beta = x_{\alpha_\beta}$, $z^*_\beta = x^*_{\alpha_\beta}$. According to (3.1) there exists $z^*_y \in T(\overline{x})$ such that

$$\langle z^*_y, \overline{x} - y\rangle \leq \limsup_{\beta} \langle z^*_\beta, z_\beta - y\rangle = \lim_{\beta} \langle z^*_\beta, z_\beta - y\rangle = \liminf_{\alpha} \langle x^*_\alpha, x_\alpha - y\rangle,$$

contradicting (3.2).

Next we present some generalized monotonicity concepts. Pseudomonotone operators play an important role in the theory of variational inequalities. Different kind of pseudomonotonicity concepts appeared in the literature, in some cases even different notions were named in the same way. In order to extend the classical concept of monotone operator, S. Karamardian [73] introduced in 1976 a class of operators called by himself *pseudomonotone maps*. This notion was introduced also for set-valued operators in several ways (see, for instance, A.N. Iusem, G. Kassay and W. Sosa [67], S. Komlosi [85] and J.C. Yao [121]). Being purely algebraical, we call it $\mathcal{A}$-pseudomonotone.

**Definition 3.1.7.** [85]. The operator $T : X \to 2^{X^*}$ is called $\mathcal{A}$-*pseudomonotone* iff for every $x, y \in X$ and $x^* \in T(x), y^* \in T(y)$, the following implication holds:

$$\langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0.$$

Another concept of generalized monotonicity for single valued maps has been already introduced in 1968 by H. Brézis [24] named pseudomonotonicity, as well. Being purely topological, in order to avoid confusion, this notion was often called topological pseudomonotonicity by some authors. Later, it has been extended for set-valued operators too, by F.E. Browder and P. Hess [27].

**Definition 3.1.8.** The operator $T : X \to 2^{X^*}$ is called *$\mathcal{B}$-pseudomonotone* iff for every $\overline{x} \in X$, for every net $\{x_\alpha\}_{\alpha \in I}$ in $X$, $x_\alpha \rightharpoonup \overline{x}$ and any $x_\alpha^* \in T(x_\alpha)$ satisfying $\limsup_\alpha \langle x_\alpha^*, x_\alpha - \overline{x}\rangle \leq 0$ implies that for every $y \in X$ there exists $x_y^* \in T(\overline{x})$ such that

$$\langle x_y^*, \overline{x} - y\rangle \leq \liminf_\alpha \langle x_\alpha^*, x_\alpha - y\rangle.$$

Recently D. Inoan and J. Kolumbán [63] introduced a new notion of pseudomonotonicity (initially defined in A. Domokos and J. Kolumbán [40] for single valued mappings) which can be seen (under some mild additional conditions) as a common generalization of both $\mathcal{A}$ and $\mathcal{B}$-pseudomonotonicities.

**Remark 3.1.9.** It is obvious that if an operator is $F$-hemicontinuous, then it is also $\mathcal{B}$-pseudomonotone.

**Definition 3.1.10.** [63]. The operator $T : X \to 2^{X^*}$ is called *$\mathcal{C}$-pseudomonotone* iff for every $x, y \in X$ and for every net $\{x_\alpha\}_{\alpha \in I}$ in $X$ with $x_\alpha \rightharpoonup \overline{x}$,

$$\sup_{x_\alpha^* \in T(x_\alpha)} \langle x_\alpha^*, (1-t)x + ty - x_\alpha\rangle \geq 0, \ \forall\, t \in [0,1], \ \forall\, \alpha \in I$$

implies $\sup_{x^* \in T(\overline{x})} \langle x^*, y - \overline{x}\rangle \geq 0.$

Later, in Section 3.4, we prove some results related to $\mathcal{C}$ and $\mathcal{B}$-pseudomonotone operators (cf. Definition 3.1.10). D. Inoan and J. Kolumbán [63] have shown by a very simple example that $\mathcal{C}$-pseudomonotonicity is indeed weaker than the other two (algebraic and topological) pseudomonotonicity concepts. Here we shall need the following result.

**Lemma 3.1.11.** (Particular case of Theorem 13 in [63]). *Let $T : X \to 2^{X^*}$ be an operator. If the conditions*
*$(i)$ $T$ is $\mathcal{B}$-pseudomonotone;*
*$(ii)$ $T(x)$ is convex, bounded and closed for every $x \in X$;*
*are satisfied, then $T$ is $\mathcal{C}$-pseudomonotone.*

Given an operator $T : X \to 2^{X^*}$, we define, for each $z \in X$, the set

$$F(z) = \{x \in X \mid \sup_{x^* \in T(x)} \langle x^*, z - x\rangle \geq 0\}.$$

In [63] the following characterization of $\mathcal{C}$-pseudomonotonicity was established.

**Lemma 3.1.12.** [63]. *The operator $T : X \to 2^{X^*}$ is $\mathcal{C}$-pseudomonotone if and only if, for every $x, y \in X$,*

$$\mathrm{cl}\Big(\bigcap_{z \in [x,y]} F(z)\Big) \cap [x,y] = \Big(\bigcap_{z \in [x,y]} F(z)\Big) \cap [x,y],$$

*where $[x,y]$ denotes the line segment joining $x$ and $y$.*

The following concept of generalized monotonicity for a set-valued operator has been considered by A. Daniilidis and N. Hadjisavvas in [39].

**Definition 3.1.13.** An operator $T : X \to 2^{X^*}$ is said to be *properly quasimonotone* iff for all $x_1, ..., x_n \in D(T)$, for all $\lambda_1, ..., \lambda_n \geq 0$ such that $\sum\limits_{i=1}^{n} \lambda_i = 1$, there exists $i \in \{1, 2, ..., n\}$ such that

$$\forall x^* \in T(x_i) : \ \langle x^*, \sum_{j=1}^{n} \lambda_j x_j - x_i \rangle \leq 0.$$

It is immediate to see the following connection between proper quasimonotonicity for an operator and for a bifunction, respectively. Recall that proper quasimonotonicity for bifunctions was defined in Definition 1.2.7.

**Lemma 3.1.14.** [78] *Let $T : X \to 2^{X^*}$ such that its domain $D(T)$ is convex and consider the bifunction $f : D(T) \times D(T) \to \mathbb{R}$, given by*

$$f(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle.$$

*If $T$ is properly quasimonotone, then $f$ is properly quasimonotone.*

## 3.2 Variational inequalities and generalized equations given by properly quasimonotone operators

In this section we are able to obtain surjectivity results for properly quasimonotone operators using Theorem 2.1.14.

The following theorem gives sufficient conditions for the solvability of variational inequalities given by properly quasimonotone and quasicoercive operators. In the special case when the domain of the operator is the whole space, it shows the existence of zeroes for such kind of operators.

**Theorem 3.2.1.** [78] *Let $X$ be a reflexive Banach space. Assume that $T : X \to 2^{X^*}$ is locally bounded, convex-valued, hemiclosed and that $D(T)$ is closed and convex. If $T$ is properly quasimonotone and quasicoercive, then the variational inequality* (VI) *defined on $D(T)$ has a solution. If, in addition, $D(T) = X$, then the generalized equation $0 \in T(x)$ admits a solution.*

**Remark 3.2.2.** If we suppose in Theorem 3.2.1 that $T$ is $\mathcal{A}$-pseudomonotone instead of being properly quasimonotone, then the conclusions remain valid. Indeed, every $\mathcal{A}$-pseudomonotone operator whose domain is a convex set is also properly quasimonotone (see, for instance, N. Hadjisavvas [58]).

**Remark 3.2.3.** It is well-known that the classical monotonicity, or even more, some kind of generalized monotonicities imply the local boundedness of the operator at any interior point of its domain (see, for instance, A.N. Iusem, G. Kassay and W. Sosa [66] for the class of premonotone operators). Then the question whether proper quasimonotonicity also implies local boundedness arises naturally. If so, then the assumption of local boundedness within Theorem 3.2.1 would be redundant. As the next example shows, it is not the case.

**Example 3.2.4.** [78] Let $F : \mathbb{R} \to \mathbb{R}$,

$$F(x) = \begin{cases} \frac{x^2+1}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

It is easy to check that the function above is $\mathcal{A}$-pseudomonotone and hence, as just mentioned before, properly quasimonotone as well. On the other hand it is not locally bounded at the origin. Observe that all hypotheses of Theorem 3.2.1 are satisfied except the local boundedness. Since $0 = F(x)$ has no solution, it follows that the latter is an essential condition.

## 3.3 A surjectivity result without monotonicity

In some situations, $F$-hemicontinuity might be difficult to verify. In the next definition we introduce a stronger but apparently useful property, which is satisfied by a relatively large class of operators.

**Definition 3.3.1.** [78] Let $X$ be a Banach space and $T : X \to 2^{X^*}$. We say that an operator $T$ satisfies *condition (P)* iff for any $\overline{x} \in D(T)$, any net $\{x_\alpha\}_{\alpha \in I} \subseteq D(T)$, $x_\alpha \rightharpoonup \overline{x}$ and any $x^*_\alpha \in T(x_\alpha)$ there exist $\{x^*_{\alpha_\beta}\} \subseteq \{x^*_\alpha\}$ and $\overline{x}^* \in T(\overline{x})$ such that $x^*_{\alpha_\beta} \to \overline{x}^*$.

Note that any single valued linear and compact operator satisfies condition (P), (see, for instance, W. Rudin [114], Definition 4.16, page 97 and Exercise 18, page 107). Many of the operators that arise in the study of integral equations are compact.

**Remark 3.3.2.** [78] If an operator satisfies condition (P), then it is $F$-hemicontinuous.

Indeed, let us take $\overline{x} \in D(T)$, $\{x_\alpha\}_{\alpha \in I} \subseteq D(T)$, $x_\alpha \rightharpoonup \overline{x}$, $x^*_\alpha \in T(x_\alpha)$ and arbitrary $y \in D(T)$. Then there exist $\{x^*_{\alpha_\beta}\} \subseteq \{x^*_\alpha\}$ and $\overline{x}^* \in T(\overline{x})$ such that $x^*_{\alpha_\beta} \to \overline{x}^*$. It follows that

$$\langle x^*_{\alpha_\beta}\, x_{\alpha_\beta} - y \rangle \to \langle \overline{x}^*, \overline{x} - y \rangle. \tag{3.3}$$

On the other hand,

$$\langle \overline{x}^*, \overline{x} - y \rangle = \lim_\beta \langle x^*_{\alpha_\beta}\, x_{\alpha_\beta} - y \rangle = \limsup_\beta \langle x^*_{\alpha_\beta}, x_{\alpha_\beta} - y \rangle \leq \limsup_\alpha \langle x^*_\alpha, x_\alpha - y \rangle.$$

According to Remark 3.1.6, this proves the statement.

Next, we construct an example with $T$ $F$-hemicontinuous which does not satisfies condition (P).

**Example 3.3.3.** [78] Let $X = l^2 = \{x = (x^i)_{i \in \mathbf{N}} : \sum\limits_{i=1}^{\infty} \mid x^i \mid^2 < \infty\}$ and $T$: $l^2 \to l^2$ be the identity operator. Take an arbitrary sequence $\{x_n\} \subseteq l^2$ converging weakly to $\overline{x}$. Since the function $x \longmapsto \|x\|^2$ is continuous and convex it is also weakly lower semicontinuous. Hence,

$$\|\overline{x}\|^2 \leq \liminf_{n \to \infty}\ \|x_n\|^2,$$

which clearly implies

$$\langle \overline{x}, \overline{x} - y \rangle \leq \liminf_{n \to \infty}\ \langle x_n, x_n - y \rangle,$$

for all $y \in l^2$, i.e., $T$ is $F$-hemicontinuous.

On the other hand, we take $x_n = e_n = (0, 0, ..., 0, 1, 0, ...)$ with 1 in the $n^{th}$ position. It is obvious that $e_n \rightharpoonup 0$, but $\{e_n\}$ does not have any strongly convergent subsequence, as $\| e_n - e_m \| = \sqrt{2}$, where $n \neq m$. Therefore, $T$ does not satisfy (P).

Now we can state a surjectivity result.

**Theorem 3.3.4.** [78] *Assume that $T : X \to 2^{X^*}$ is such that for each $x \in X$, $T(x)$ is convex, closed, bounded and that $D(T) = X$. If $T$ is $F$-hemicontinuous and quasicoercive, then it is surjective.*

**Corollary 3.3.5.** [78] *Let $T : X \to 2^{X^*}$. Assume that for each $x \in X$, $T(x)$ is convex, closed, bounded and that $D(T) = X$. If $T$ has the property (P) and it is quasicoercive, then $T$ is surjective.*

Next, we construct an example of $T$ which is $F$-hemicontinuous and quasicoercive and there exists an $x$ for which $T(x)$ is not bounded.

**Example 3.3.6.** [78] Let $X = \mathbb{R}$ and $T : \mathbb{R} \to 2^{\mathbb{R}^*}$ defined by

$$T(x) = \begin{cases} \{x\}, & x \neq 1, \\ [1, \infty), & x = 1. \end{cases}$$

It obvious that for any sequence $\{x_n\} \subseteq \mathbb{R}$, $x_n \to 1$, $x_n \neq 1$, any $x_n^* \in T(x_n)$ and any $y \in \mathbb{R}$, $x_y^* = 1$ satisfies

$$1 - y \leq \liminf_n (x_n - y).$$

Therefore $T$ is $F$-hemicontinuous and $T(1) = [1, \infty)$ is unbounded.

## 3.4 Results on variational inequalities and surjectivity given by $\mathcal{C}$ and $\mathcal{B}$-pseudomonotone operators

In this section we deal with $\mathcal{C}$-pseudomonotone operators, introduced in Definition 3.1.10. First we establish a result concerning variational inequalities (introduced at the beginning of this chapter) given by $\mathcal{C}$-pseudomonotone operators, which can be seen in the framework of a reflexive Banach space a generalization of a similar result by D. Inoan and J. Kolumbán [63].

**Theorem 3.4.1.** [78] *Let $T : X \to 2^{X^*}$ with $D(T) = A$ a nonempty, closed, convex set. Suppose that $T(x)$ is convex, closed and bounded subset of $X^*$ for every $x \in X$ and*

$(i)$ *$T$ is $\mathcal{C}$-pseudomonotone;*

$(ii)$ *For every finite dimensional subspace $Z$ of $X$, $T$ is upper semicontinuous on $A \cap Z$ to the weak topology in $X^*$;*

$(iii)$ *There exists $r > 0$ such that for each $x \in A \backslash A_r$, there exists $y \in A$ with $\|y\| < \|x\|$ such that $\sup_{x^* \in T(x)} \langle x^*, y - x \rangle \leq 0$.*

*Then $(VI)$ admits a solution.*

The advantage of $\mathcal{C}$-pseudomonotonicity comparing to proper quasimonotonicity is that the sum of two $\mathcal{C}$-pseudomonotone operators preserves the property, and, in such a way, allows us to obtain a surjectivity result for $\mathcal{C}$-pseudomonotone operators.

**Theorem 3.4.2.** [78] *Let $T : X \to 2^{X^*}$ with $D(T) = A$ a nonempty, closed, convex set. Suppose that $T(x)$ is convex, closed and bounded subset of $X^*$ for every $x \in X$ and*

$(i)$ *$T$ is $\mathcal{B}$ -pseudomonotone;*

$(ii)$ *For every finite dimensional subspace $Z$ of $X$, $T$ is upper semicontinuous on $A \cap Z$ to the weak topology in $X^*$;*

$(iii)$ *$T$ is quasicoercive.*

*Then $T$ is surjective.*

## 3.5 On set-valued optimization problems and vector variational-like inequalities

In 1998, Giannessi [54] first used the so called *Minty-type vector variational inequality* (in short, $MVVI$) to establish the necessary and sufficient conditions for a point to be an efficient solution of a vector optimization problem (in short, $(VOP)$) for differentiable and convex functions. Since then, several researchers have studied $(VOP)$ by using different kinds of $MVVI$ under different assumptions, see S. Al-Homidan and Q.H. Ansari [1], Q.H. Ansari and G.M. Lee [3], Q.H. Ansari, M. Rezaie and J. Zafarani [5], Q.H. Ansari and J.-K. Yao [6], G.R. Garzon, R.O. Gomez and A.R. Lizana [53], S.K. Mishra and S.Y. Wang [101] and the references therein. Consequently, vector variational inequalities have been generalized in various directions, in particular, vector variational-like inequality problems, see S. Al-Homidan and Q.H. Ansari [1], F. Giannessi, A. Maugeri and P.M. Pardalos [56], T. Jabarootian and J. Zafarani [69], S.K. Mishra and S.Y. Wang [101], M. Rezaie and J. Zafarani [110], X.M. Yang and X.Q. Yang [122] and the references therein. The vector variational-like inequalities are closely related to the concept of the invex and preinvex functions which generalize the notion of convexity of functions. The concept of invexity was first introduced by M.A. Hanson [60]. R. Pini [108] introduced prepseudoinvex and prequasiinvex functions and established some relationships between invexity and generalized convexity. More recently, the characterization and applications for generalized invexity were studied by many authors, see Q.H. Ansari and J.-K. Yao [6], A. Cambini and L. Martein [29], A. Chinchuluun and P.M. Pardalos [37], T. Jabarootian and J. Zafarani [69], G.R. Garzon, R.O. Gomez and A.R. Lizana [52], M. Soleimani-damaneh [116], M. Soleimani-damaneh [117], X.M. Yang, X.Q. Yang and K.L. Teo [123], X.M. Yang, X.Q. Yang and K.L. Teo [126] and the references therein.

The relation between the vector variational inequality and the smooth vector optimization problem has been studied by many authors (see, for example, F. Giannessi [54], X.M. Yang and X.Q. Yang [122], X.M. Yang, X.Q. Yang and K.L. Teo [125] and the references therein). Yang et al. X.M. Yang, X.Q. Yang and K.L. Teo [125] extended the result of F. Giannessi [54] for differentiable but pseudoconvex functions. X.M. Yang and X.Q. Yang [122] gave some relations between Minty variational-like inequalities and the vector optimization problems for differentiable but pseudo-invex vector-valued functions. X.M. Yang, X.Q. Yang and K.L. Teo [124], [125] and Garzon et al. [52], [53] studied the relations between generalized invexity of a differentiable function and generalized monotonicity of its gradient mapping. Very recently, M. Rezaie and J. Zafarani [110] showed some relations between the vector variational-like inequalities and vector optimization problems for nondifferential functions under generalized monotonicity. S. Al-Homidan and Q.H. Ansari [1] studied the relation among the generalized Minty vector variational-like inequality, generalized Stampacchia vector variational-like inequality and vector optimization problems for nondifferential and nonconvex functions with Clarke's generalized directional derivative and then, Q.H. Ansari and G.M. Lee [3] showed that for pseudoconvex functions with upper Dini directional derivative, similar results hold. Q.H. Ansari, M. Rezaie and J. Zafarani [5] considered generalized Minty vector variational-like inequality problems, Stampacchia vector variational-like inequality problems and nonsmooth vector optimization problems under nonsmooth pseudo-invexity assumptions. They also considered the weak formulations of generalized Minty vector variational-like inequality problems and generalized Stampacchia vector variational-like inequality problems in a very general setting and established the existence result for their solutions.

J. Zeng and S.J. Li [129] considered several kinds of generalized invexity for set-valued mappings and established some solution relationships between set-valued optimization problems and vector variational-like inequalities.

Farajzadeh et al. [49] considered generalized variational-like inequalities with set-valued mappings in topological spaces, which include as a special case the strong vector variational-like inequalities.

There are some papers discussing solution relationships between set-valued optimization problems and vector variational-like inequalities. Motivated by the works T.Q. Bao, B.S. Mordukhovich [13], Y.P. Fang, N.J. Huang [47], Y.P. Fang, N.J. Huang [48], A.P. Farajzadeh, A.A. Harandi, K.R. Kazmi [49], J. Zeng, S.J. Li [129], in this section we introduce several kinds of generalized invexity for set-valued mappings and study relationships among them. Then, we establish solution relationships between several kind of vector variational-like inequality problems and a set-valued optimization problem by means of weak contingent generalized subdifferential defined for set-valued mappings.

Throughout this section, unless otherwise specified, $X$ is a Banach space, $A$ is a nonempty subset of $X$ and $Y$ is a Banach space partially ordered by a convex pointed cone $K \subseteq Y$ with a nonempty interior int$K$. Let $K^0 := K\backslash\{0\}$ and $\eta : A \times A \to X$ be a bifunction.

Let $y_1,\, y_2 \in Y$ and $C,\, D \subseteq Y$. The orderings are defined as follows:

$$\begin{aligned} y_1 \quad \leq_K \quad y_2 &\iff y_2 - y_1 \in K, \\ C \quad \leq \quad D &\iff x \leq_K y \ \text{ for all } \ x \in C,\ y \in D. \end{aligned}$$

Let $T : A \to 2^Y$ be a set-valued mapping. We define $(T + K)(x) = T(x) + K$ for $x \in A$.

**Definition 3.5.1.** (see, for instance, [68]) A point $x_0 \in C$ is called a *minimal point* of $C$ with respect to $K$ iff

$$(C - \{x_0\}) \cap (-K^0) = \emptyset.$$

The set of all minimal points of $C$ is denoted by $\mathrm{Min}\, C$.

**Definition 3.5.2.** (see, for instance, [68]) A point $x_0 \in C$ is called a *weak minimal* point of $C$ with respect to $K$ iff

$$(C - \{x_0\}) \cap (-\mathrm{int}K) = \emptyset.$$

The set of all weak minimal points of $C$ is denoted by $\mathrm{WMin}\, C$.

For the convenience of investigation of optimality conditions of set-valued optimization problems, we present the following helpful definition of contingent cone, see J.-P. Aubin and H. Francowska [11].

**Definition 3.5.3.** Let $A$ be a nonempty subset of a normed vector space $X$ and $\overline{x} \in \mathrm{cl}\, A$. The *contingent cone* to $A$ at $\overline{x}$ is defined by

$$C_A(\overline{x}) = \left\{ \begin{array}{l} v \mid \exists\ \lambda_n \to 0,\ \exists\ v_n \to v, \\ \ \text{such that} \\ \forall n, \overline{x} + \lambda_n v_n \in A. \end{array} \right\}$$

According to J. Zeng and S.J. Li, we present the following definitions for contingent cone and for contingent derivative, respectively.

**Definition 3.5.4.** [129] Let $A$ be a nonempty subset of $X$ and $\overline{x} \in \mathrm{cl}\, A$. The *contingent cone* to $A$ at $\overline{x}$ is defined by

$$C_A(\overline{x}) = \left\{ \begin{array}{l} v \mid \exists\ (\lambda_n)_{n\in\mathbf{N}} > 0,\ \exists\ (x_n)_{n\in\mathbf{N}} \in A, \\ \ \text{such that} \\ \overline{x} = \lim_{n\to\infty} x_n \text{ and } v = \lim_{n\to\infty} \lambda_n (x_n - \overline{x}). \end{array} \right\}$$

**Definition 3.5.5.** [129] Let $(x_0, y_0) \in \mathrm{gph}\, T$. The *contingent derivative* $DT(x_0, y_0)$ of $T$ at $(x_0, y_0)$ is the set-valued mapping from $X$ to $Y$ defined by

$$\mathrm{gph}\ (DT(x_0, y_0)) := C_{\mathrm{gph}\, T}(x_0, y_0).$$

Let $\overline{D}T(x_0, y_0)(x) = D(T + K)(x_0, y_0)(x)$, for all $x \in X$, i.e.

$$y \in \overline{D}T(x_0, y_0)(x) \iff (x, y) \in C_{\mathrm{epi}\, T}(x_0, y_0).$$

According to Definition 4 of J. Zeng and S.J. Li [129], we present the following definition for set-valued mappings.

**Definition 3.5.6.** [129] Let $T : A \to 2^Y$ be a set-valued mapping and $(x_0, y_0) \in \text{gph } T$. The set $\partial T(x_0, y_0)$ defined by

$$\partial T(x_0, y_0) = \{H \in L(X, Y) \mid H(x) \le \overline{D}T(x_0, y_0)(x), \ \forall \ x \in \text{dom } \ \overline{D}T(x_0, y_0)\}$$

is called *weak contingent generalized subdifferential* of $T$ at $(x_0, y_0)$.

**Definition 3.5.7.** [96] Let $x_0 \in A$. A bifunction $\eta : A \times A \to X$ is said to be *skew* at $x_0$ iff for any $x \in A$, $x \neq x_0$,

$$\eta(x, x_0) + \eta(x_0, x) = 0.$$

**Remark 3.5.8.** Next we recall the following definitions of generalized invexities for set-valued mappings, given in J. Zeng and S.J. Li [129].

Let $T : A \to 2^Y$ a set valued mapping.

**Definition 3.5.9.** [129] $T$ is said to be $(\text{int}K, \text{int}K)$-*pseudoinvex* at $(x_0, y_0)$ with respect to $\eta$ iff for any $x \in A, \quad y \in T(x), \quad \xi \in \partial T(x_0, y_0)$,

$$y - y_0 \in -\text{intK} \ \Rightarrow \langle \xi, \eta(x, x_0) \rangle \in -\text{intK}.$$

$T$ is said to be $(\text{int}K, \text{int}K)$-*pseudoinvex* with respect to $\eta$ on $A$ iff for any pair $(x, y)$, $x \in A$, $y \in T(x)$, $T$ is $(\text{int}K, \text{int}K)$-*pseudoinvex* at $(x, y)$ with respect to $\eta$ on $A$.

**Definition 3.5.10.** [129] $T$ is said to be $(K^0, \text{int}K)$-*pseudoinvex* at $(x_0, y_0)$ with respect to $\eta$ iff for any $x \in A, \quad y \in T(x), \quad \xi \in \partial T(x_0, y_0)$,

$$y - y_0 \in -K^0 \Rightarrow \langle \xi, \eta(x, x_0) \rangle \in -\text{intK}.$$

$T$ is said to be $(K^0, \text{int}K)$-*pseudoinvex* with respect to $\eta$ on $A$ iff for any pair $(x, y)$, $x \in A$, $y \in T(x)$, $T$ is $(K^0, \text{int}K)$-*pseudoinvex* at $(x, y)$ with respect to $\eta$ on $A$.

**Definition 3.5.11.** [129] $T$ is said to be $(K^0, K^0)$-*pseudoinvex* at $(x_0, y_0)$ with respect to $\eta$ iff for any $x \in A, \quad y \in T(x), \quad \xi \in \partial T(x_0, y_0)$,

$$y - y_0 \in -K^0 \Rightarrow \langle \xi, \eta(x, x_0) \rangle \in -K^0.$$

$T$ is said to be $(K^0, K^0)$-*pseudoinvex* with respect to $\eta$ on $A$ iff for any pair $(x, y)$, $x \in A$, $y \in T(x)$, $T$ is $(K^0, K^0)-$ *pseudoinvex* at $(x, y)$ with respect to $\eta$ on $A$.

**Definition 3.5.12.** [129] $T$ is said to be $(K, \text{intK})$-*pseudoinvex* at $(x_0, y_0)$ with respect to $\eta$ iff for any $x \in A$, $x \neq x_0, \quad y \in T(x), \quad \xi \in \partial T(x_0, y_0)$,

$$y - y_0 \in -K \Rightarrow \langle \xi, \eta(x, x_0) \rangle \in -\text{intK}.$$

$T$ is said to be $(K, \text{intK})$-*pseudoinvex* with respect to $\eta$ on $A$ iff for any pair $(x, y)$, $x \in A$, $y \in T(x)$, $T$ is $(K, \text{intK})$-*pseudoinvex* at $(x, y)$ with respect to $\eta$ on $A$.

We introduce now the following definition of generalized invexities for set-valued mappings in a unified approach form.

**Definition 3.5.13.** [96] Let $K_1, K_2 \subseteq Y$ be two pointed convex cones with nonempty interiors. $T$ is said to be $(K_1, K_2)$-*weakly* $-$ *quasi* $-$ *invex* at $(x_0, y_0) \in \operatorname{gph} F$ with respect to $\eta$ iff for any $(x, y) \in \operatorname{gph} F$ one has

$$\langle \xi, \eta(x, x_0)\rangle \in K_1 \quad \text{for all} \quad \xi \in \partial T(x_0, y_0) \Longrightarrow y - y_0 \in K_2.$$

$T$ is said to be $(K_1, K_2)$-*weakly* $-$ *quasi* $-$ *invex* with respect to $\eta$ on $A$ iff for any $(x, y) \in \operatorname{gph} F$, $T$ is $(K_1, K_2)$-*weakly* $-$ *quasi* $-$ *invex* at $(x, y)$ with respect to $\eta$ on $A$.

In what follows we use Definition 3.5.13 in the case when $K_i = K^0$ or intK, $i = 1, 2$. Next result show some relationships among this generalized invexities which was defined before.

**Proposition 3.5.14.** [96] *Let $K_1, K_2, L_1, L_2 \subseteq Y$ be four pointed convex cones with nonempty interiors. The following results hold:*

1. *If $T$ is $(K_1, L_1)$-weakly $-$ quasi $-$ invex with respect to $\eta$ on $A$ then $T$ is $(K_2, L_1)$-weakly $-$ quasi $-$ invex with respect to $\eta$ on $A$, whenever $K_2 \subseteq K_1$.*
2. *If $T$ is $(K_1, L_1)$-weakly $-$ quasi $-$ invex with respect to $\eta$ on $A$ then $T$ is $(K_1, L_2)$-weakly $-$ quasi $-$ invex with respect to $\eta$ on $A$, whenever $L_1 \subseteq L_2$.*

Now we give an example to illustrate that we can find $T$ such that is $(K^0, K^0)-$ weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ and which is not $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ on $A$.

**Example 3.5.15.** [96] Let $X = \mathbf{R}$, $Y = \mathbf{R}^2$, $A = [-1, 1] \subseteq X$, $K = \mathbf{R}^2_+ = \{y = (y_1, y_2) \in \mathbf{R}^2 | \; y_1 \geq 0, \; y_2 \geq 0\}$. Let $T : A \to 2^Y$ and $\eta : A \times A \to A$ be defined as

$$T(x) = \{y = (y_1, y_2) \in \mathbf{R}^2 \mid \; y_1 = -x, \; y_2 \in [0, 1]\}$$

and

$$\eta(x, y) = \begin{cases} x, & y = 0, \\ 0, & y \neq 0. \end{cases}$$

Let us take $x_0 = 0$ and $y_0 \in T(x_0)$, $y_0 = (0, 0)$. Therefore,

$$C_{\operatorname{gph}(T+K)}(x_0, y_0) = \{(x, y_1, y_2) \in \mathbf{R}^3 \mid \; x \in \mathbf{R}, \; y_1 \geq -x, \; y_2 \geq 0\}$$

and

$$\partial T(x_0, y_0) = \{H\},$$

where

$$H(x) = \begin{pmatrix} -x \\ 0 \end{pmatrix}, \forall x \in \mathbb{R}.$$

It is not difficult to see that $T$ is $(K^0, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ but not $(K^0, \operatorname{int} K)$-weakly-quasi-invex with respect to $\eta$ because for $x = -1$, $y = (1, 0) \in T(-1)$ we obtain

$$\langle \partial T(x_0, y_0), \eta(x, x_0)\rangle \subseteq K^0, \quad y - y_0 \notin \operatorname{int} K.$$

The next example shows that we can find $T$ which is $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ which is not $(K^0, \operatorname{int} K)$-pseudoinvex at $(x_0, y_0)$ with respect to $\eta$ on $A$, see [129].

**Example 3.5.16.** [96] Let $X = \mathbf{R}$, $Y = \mathbf{R}^2$, $A = [-1, 1] \subseteq X$, $K = \mathbf{R}^2_+ = \{y = (y_1, y_2) \in \mathbf{R}^2 | \ y_1 \geq 0, \ y_2 \geq 0\}$. Let $T : A \to 2^Y$ and $\eta : A \times A \to A$ be defined as

$$T(x) = \begin{cases} \{y = (y_1, y_2) \in \mathbf{R}^2 \mid \ y_1 = -x, \ y_2 \in [0, 1]\}, & x \in [0, 1], \\ \{y = (y_1, y_2) \in \mathbf{R}^2 \mid \ y_1 = -x, \ y_2 \in (0, 1]\}, & x \in [-1, 0) \end{cases}$$

and

$$\eta(x, y) = \begin{cases} x, & y = 0, \ x \neq 1, \\ 0, & y \neq 0 \text{ or } x = 1. \end{cases}$$

Let us take $x_0 = 0$ and $y_0 \in T(x_0)$, $y_0 = (0, 0)$. Therefore,

$$C_{\mathrm{gph}(T+K)}(x_0, y_0) =$$

$$= \{(x, y_1, y_2) \in \mathbf{R}^3 \mid \ x \in \mathbf{R}, \ y_1 \geq -x, \ y_2 \geq 0\} \backslash \{(x, -x, 0) \in \mathbf{R}^3 \mid \ x < 0\}$$

and

$$\partial T(x_0, y_0) = \{H\},$$

where

$$H(x) = \begin{pmatrix} -x \\ 0 \end{pmatrix}, \forall x \in \mathbb{R}.$$

$T$ is $(K^0, \mathrm{int}K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ and is not $(K^0, \mathrm{int}K)$-pseudoinvex at $(x_0, y_0)$ with respect to $\eta$ because, for $x = 1$, $y = (-1, 0) \in T(1)$, we have

$$y - y_0 \in -K^0 \text{ and } \langle \xi, \eta(x, 0) \rangle = (0, 0) \notin -\mathrm{int}K \text{ for all } \xi \in \partial T(x_0, y_0).$$

**Remark 3.5.17.** For more detalies about generalized invexities of set valued mappings, see [129].

A.P. Farajzadeh, A.A. Harandi and K.R. Kazmi [49] introduced the following definitions for set-valued maps. Let $x \in A$. $K(x) \subseteq X$ denotes a solid, pointed, closed and convex cone.

**Definition 3.5.18.** [49] A set valued mapping $T : A \to 2^{L(X,Y)}$ is said to be *$K$-$\eta$-strong pseudomonotone* iff for any given $x$, $y \in A$,

$$\langle T(x), \eta(x, y) \rangle \nsubseteq -K(x) \backslash \{0\} \Longrightarrow \langle T(y), \eta(y, x) \rangle \subseteq -K(y).$$

**Definition 3.5.19.** [49] A set valued mapping $T : A \to 2^{L(X,Y)}$ is said to be *$K$-$\eta$-pseudomonotone* iff for any given $x$, $y \in A$,

$$\langle T(x), \eta(x, y) \rangle \nsubseteq -\mathrm{int}K(x) \Longrightarrow \langle T(y), \eta(y, x) \rangle \subseteq -\mathrm{int}K(y).$$

Let us consider $C$, $K \subseteq Y$ two convex, solid, pointed cones.

**Definition 3.5.20.** [96] A set valued mapping $T : A \to 2^{L(X,Y)}$ is said to be *$(C, K)$-$\eta$-strong pseudomonotone* iff for any $x$, $y \in A$,

$$\langle T(x), \eta(x, y) \rangle \nsubseteq C \Longrightarrow \langle T(y), \eta(y, x) \rangle \subseteq K.$$

**Definition 3.5.21.** [96] A set valued mapping $T : A \to 2^{L(X,Y)}$ is said to be *strictly $(C, K)$-$\eta$-strong pseudomonotone* iff for any $x$, $y \in A$, $x \neq y$,

$$\langle T(x), \eta(x, y) \rangle \nsubseteq C \Longrightarrow \langle T(y), \eta(y, x) \rangle \subseteq K.$$

**Theorem 3.5.22.** [96] *If $T : A \to 2^{X^*}$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$- strong pseudomonotone, then $T$ is strictly $(-K^0, -K^0)$-$\eta$-strong pseudomonotone.*

The next example shows that the converse of the Theorem 3.5.22 does not hold: we can find $T$ which is strictly $(-K^0, -K^0)$-$\eta$-strong pseudomonotone but not strictly $(-\text{int}K, -\text{int}K)$-$\eta$-strong pseudomonotone.

**Example 3.5.23.** [96] Let $X = \mathbf{R}$, $Y = \mathbf{R}^2$, $A = [-1, 1] \subseteq X$, $K = \mathbf{R}^2_+ = \{y = (y_1, y_2) \in \mathbf{R}^2 |\ y_1 \geq 0,\ y_2 \geq 0\}$. Let $T : A \to 2^Y$ and $\eta : A \times A \to A$ be defined as

$$T(x) = \{y = (y_1, y_2) \in \mathbf{R}^2 \mid\ y_1 = -x,\ y_2 \in [0, 1], x \in [-1, 1]\}$$

and

$$\eta(x, y) = \begin{cases} x, & x < 0,\ y = 0, \\ -x, & x > 0,\ y = 0, \\ y, & x = 0,\ y > 0, \\ -y, & x = 0,\ y < 0, \\ +1, & \text{otherwise.} \end{cases}$$

Let us take $x_0 = 0$ and $y_0 \in T(x_0)$, $y_0 = (0, 0)$. Therefore,

$$T_{\text{gph}(T+K)}(x_0, y_0) = \{(x, y_1, y_2) \in \mathbf{R}^3 \mid\ x \in \mathbf{R},\ y_1 \geq -x,\ y_2 \geq 0\}$$

and

$$\partial T(x_0, y_0) = \{H\},$$

where

$$H(x) = \begin{pmatrix} -x \\ 0 \end{pmatrix}, \forall x \in \mathbb{R}.$$

It is not difficult to verify that $\partial T$ is strictly $(-K^0, -K^0)$-$\eta$-strong pseudo-
monotone and is not strictly $(-\text{int}K, -\text{int}K)$-$\eta$-strong pseudomonotone because for $x \in A$, $x < 0$, $y \in T(x)$ we have

$$\langle \partial T(x, y), \eta(x, x_0) \rangle \nsubseteq -\text{int}K \text{ and } \langle \partial T(x_0, y_0), \eta(x_0, x) \rangle \nsubseteq -\text{int}K.$$

**Remark 3.5.24.** In Example 3.5.23 $\eta$ is skew at $x_0 = 0$.

### 3.5.1 Relationships between $(SOP)$ and $W(SSVI)$

In this subsection, we discuss solution relationships between a Stampacchia-type vector variational-like inequality problem and a set-valued optimization problem.

A *Stampacchia-type vector variational-like inequality problem* $(SSVI)$ is to find $(x_{0,}y_0)$ with $x_0 \in A$ and $y_0 \in T(x_0)$, such that for all $\xi \in \partial T(x_0, y_0)$

$$\langle \xi, \eta(x, x_0) \rangle \notin -K^0.$$

A *weak vector variational-like inequality problem* $(WSSVI)$ is to find $(x_0,y_0)$ with $x_0 \in A$ and $y_0 \in T(x_0)$, such that for all $\xi \in \partial T(x_0, y_0)$

$$\langle \xi, \eta(x, x_0) \rangle \notin -\text{int}K.$$

A set-valued optimization problem $(SOP)$ is

$$\min_{x \in A} T(x).$$

**Definition 3.5.25.** (see, for instance, [129]) Let $T(A) := \cup_{x \in A} T(x)$ denote the image set of $T$. A pair $(\overline{x}, \overline{y})$ with $\overline{x} \in A$ and $\overline{y} \in T(\overline{x})$ is called a *minimizer* of the set-valued optimization problem $(SOP)$, iff $\overline{y} \in \operatorname{Min} T(A)$. A pair $(\overline{x}, \overline{y})$ with $\overline{x} \in A$ and $\overline{y} \in T(\overline{x})$ is called a *weak minimizer* of the set-valued optimization problem $(SOP)$, iff $\overline{y} \in \operatorname{WMin} T(A)$.

**Remark 3.5.26.** J. Zeng and S.J. Li obtained some relationships between $((W)SSVI)$ and $(SOP)$ under generalized pseudoinvexity.

The next result gives sufficient conditions for the existence of weak minimizers of $(SOP)$.

**Theorem 3.5.27.** [129] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves $(WSSVI)$ and $T$ is $(K^0, K^0)$-pseudoinvex at $(x_0, y_0)$ with respect to $\eta$, then $(x_0, y_0)$ is a weak minimizer of the $(SOP)$.*

Due to the relationship between pseudoinvexities, the next two corollaries become evident.

**Corollary 3.5.28.** [129] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves $(WSSVI)$ and $T$ is $(K^0, intK)$-pseudoinvex at $(x_0, y_0)$ with respect to $\eta$, then $(x_0, y_0)$ is a weak minimizer of the $(SOP)$.*

**Corollary 3.5.29.** [129] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves $(WSSVI)$ and $T$ is $(K, intK)$-pseudoinvex at $(x_0, y_0)$ with respect to $\eta$, then $(x_0, y_0)$ is a weak minimizer of the $(SOP)$.*

The statement below provides sufficient conditions for the existence of minimizers of $(SOP)$.

**Theorem 3.5.30.** [129] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves $(SSVI)$ and $T$ is $(K^0, K^0$-pseudoinvex at $(x_0, y_0)$ with respect to $\eta$, then $(x_0, y_0)$ is a minimizer of the $(SOP)$.*

In what follows, we deal with a kind of converse of the two previous theorem: namely, we obtain sufficient conditions such that a solution of $(SOP)$ to be a solution of $(SSVI)$ under assumption of weak-quasi-invexity.

**Theorem 3.5.31.** [95] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(K^0, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\partial T$ is strictly $(-K^0, -K^0)$-$\eta$-strong pseudomonotone, $\langle \partial T(x_0, y_0), \eta(x_0, x_0) \rangle \subseteq K^0$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(SSVI)$.*

*Proof.* Suppose the pair $(x_0, y_0)$ solves the $(SOP)$ and is not a solution of $(SSVI)$. Then there exist $\overline{x} \in A$, $\xi^0 \in \partial T(x_0, y_0)$ such that

$$\langle \xi^0, \eta(\overline{x}, x_0) \rangle \in -K^0.$$

Therefore,

$$\langle \xi^0, \eta(\overline{x}, x_0) \rangle \notin K^0.$$

Since $\eta$ is skew at $x_0$ and $\langle \partial T(x_0, y_0), \eta(x_0, x_0) \rangle \subseteq K^0$ we obtain

$$\langle \xi^0, \eta(x_0, \overline{x}) \rangle \notin -K^0, \ \overline{x} \neq x_0. \tag{3.4}$$

By (3.4), since $\partial T$ is strictly $(-K^0, -K^0)$-$\eta$-strong pseudomonotone it follows that

$$\langle \partial T(\overline{x}, \overline{y}), \eta(\overline{x}, x_0)\rangle \subseteq -K^0.$$

Since $\eta$ is skew at $x_0$ we obtain

$$\langle \partial T(\overline{x}, \overline{y}), \eta(x_0, \overline{x})\rangle \subseteq K^0. \tag{3.5}$$

Now using that $T$ is $(K^0, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, by (3.5) it follows that

$$y_0 - \overline{y} \in K^0.$$

Hence,

$$\overline{y} - y_0 \in -K^0,$$

which contradicts the fact that $(x_0, y_0)$ is a minimizer of $(SOP)$. Therefore, $(x_0, y_0)$ is a solution of $(SSVI)$.

**Theorem 3.5.32.** [95] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(\operatorname{int} K, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone, $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(SSVI)$.*

*Proof.* The proof is similar to that of Theorem 3.5.31.

The following corollaries are consequences of Theorem 3.5.31 and Theorem 3.5.32, based on Proposition 3.5.14.

**Corollary 3.5.33.** [95] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(\operatorname{int} K, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone, $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(SSVI)$.*

**Corollary 3.5.34.** [95] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\partial T$ is $(-K^0, -K^0)$-$\eta$-strong pseudomonotone, $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(SSVI)$.*

**Corollary 3.5.35.** [95] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a weak minimizer of $(SOP)$, $T$ is $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\partial T$ is $(-K^0, -K^0)$-$\eta$-strong pseudomonotone, $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(WSSVI)$.*

### 3.5.2 Relationships between $(SOP)$ and $W(SMVI)$

In this subsection, we discuss solution relationships between a Minty-type vector variational-like inequality problem and a set-valued optimization problem.

A *Minty-type vector variational-like inequality problem* $(SMVI)$ is to find $(x_0,y_0)$ with $x_0 \in A$ and $y_0 \in T(x_0)$, such that for all $x \in A$, $y \in T(x)$ there exists $\xi \in \partial T(x, y)$ such that

$$\langle \xi, \eta(x, x_0)\rangle \notin -K^0.$$

A *weak Minty-type vector variational-like inequality problem* $(WSMVI)$ is to find $(x_0,y_0)$ with $x_0 \in A$ and $y_0 \in T(x_0)$, such that for all $x \in A$, $y \in T(x)$ there exists $\xi \in \partial T(x, y)$ such that

$$\langle \xi, \eta(x, x_0)\rangle \notin -\text{int} K.$$

The next result gives necessay conditions for minimizers of $(SOP)$.

**Theorem 3.5.36.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(K^0, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(SMVI)$.*

The next corollary is stated under stronger assumptions that those of Theorem 3.5.36, see Proposition 3.5.14.

**Corollary 3.5.37.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(SMVI)$.*

The result below provides an alternative necessary condition for weak minimizers of $(SOP)$.

**Theorem 3.5.38.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(\operatorname{int} K, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(WSMVI)$.*

The next results are consequences of Theorem 3.5.38, based on Proposition 3.5.14.

**Corollary 3.5.39.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(\operatorname{int} K, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(WSMVI)$.*

**Corollary 3.5.40.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(K^0, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(WSMVI)$.*

**Corollary 3.5.41.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ is a minimizer of $(SOP)$, $T$ is $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$ and $\eta$ is skew at $x_0$, then $(x_0, y_0)$ is a solution of $(WSMVI)$.*

The following result gives sufficient conditions that a solution of $(SMVI)$ to be a solution of $(SOP)$.

**Theorem 3.5.42.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(SMVI)$, $T$ is $(\operatorname{int} K, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq \operatorname{int} K$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

The next results are consequences of Theorem 3.5.42, based on Proposition 3.5.14.

**Corollary 3.5.43.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(SMVI)$, $T$ is $(\operatorname{int} K, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq \operatorname{int} K$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

**Corollary 3.5.44.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(SMVI)$, $T$ is $(K^0, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

**Corollary 3.5.45.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(SMVI)$, $T$ is $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

The following theorem gives sufficient conditions, under $(K^0, K^0)$-weakly-quasi-invexity, such that a solution of $(SMVI)$ to be a solution of $(SOP)$.

**Theorem 3.5.46.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(SMVI)$, $T$ is $(K^0, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is $(-K^0, -K^0)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

**Corollary 3.5.47.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(SMVI)$, $T$ is $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-K^0, -K^0)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

The following theorem give sufficient conditions, under $(\operatorname{int} K, K^0)$-weakly-quasi-invexity, such that a solution of $(WSMVI)$ to be a solution of $(SOP)$.

**Theorem 3.5.48.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(WSMVI)$, $T$ is $(\operatorname{int} K, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq \operatorname{int} K$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

The following corollaries are consequences of Theorem 3.5.48, based on Proposition 3.5.14.

**Corollary 3.5.49.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(WSMVI)$, $T$ is $(\operatorname{int} K, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq \operatorname{int} K$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

**Corollary 3.5.50.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(WSMVI)$, $T$ is $(K^0, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

**Corollary 3.5.51.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(WSMVI)$, $T$ is $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-\operatorname{int} K, -\operatorname{int} K)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

**Theorem 3.5.52.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(WSMVI)$, $T$ is $(K^0, K^0)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-K^0, -K^0)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

**Corollary 3.5.53.** [96] *Let $T : A \to 2^Y$ be a set-valued mapping, $x_0 \in A$ and $y_0 \in T(x_0)$. If the pair $(x_0, y_0)$ solves the $(WSMVI)$, $T$ is $(K^0, \operatorname{int} K)$-weakly-quasi-invex at $(x_0, y_0)$ with respect to $\eta$, $\eta$ is skew at $x_0$, $\partial T$ is strictly $(-K^0, -K^0)$-$\eta$-strong pseudomonotone and $\langle \partial T(x_0, y_0), \eta(x_0, x_0)\rangle \subseteq K^0$, then $(x_0, y_0)$ is a minimizer of $(SOP)$.*

## 3.6 Vector variational-like inequalities

Throughout this section, let $X$ be a Banach space endowed with a norm $\|.\|$ and $X^*$ its dual space with a norm $\|.\|_*$. Denote $[x,y]$, $]x,y[$, the line segment for $x,y \in X$ and $[x,y]\backslash\{x,y\}$, respectively. Recall that $\langle\cdot,\cdot\rangle$ is the duality pairing between $X$ and $X^*$. Let $A$ be a nonempty open subset of $X$.

Let $F = (F_1, ..., F_n) : A \to \mathbb{R}^n$ be a vector-valued function, where
$F_i : A \to \mathbb{R}$ $(i = 1, ..., n)$ is locally Lipschitz functions on $A$.

We deal with the following vector optimization problem:
$(VOP)$ Minimize $F(x) = (F_1(x), ..., F_n(x))$ subject to $x \in A$.

**Definition 3.6.1.** (see, for instance, [68]) A point $x_0 \in A$ is said to be an *efficient solution (or Pareto)* of $(VOP)$ iff for all $x \in A$,

$$F(x) - F(x_0) \notin -\mathbb{R}^n_+\backslash\{0\},$$

where $\mathbb{R}^n_+$ is the nonnegative orthant of and 0 is the origin of $\mathbb{R}^n$.

**Definition 3.6.2.** (see, for instance, [68]) A point $x_0 \in A$ is said to be a *weak efficient solution* of $(VOP)$ iff for all $x \in A$,

$$F(x) - F(x_0) \notin -\operatorname{int}\mathbb{R}^n_+.$$

Also, in this section we deal with limiting subdifferential in Asplund spaces and the following definitions are presented for this purpose.

**Definition 3.6.3.** [102] Let $x \in A$ and $\varepsilon \geq 0$. The set of $\varepsilon-$*normals* to $A$ at $x$ is defined by

$$\widehat{N}_\varepsilon(x, A) = \{x^* \in X^* \mid \limsup_{u \xrightarrow{A} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \leq \varepsilon\}.$$

Let $\overline{x} \in A$, the *limiting normal cone* to $A$ at $\overline{x}$ is

$$N(\overline{x}, A) = \limsup_{\substack{x \to \bar{x} \\ \varepsilon \searrow 0}} \widehat{N}_\varepsilon(x, A).$$

**Definition 3.6.4.** [102] Let $F : A \to \overline{\mathbb{R}}$ be finite at $\overline{x} \in A$. The *limiting subdifferential* of $F$ at $\overline{x}$ is defined as follows

$$\partial_L F(\overline{x}) = \{x^* \in X^* \mid (x^*, -1) \in N((\overline{x}, F(\overline{x})), \text{epi } F)\}.$$

**Definition 3.6.5.** [102] A Banach space $X$ is Asplund, or it has the Asplund property, iff every convex continuous function $F : A \to X$ defined on an open convex subset $A$ of $X$ is Frèchet differentiable on a dense subset of $A$.

**Remark 3.6.6.** One of the most popular Asplund spaces is any reflexive Banach space.

For more details and applications, see B. Mordukhovich [102], M. Miholca [99].

In this section we also consider the Minty vector variational-like inequality and (weak) Stampacchia vector variational-like inequality, defined by means of Mordukhovich limiting subdifferentials in Asplund spaces.

Let $F = (F_1, ..., F_n) : A \to \mathbb{R}^n$ be a vector-valued function, where
$F_i : A \to \mathbb{R}$ $(i = 1, ..., n)$ is locally Lipschitz functions on $A$.

We consider the following *Minty vector variational-like inequality problem* and *(weak) Stampacchia vector variational-like inequality problem.*

$(MVVI)$ Find $x_0 \in A$ such that, for all $x \in A$ there exists $\xi_i^* \in \partial_L F_i(x)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi_1^*, \eta(x, x_0)\rangle, ..., \langle \xi_n^*, \eta(x, x_0)\rangle) \notin -\mathbb{R}_+^n \backslash \{0\}.$$

$((W)SVVI)$ Find $x_0 \in A$ such that, for all $x \in A$ there exists $\xi_i^* \in \partial_L F_i(x_0)$ $(i = 1, ..., n)$,

$$\langle \xi^*, \eta(x, x_0)\rangle = (\langle \xi_1^*, \eta(x, x_0)\rangle, ..., \langle \xi_n^*, \eta(x, x_0)\rangle) \notin -\mathbb{R}_+^n \backslash \{0\} \;\; (\notin -\mathrm{int}\mathbb{R}_+^n).$$

We obtain some relationships between a solution of $(VOP)$ and these vector variational-inequalities using the concept of generalized invexity for vector valued functions. The main results in S. Al-Homidan and Q.H. Ansari [1], Q.H. Ansari, M. Rezaie and J. Zafarani [5], M. Rezaie and J. Zafarani [110] were obtained in the setting of Clarke subdifferential. Since the class of Clarke subdifferential is larger than the class of Mordukhovich subdifferential (see B. Mordukhovich [102]), some authors studied the vector variational-like inequalities and vector optimization problems in the setting of Mordukhovich subdifferential in order to obtain stronger results. M. Oveisiha and J. Zafarani [105] established some relations between vector variational-like inequalities and vector optimization problems, dealing with Mordukhovich subdifferential. B. Chen and N.J. Huang [34] considered (weak) Minty vector variational-like inequality and (weak) Stampacchia vector variational-like inequality defined by means of Mordukhovich limiting subdifferentials in Asplund spaces and established some relations between the vector variational-like inequalities and vector optimization problems.

By means of several examples we show that our results are stronger than the similar statements in the literature.

**Definition 3.6.7.** (see, for instance, [116]) Let $\eta : A \times A \to X$. A subset $A$ of $X$ is said to be *invex* with respect to $\eta$ iff, for any $x, y \in A$ and $\lambda \in [0, 1]$ we have $y + \lambda\eta(x, y) \in A$.

Hereafter, unless otherwise specified, we assume that $X$ is an Asplund space and $A \subseteq X$ is a nonempty open invex set with respect to the mapping $\eta : A \times A \to X$.

**Definition 3.6.8.** (see, for instance, [105], [116]) Let $F : A \to \mathbb{R}$ be a function. $F$ is said to be

(i) *quasi-preinvex* with respect to $\eta$ on $A$ iff for any $x, y \in A$, $\lambda \in [0, 1]$,

$$F(x) \leq F(y) \Rightarrow F(y + \lambda\eta(x, y)) \leq F(y);$$

(ii) *strictly-quasi-preinvex* with respect to $\eta$ on $A$ iff for any $x, y \in A$, $\lambda \in (0, 1)$,

$$F(x) < F(y) \Rightarrow F(y + \lambda\eta(x, y)) < F(y);$$

(iii) *prequasi-invex* with respect to $\eta$ on $A$ iff for any $x, y \in A$, $\lambda \in [0, 1]$,

$$F(y + \lambda\eta(x, y)) \leq \max\{F(x), F(y)\}.$$

It is obvious that $(iii) \Rightarrow (i)$ but the converse is not true as the next example shows.

**Example 3.6.9.** [97] Let $X = \mathbf{R}$, $A = ]1;3[$, $F : A \to \mathbf{R}$, $F(x) = e^x$,
$\eta : A \times A \to \mathbf{R}$,

$$\eta(x,y) = \begin{cases} 2x - y, \ x > y, \ x, y \in (1; \frac{3}{2}), \\ x - y, \ otherwise. \end{cases}$$

The function $F$ is quasi-preinvex with respect to $\eta$ on $A$ and is not prequasi-invex with respect to $\eta$ on $A$ because if we take $x = \frac{5}{4}$, $y = \frac{11}{10}$, $\lambda = 1$, we have

$$F(x) > F(y) \text{ and } F(y + \lambda\eta(x,y)) > F(x).$$

It easy to verify that $A$ is invex with respect to $\eta$ on $A$.

Next, we introduce the concept of generalized invexity for vector-valued functions.

**Definition 3.6.10.** [97] Let $F = (F_1, ..., F_n) : A \to \mathbb{R}^n$ be a vector-valued function and $x_0 \in A$. $F$ is said to be

(i) *quasi-preinvex* with respect to $\eta$ on $A$ iff for any $x, y \in A, \lambda \in [0,1]$,

$$F(x) - F(y) \in -\mathbb{R}^n_+ \implies F(y + \lambda\eta(x,y)) - F(y) \in -\mathbb{R}^n_+;$$

(ii) *strictly-quasi-preinvex* with respect to $\eta$ on $A$ iff for any $x, y \in A, \lambda \in (0,1)$,

$$F(x) - F(y) \in -\mathbb{R}^n_+ \backslash \{0\} \implies F(y + \lambda\eta(x,y)) - F(y) \in -\mathbb{R}^n_+ \backslash \{0\};$$

(iii) *strictly-pseudo-invex* with respect to $\eta$ on $A$ iff for any $x, y \in A$,

$$F(x) - F(y) \in -\mathbb{R}^n_+ \implies \langle \partial_L F(y), \eta(x,y) \rangle \subseteq -\mathbb{R}^n_+ \backslash \{0\};$$

(iv) *weakly-quasi-invex* with respect to $\eta$ on $A$ iff for any $x, y \in A$,

$$\langle \partial_L F(y), \eta(x,y) \rangle \subseteq \mathbb{R}^n_+ \backslash \{0\} \implies F(x) - F(y) \in \mathbb{R}^n_+ \backslash \{0\}.$$

**Remark 3.6.11.** (see, for instance, [110]) The definition of generalized limiting subdifferential of $F$ at $x \in A$ can be extended to vector-valued functions in the following way:

$$\partial_L F(x) = \partial_L F_1(x) \times \partial_L F_2(x) \times ... \times \partial_L F_n(x).$$

Recall the following generalized monotonicity concepts for set-valued maps with respect to $\eta$. Let $K$ be a solid pointed convex cone of $\mathbb{R}^n$ such that $K \subseteq \mathbb{R}^n \setminus \{0\}$.

**Definition 3.6.12.** *(*see, for instance, [110], [116]) A set valued mapping
$F : A \to 2^{X^*}$ is said to be

(i) *K-pseudomonotone* with respect to $\eta$ on $A$ iff for any $x, y \in A$,

$$\langle F(x), \eta(y,x) \rangle \subseteq K \Longrightarrow \langle F(y), \eta(x,y) \rangle \subseteq -K;$$

(ii) *K-strictly pseudomonotone* with respect to $\eta$ on $A$ iff for any $x, y \in A$,
$x \neq y$,

$$\langle F(x), \eta(y,x) \rangle \subseteq K \Longrightarrow \langle F(y), \eta(x,y) \rangle \subseteq -\operatorname{int} K.$$

In some results of this section we need to consider further assumptions on $\eta$, like *condition C*:

$$\eta(x + \lambda\eta(x, y), y) = \lambda\eta(x, y), \quad \forall x, y \in A, \forall\lambda \in [0, 1].$$

These assumptions are known in the literature of invexity, see T. Jabarootian and J. Zafarani [69].

**Definition 3.6.13.** [97] We say that $\eta : A \times A \to X$ satisfies *condition C at $x_0 \in A$* iff for any $x \in A$, $\lambda \in [0, 1], x \neq x_0$,

$$\eta(x_0 + \lambda\eta(x, x_0), x_0) = \lambda\eta(x, x_0).$$

Note that mean-value theorems are important and useful tools in nonsmooth analysis and we mention here one of these results for limiting subdifferential.

**Theorem 3.6.14.** [102] *Let $F$ be Lipschitz continuous on an open set containing $[x, y]$ in $X$. Then*

$$F(y) - F(x) \leq \langle\xi^*, y - x\rangle,$$

*for some $c \in [x, y)$, and one $\xi^* \in \partial_L F(c)$.*

The next statement refers to a property of prequasi-invex, locally Lipschitz functions.

**Theorem 3.6.15.** [105] *Let $F : A \to \mathbb{R}$ be a locally Lipschitz function on $A$. Assume that $F$ is prequasi-invex with respect to $\eta$ on $A$, $\eta$ is continuous with respect to the second argument and satisfies condition C. If $\langle\xi^*, \eta(y, x)\rangle > 0$ for a $\xi^* \in \partial_L F(x)$, then $F(y) > F(x)$.*

Using a similar argument as in the proof of Theorem 3.6.15, the property above can easily be extended to quasi-preinvex functions.

**Theorem 3.6.16.** [97] *Let $F : A \to \mathbb{R}$ be a locally Lipschitz function on $A$. Assume that $F$ is quasi-preinvex with respect to $\eta$ on $A$, $\eta$ is continuous with respect to the second argument satisfying condition C. If $\langle\xi^*, \eta(y, x)\rangle > 0$ for a $\xi^* \in \partial_L F(x)$, then $F(y) > F(x)$.*

The next result gives sufficient conditions that a solution of $(MVVI)$ to be solution of $(VOP)$.

**Theorem 3.6.17.** [97] *Assume that $F : A \to \mathbb{R}^n$ is strictly-quasi-preinvex at $x_0$ with respect to $\eta$ on $A$, $\partial_L F_i$ $(i = 1, ..., n)$ is strictly pseudomonotone with respect to $\eta$ on $A$. If $x_0 \in A$ is a solution of $(MVVI)$ and $\eta$ satisfies condition C at $x_0$, then $x_0$ is an efficient solution of $(VOP)$.*

**Remark 3.6.18.** In [105] (Theorem 13), the authors proved that if $x_0$ is a solution of $(MVVI)$, then $x_0$ is an efficient solution of $(VOP)$ by assuming that $F_i$ $(i = 1, \ldots, n)$ is pseudo-invex with respect to $\eta$ on $A$ and $\eta$ is continuous with respect to the second argument satisfying condition C. The following example shows an instance when our Theorem 3.6.17 applies, but Theorem 13 [105] not. The function $F : A \to \mathbb{R}^2$ is strictly-quasi-preinvex at a certain $x_0$ with respect to some $\eta$ on $A$, $\partial_L F_i$ $(i = 1, ..., n)$ is strictly pseudomonotone with respect to $\eta$ on $A$, $\eta$ satisfies condition C at $x_0$ but there exists $k, 1 \leq k \leq n$, such that $F_k$ is not pseudo-invex with respect to $\eta$ on $A$.

**Example 3.6.19.** [97] Let us consider $X = \mathbb{R}$, $A = [0, 1]$, $F : A \to \mathbb{R}^2$, $F = (F_1, F_2)$ defined as

$$F(x) = (F_1(x), F_2(x)) = (x, \frac{1}{2}x),$$

$x_0 = 0$ and $\eta : A \times A \rightarrow \mathbf{R}$ defined as

$$\eta(x,y) = \begin{cases} x-y, & \begin{array}{l}(x>0, y=0) \text{ or}\\ (x=0, y>0),\end{array} \\ 0, & x=y, \\ 1-y, & y>x>0, \\ -y, & 0<y<x. \end{cases}$$

We obtain that

$$\partial_L F(x) = \{(1, \frac{1}{2})\}, \quad x \in A.$$

It is easy to check that $F$ is strictly quasi-preinvex at $x_0 = 0$ with respect to $\eta$ on $A$ and $F_2$ is not pseudo-invex with respect to $\eta$ on $A$ because for $y > x > 0$ we obtain

$$\exists\ \xi^* = \frac{1}{2} \in \partial_L F_2(y),\ \langle \xi^*, \eta(x,y)\rangle > 0 \ \text{ and } F_2(x) < F_2(y).$$

On the other hand for any $x, y \in A$, $x \neq y$,

$$\langle \partial_L F_i(x), \eta(y,x)\rangle \subseteq \mathbb{R}_+ \implies \langle \partial_L F_i(y), \eta(x,y)\rangle \ \subseteq -\text{int}\mathbb{R}\ (i=1,2).$$

Therefore, $\partial_L F\ _1$, $\partial_L F\ _2$ are strictly pseudomonotone with respect to $\eta$ on $A$ and $\eta$ satisfies condition C at $x_0$. It easy to see that $x_0 = 0$ is a solution of $(MVVI)$ and an efficient solution of $(VOP)$.

**Remark 3.6.20.** The function $\eta : A \times A \rightarrow \mathbf{R}$ defined in Exemple 3.6.19 does not fulfill condition C. Set, for example, $x = 2$, $y = 1$, and $\lambda = \frac{1}{2}$.

**Theorem 3.6.21.** [97] *Assume that $F : A \to \mathbb{R}^n$ is weakly quasi-invex with respect to $\eta$ on $A$ and $\partial_L F$ is $-\mathbb{R}^n_+ \backslash \{0\}-$pseudomonotone with respect to $\eta$ on $A$. If $x_0 \in A$ is an efficient solution of $(VOP)$, then $x_0$ is a solution of $(SVVI)$.*

**Corollary 3.6.22.** [97] *Assume that $F : A \to \mathbb{R}^n$ is weakly quasi-invex with respect to $\eta$ on $A$ and $\partial_L F$ is $-\mathbb{R}^n_+ \backslash \{0\}-$pseudomonotone with respect to $\eta$ on $A$. If $x_0 \in A$ is an efficient solution of $(VOP)$, then $x_0$ is a solution of $(WSVVI)$.*

**Remark 3.6.23.** In [34] (Theorem 4.2) the authors proved that $x_0$ weak solution of $(VOP)$ implies $x_0$ solution of $(WSVVI)$ by assuming that $F_i$ $(i = 1, \ldots, n)$ is invex with respect to $\eta$ on $A$. The following example shows that we can obtain a result with $F : A \to \mathbb{R}^n$ weakly quasi-invex with respect to $\eta$ on $A$, $\partial_L F$ $-\mathbb{R}^n_+ \backslash \{0\}-$pseudomonotone and there exists $k, 1 \leq k \leq n$, such that $F_k$ is not invex with respect to $\eta$ on $A$.

**Example 3.6.24.** [97] Let us consider $X = \mathbb{R}$, $A = [0, 1]$, $F = (F_1, F_2) : A \to \mathbf{R^2}$ defined as

$$F(x) = (F_1(x), F_2(x)) = (\sqrt{x}, \frac{1}{2}\sqrt{x}),$$

$x_0 = 0$ and $\eta : A \times A \rightarrow \mathbf{R}$ defined as

$$\eta(x,y) = \begin{cases} 0, & (0<x<y) \text{ or } (x=y), \\ x, & x>0,\ y=0, \\ x-y, & x>y>0, \\ 0, & x=0,\ y>0. \end{cases}$$

We obtain that

$$\partial_L F(x) = \begin{cases} (\frac{1}{2\sqrt{x}}, \frac{1}{4\sqrt{x}}), & x > 0, \\ [0,\infty[\times[0,\infty[, & x = 0. \end{cases}$$

The function $F$ is weakly quasi-invex with respect to $\eta$ on $A$ because

$$\langle \partial_L F(y), \eta(x,y) \rangle \subseteq \mathbb{R}^2_+ \backslash \{0\} \implies F(x) - F(y) \in \mathbb{R}^2_+ \backslash \{0\}.$$

The function $F_2$ is not invex with respect to $\eta$ on $A$ because for $x > 0$, $y = 0$ we obtain

$$F_2(x) - F_2(0) < \langle \xi^*, \eta(x,0) \rangle,$$

for $\xi^* = 2$. It is not difficult to verify that $\partial_L F$ is $-\mathbb{R}^2_+ \backslash \{0\}-$pseudomonotone with respect to $\eta$ on $A$, $x_0$ is an efficient solution of $(VOP)$ and a solution of $((W)SVVI)$.

**Theorem 3.6.25.** [97] *Suppose*

*(i)* $\partial_L F_i$ $(i = 1, ..., n)$ *is strictly pseudomonotone with respect to* $\eta$ *on* $A$*;*

*(ii)* $\eta$ *satisfies condition* $C$ *at* $x_0$*;*

*(iii) for each* $x \neq u$ *and for each* $i = 1, ..., n$,

$$[F_i(x) \geq F_i(u) \Rightarrow \tag{3.6}$$

$$\exists\ c_i = x + \lambda_i \eta(u,x) \in A, \lambda_i \in (0,1), \exists\ \xi_i \in \partial_L F_i(c_i)$$

*such that*

$$\langle \xi_i, \eta(x, c_i) \rangle \geq 0].$$

*If* $x_0 \in A$ *is a solution of* $(WSVVI)$*, then* $x_0$ *is an efficient solution of* $(VOP)$*.*

**Corollary 3.6.26.** [97] *Suppose*

1. $\partial_L F_i$ $(i = 1, ..., n)$ *is strictly pseudomonotone with respect to* $\eta$ *on* $A$*;*

2. $\eta$ *satisfies condition* $C$ *at* $x_0$*;*

3. *for each* $x \neq u$ *and for each* $i = 1, ..., n$,

$$[F_i(x) \geq F_i(u) \Rightarrow$$

$$\exists\ c_i = x + \lambda_i \eta(u,x) \in A, \lambda_i \in (0,1), \exists\ \xi_i \in \partial_L F_i(c_i)$$

*such that*

$$\langle \xi_i, \eta(x, c_i) \rangle \geq 0].$$

*If* $x_0 \in A$ *is a solution of* $(SVVI)$*, then* $x_0$ *is an efficient solution of* $(VOP)$*.*

Ansari, Rezaie and Zafarani (2011) [5] considered the perturbed form of the generalized weak Stampacchia vector variational-like inequality problem.

Motivated by [5], let us consider the following generalized Stampacchia vector variational-like inequality problem.

$(PSVVI)$ Find $x_0 \in A$ for which there exists $t_0 \in (0,1)$ such that for all $x \in A$ and all $t \in (0, t_0]$, there exists $\xi_i^t \in \partial_L F_i(x_0 + t\eta(x, x_0))$ $(i = 1, ..., n)$ satisfying

$$\langle \xi^t, \eta(x, x_0) \rangle = (\langle \xi_1^t, \eta(x, x_0) \rangle, ..., \langle \xi_n^t, \eta(x, x_0) \rangle) \notin -\mathbb{R}_+^n \backslash \{0\}.$$

The following result provides the relationship between a solution of $(PSVVI)$ and $(WSVVI)$.

**Theorem 3.6.27.** [97] *If $x_0 \in A$ is a solution of $(PSVVI)$, then it is a solution of $(WSVVI)$.*

**Remark 3.6.28.** The reverse of Theorem 3.6.27 is not true. The next example shows that a solution of $(WSVVI)$ might not be a solution of $(PSVVI)$.

**Example 3.6.29.** [97] Let us consider $X = \mathbb{R}$, $A = [-1, 1]$, $F : A \to \mathbb{R}$ defined as

$$F(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ x, & x < 0, \end{cases}$$

and $\eta : A \times A \to \mathbb{R}$ defined as

$$\eta(x, y) = x - y.$$

We have

$$\partial_L F(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & x > 0, \\ [0, \infty[, & x = 0, \\ 1, & x < 0. \end{cases}$$

It is not difficult to observe that $x_0 = 0$ is a solution of $(WSVVI)$. On the other hand, for each $t_0 \in (0,1)$ there exist $x = -1 \in A$ and $t = t_0 \in (0, t_0]$ such that for all $\xi^t \in \partial_L F(-t)$,

$$\langle \xi^t, \eta(x, 0) \rangle \in -\mathbf{R}_+ \setminus \{0\}.$$

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