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Ph.D. Thesis Summary

Brauer-Clifford groups and characters of G-graded algebras

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Introduction

A special place in abstract algebra is taken by Representation Theory. Let us consider a group (taken from nature, for example). We do this by considering sets of symmetries of an object, which are closed under composition and under inverses. What Representation Theory tries to find out is on what objects does this group act on? Informally talking, if we take G to be the group we are talking about, for what objects X, does there exist a map

$$\alpha : G \times X \to X$$

compatible with the group law? The work is much easier when we consider X to be a vector space and the action of G to be linear.

History says that this whole branch started in 1896 when the German mathematician Richard J.W. Dedekind wrote a letter to his friend Ferdinand G. Frobenius asking for his help. He observed that taking the multiplication table of a finite group G and turning it into a matrix X_G by replacing every entry g of this table with a variable x_g the determinant of X_G factors into a product of irreducible polynomials in $\{x_g\}$ with multiplicity equal to its degree. Dedekind asked Frobenius to prove this for the general case, not just for some special cases, as Dedekind did.

A number of mathematicians such as Ferdinand G. Frobenius, William Burnside, Issai Schur (who was actually a student of Frobenius) and Richard Brauer in their work about 100 years ago, were interested in Representation Theory. It wasn't until 1937 when the American mathematician Alfred H.Clifford introduced Clifford Theory in [5] describing the relation between representations of a group and the representations of a normal subgroup. Generally speaking the aim of this thesis is to study Clifford Theory and field extensions.

Our interest in this field of study was triggered by some articles by Everett C. Dade ([6, 7, 8, 9, 10, 11, 12, 13, 14, 15]) and Alexandre Turull ([61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72]).

Alexandre Turull's research on Clifford theory, and specifically the one in combination with Schur indices, offered a motivation for the research behind this thesis. A. Turull introduced Clifford classes in [61] in order to describe the Schur indices from families of groups closely connected to finite simple groups and in order to study some general properties of representations of finite groups. With these notions he proved a strengthening of the McKay conjecture for solvable groups in [64]. He also noticed that the Clifford classes do not form a group. When fixing their centers, things become more interesting. Let G be a finite group. Unlike the previous attempt made by A. Turull (in 1994) the set of equivalence classes of these central simple G-algebras with given fixed center form a group called the Brauer-Clifford group. He introduced this notion in [67]. The definition he gives is similar to the definition of the Brauer group. The last group has been studied greatly. Some basic properties are found in [49].

Strongly graded algebras are natural for Clifford Theory. That is why the entire thesis is build around this notion. We also use Morita equivalences, more specifically, graded Morita equivalences, because in the recent years it has proven to be of great relevance for Group Representation Theory.

Why do we wish to work with G-graded algebras? There are a number of reasons. Probably the most noticeable one is that theorems such as Theorem 4.3.13 and a number of Theorems found in Turull's articles ([61, Theorem 3.5][70, Theorem 4.7], [71, Theorem 4.9], [70, Theorem 7.5], [71, Theorem 7.5]) look like consequences of G-graded Morita equivalences or Rickard equivalences.

We now give a presentation of the content of this thesis. The present work uses as starting points important theorems such as those of: Jordan-Hölder, Krull-Schmidt, Noether and Schur (Theorem 1.1.4), Schur-Zassenhaus, Glauberman-Isaacs (Theorem 6.2.3), Watanabe and many others.

Let R be a finite dimensional strongly G-graded F-algebra, where F is a field. The present research started by trying to find an answer to a number of questions, such as:

(a) Can the Brauer-Clifford group be characterized using Morita equivalences? How about the Clifford classes?

(b) Let K/F be an algebraic field extension. Then the R-module induced from the R_1-module generates a category that is equivalent to the category of modules over its endomorphism algebra. What happens to the G-graded derived equivalences over F? Do they preserve Clifford Theory defined by corresponding simple modules? How about Galois actions and Schur indices?

(c) Can Turull's results from [61, 67] about the Brauer-Clifford group be generalized for the case of strongly group graded algebras? These results could then characterize the Clifford Theory.

(d) Do we have good compatibility properties for endoisomorphisms between endomorphisms of G-algebras of modules over strongly G-graded algebras?

(e) Can we use graded Morita equivalences to deduce Turull's observations from [66] and

L.Puig's from [53]. More precisely, can the Morita equivalence from [75] be extended to a graded Morita equivalence?

The results of the study of these questions are presented throughout this thesis and the overall view is as follows. In Chapter 2 we investigate the G-graded algebras, the G-acted algebras and the Morita equivalences over a commutative G-ring Z. We also study the particular case of skew-group algebras and the relation between G-graded Morita equivalences and G-equivariant Morita equivalences. The authors results are presented in: Lemma 2.2.3, Lemma 2.2.4, Lemma 2.2.5, Theorem 2.2.6, Lemma 2.3.2 and Corollary 2.3.3.

The next chapter, Chapter 3 we study the Clifford Theory in connection with the action of the Galois group of a field extension in the context of group graded algebras. Original results of the author and A. Mărcuş are given by: Theorem 3.4.2, Corollary 3.4.4 and Theorem 3.4.5.

In Chapter 4 we analyze the Clifford Theory for a finite group using the Brauer-Clifford group. Original results are given in Theorem 4.3.13 and Proposition 4.5.1 continuing the results of Chapter 2 and Chapter 3.

Chapter 5 is devoted to research triggered by Turull's articles [70], [71]. Using p-modular systems we prove some equivalences of categories and give some good compatibility properties. Results of the author are given in Theorem 5.5.1 and Theorem 5.6.1.

The last chapter, presents joined work of the author with A. Mărcuş. In order to answer the last question in our plan, we started from the Glauberman correspondence and we present some special graded equivalences. The main results are given by: Theorem 6.5.9 and Theorem 6.6.5.

For a more easier reading, the new concepts will be introduced gradually, as needed, except for the general notions presented in Chapter 1.

Key words

Clifford theory, field extensions, center simple algebras, strongly graded algebras, the Brauer-Clifford group, group graded algebras, G-algebras, Morita equivalences, characters, Galois action, endoisomorphisms, Glauberman correspondence.

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Chapter 1

Preliminaries

1.1 Field extensions and the Galois action

We present a theorem derived from the work of Noether (1933) and Schur(1909) which will be used later. In order to do this we introduce first some well known concepts. For general terminology we refer the reader to the book by Grigory Karpilovsky [36]. By a ring we understand a associative ring with unity. All modules will be assumed to be left modules, unless otherwise specified.

1.1.1. Semisimple algebras.

Let F be a field, A a finite dimensional F-algebra. We suppose that A is defined over a perfect subfield of F.

1.1.2. Let K/F be an algebraic normal field extension, and consider the Galois group $\hat{G} := \mathrm{Gal}(K/F)$. Then $\hat{G}$ acts on the set of isomorphism classes of simple $K \otimes_F A$-modules, and if W is a simple $K \otimes_F A$-module, denote

$$\hat{G}_W = \{\sigma \in \hat{G} \mid {}^{\sigma}W \simeq W \text{ as } K \otimes_F A\text{-modules}\}$$

the stabilizer of W.

Definition 1.1.3. Let A be an algebra over a field F. We say that F is a *splitting field* for A if for each simple A-module V we have $\mathrm{End}_A(V) = F$.

In this setting we have results similar to Clifford theory, due to Schur and Noether (see [36, Theorem 8.1.11]).

Theorem 1.1.4 (**Noether, Schur**)**.** *With the above notations, the following statements hold.*

1) If V is a simple A-module, then $K \otimes_F V$ is a semisimple $K \otimes_F A$-module.

2) Let W be a simple $K \otimes_F A$-module that is a direct summand of $K \otimes_F V$, where V is a simple A-module. Then

$$K \otimes_F V \simeq m \bigoplus_{\sigma \in [\hat{G}/\hat{G}_W]} {}^{\sigma}W, \quad \textit{for some positive integer } m.$$

3) For any simple $K \otimes_F A$-module W, there exists a simple A-module V, unique up to isomorphism, such that W is a summand of $K \otimes_F V$.

1.2 Graded Morita equivalences

Let G be a finite group and $\mathcal{O}$ be a commutative ring. We say that there is a *G-graded Morita equivalence* between two strongly G-graded $\mathcal{O}$-algebras R and S if there are G-graded (R, S)-bimodules M and a G-graded (S, R)-bimodule N inducing a Morita equivalence between R and S such that the bimodule isomorphism

$$\alpha : M \otimes_S N \to R \quad \text{and} \quad \beta : N \otimes_R M \to S$$

are grade preserving (that is, $\alpha(M_x \otimes_S N_y) \subseteq R_{xy}$ and $\beta(N_x \otimes_R M_y) \subseteq S_{xy}$, for all $x, y \in G$).

We present the following theorem because we consider it is important and offers a motivation for the research that follows, but the theorem is not original, as it is due to A. Mărcuş [44, Theorem 5.1.2].

Theorem 1.2.1. *Let M_1 be an (R_1, S_1)-bimodule, N_1 an (S_1, R_1)-bimodule, and denote*

$$M = R \otimes_{R_1} M_1 \quad \textit{and} \quad N = N_1 \otimes_{S_1} S.$$

The following statements are equivalent: (i) There is a structure of a G-graded (R, S)-bimodule on M and a structure of a G-graded (S, R)-bimodule on N (extending the given structure), such that M and S induce a graded Morita equivalence between R and S. (ii) M_1 and N_1 induce a Morita equivalence between R_1 and S_1, and M_1 extends to a Δ-module, where

$$\Delta = \bigoplus_{x \in G} (R_g \otimes_{\mathcal{O}} S_g^{\mathrm{op}}).$$

Chapter 2

Group graded bimodules over a commutative G-ring

The aim of this chapter is to study G-graded algebras, G-acted algebras and Morita equivalences over a commutative G-ring Z. We present a setting appropriate for the next chapter and show how to associate a central simple G-graded algebra over Z to a character of a strongly G-graded algebra over a field of characteristic zero. The results are due to D. Gliţia and can be found in [23].

2.1 Motivation

Let G be a finite group, Z a commutative G-ring, and let $F = Z^G$. Let A and B be two G-acted F-algebras such that $Z \to Z(A)$ and $Z \to Z(A)$ are G-ring homomorphisms. Then the tensor product over Z of A and B is again a G-acted F-algebra over Z. Motivated by the study of Clifford theory in combination with Schur indices, Turull has introduced an equivalence relation between simple algebras of this kind, which comes down to the notion of equivariant Morita equivalence over Z between them.

However, strongly graded algebras are natural for Clifford theory. So let R and S be two strongly G-graded algebras such that $Z \to Z(R_1)$ and $Z \to Z(S_1)$ are G-ring homomorphisms.

Turull's equivalence classes over F (see [61]) can be generalized to the case of strongly G-graded algebras (see Mărcuş [46], [47]). The problem is that the tensor product over Z of R and S is no longer an algebra. More precisely, we have:

Remark 2.1.1. 1) R and S are F-algebras and R_1 and S_1 are actually Z-algebras.

2) Many arguments in [43] where based on the fact that $R \otimes_F S^{\mathrm{op}}$ is a $G \times G$-graded F-algebra, and then the subalgebra

$$\Delta(R \otimes_F S^{\mathrm{op}}) := \bigoplus_{g \in G} (R_g \otimes_F S_g^{\mathrm{op}})$$

is a strongly G-graded F-algebra.

3) Here we may also construct $R \otimes_Z S$, but this is not a ring in general, nor is $\Delta(R \otimes_Z S)$.

2.2 G-graded F-algebras over Z and Morita equivalences

Nevertheless, we show that we can still consider G-graded Morita equivalences over Z (not only over F) between R and S. For this, we introduce the following notion.

Definition 2.2.1. We say that M is a *G-graded (R,S)-bimodule over Z* if M is a (R,S)-bimodule, M has a decomposition $M = \bigoplus_{x\in G} M_x$ such that $R_g M_x S_h = M_{gxh}$, and $m_g z =^g z m_g$ for all $g \in G$, $z \in Z$, and $m_g \in M_g$.

Remark 2.2.2. If M is a G-graded (R,S)-bimodule then $R_g M_1 R_{g^{-1}} = M_1$. But obviously, we can not say M is a $R \otimes_Z S^{\mathrm{op}}$-module because $R \otimes_Z S^{\mathrm{op}}$ is not a ring.

Lemma 2.2.3. $\Delta(R \otimes_Z S^{\mathrm{op}})$ *is a ring, moreover an G-algebra over Z and $R \otimes_Z S^{\mathrm{op}}$ is a right $\Delta(R \otimes_Z S^{\mathrm{op}})$-module.*

Let N be a Δ-module, where $\Delta = \Delta(R \otimes_Z S^{\mathrm{op}})$, then N it is also a Δ_1-module.

Lemma 2.2.4. *If N is a Δ-module, there exists an isomorphism of G-graded (R,S)-bimodules over Z:*

$$R \otimes_{R_1} N \simeq N \otimes_{S_1} S \simeq (R \otimes_Z S^{\mathrm{op}}) \otimes_\Delta N =: \widetilde{N}.$$

Lemma 2.2.5. 1) *Assume that N is a left $\Delta(R \otimes_Z S^{\mathrm{op}})$-module and N' is a left $\Delta(S \otimes_Z T^{\mathrm{op}})$-module. Then $N \otimes_{S_1} N'$ is a $\Delta(R \otimes_Z T^{\mathrm{op}})$-module with multiplication*

$$(r_g \otimes_Z t_g^{\mathrm{op}})(n \otimes_{S_1} n') = \sum_{i=1}^{l} r_g n s_i' \otimes_{S_1} s_i n t_{g^{-1}}.$$

Moreover we have the isomorphism of G-graded (R,T)-bimodules over Z:

$$\widetilde{N \otimes_{S_1} N'} \simeq \widetilde{N} \otimes_S \widetilde{N'}.$$

2) *Assume that N is a $\Delta(S \otimes_Z R^{\mathrm{op}})$-module and N a $\Delta(S \otimes_Z T^{\mathrm{op}})$-module. Then* $\mathrm{Hom}_{S_1}(N, N')$ *is a $\Delta(R \otimes_Z T^{\mathrm{op}})$-module with multiplication:*

$$(r_{g^{-1}} f t_g)(n) = \sum_{i=1}^{l} s_i' f(s_i n r_{g^{-1}}) t_g \quad \text{for } n \in N \text{ and } f \in \mathrm{Hom}_{S_1}(N, N').$$

Moreover we have the isomorphism of G-graded (R,T)-bimodules over Z:

$$\mathrm{Hom}_S(\widetilde{N},\widetilde{N'}) \simeq \widetilde{\mathrm{Hom}_{S_1}(N,N')}.$$

Theorem 2.2.6. *Let M_1 be a (R_1,S_1)-bimodule and N_1 a (S_1,R_1)-bimodule such that the bimodules M_1 and N_1 induce a Morita equivalence between R_1 and S_1. Moreover, if M_1 is a Δ_1-module then N_1 extends to a Δ-module and $\widetilde{M_1}$, $\widetilde{N_1}$ induce G-graded Morita equivalences between R and S over Z.*

2.3 Skew group algebras

In this section we discuss the particular case of skew group algebras and the relation between G-graded Morita equivalences and G-equivariant Morita equivalences.

Let G be a finite group, Z a commutative G-algebra, and let $F = Z^G$. Let A and B be G-algebras over Z. In this case, the skew group algebras $A*G$ and $B*G$ are G-graded F-algebras over Z. We denote $R = A*G$ and $S = B*G$. Note that the tensor products $A\otimes_Z B$ and $A\otimes_Z B^{\mathrm{op}}$ are both G-algebras with diagonal action.

Definition 2.3.1. A Morita equivalence over Z between A and B induced by two bimodules M and N is said to be *G-equivariant* if M is a $(A\otimes_Z B^{\mathrm{op}})*G$ module, N is a $(B\otimes_Z A^{\mathrm{op}})*G$ module and all homomorphisms involved in the Morita equivalence are ZG-linear.

By [32, Proposition 9] two central separable G-algebras are equivalent (in the sense of Turull [67]) if and only if they are equivariantly Morita equivalent and the group of equivalence classes of central separable G-algebras over a commutative G-algebra is isomorphic to $\mathrm{BrClif}(Z,G)$. We wish to relate this to strongly graded central simple algebras.

Lemma 2.3.2. *Let A and B to be G-algebras over Z and let $R = A*G$ and $S = B*G$. There is the following isomorphism of G-graded F-algebras over Z:*

$$\Delta(R\otimes_Z S^{\mathrm{op}}) \simeq (A\otimes_Z B^{\mathrm{op}})*G.$$

Corollary 2.3.3. Let M_1 be a (A,B)-bimodule. If M_1 induces a G-equivariant Morita equivalence between A and B then $R\otimes_{R_1} M_1$ induces a G-graded Morita equivalence between R and S. Conversely, let M be a G-graded (R,S)-bimodule. If M induces a G-graded Morita equivalence over Z between R and S, then M_1 induces a G-equvariant Morita equivalence over Z between A and B.

Chapter 3

Field extensions and Clifford theory

We go on to study Clifford theory in connection with the action of the Galois group of a field extension in the context of group graded algebras (see [25]).

3.1 Motivation

Let G be a finite group, let K/F be an algebraic field extension, and let $R = \bigoplus_{g\in G} R_g$ be a finite dimensional strongly G-graded F-algebra. A simple R_1-module, as well as a simple $K \otimes_F R_1$-module, define a "Clifford theory". The main idea is that the R-module induced from a simple R_1-module generates an abelian subcategory of the category of R-module which is equivalent to the category of modules over its endomorphism algebra. We investigate the relationships between these theories. One of the main results below says that a G-graded derived equivalence over F preserves the Clifford theory defined by corresponding simple modules, and also preserves Galois actions and Schur indices.

An important motivation for this chapter is Turull's approach to Clifford theory and Schur indices via G-algebras. He considers the Clifford theory defined by an R-module lying over a simple (or semisimple) R_1-module and introduces the notion of endoisomorphism to formalize the idea of two modules determining the same Clifford theory. We show in Section 3.4 that a G-graded derived equivalence over F induces an endoisomorphism between two corresponding simple R-modules. This is related to the results of [47].

In what follows, groups are finite and algebras and modules are finite dimensional. We consider only algebras over fields, but this is enough for our purposes, as we essentially deal with simple modules. Our notations are standard.

3.2 Clifford theory for strongly G-graded algebras

We present Dade's treatment of Clifford theory for strongly G-graded F-algebras. The results presented in this section are Dade's version [10], [11] of the Clifford correspondence for group graded algebras.

3.2.1. As in the introduction, let G be a finite group, F a field, and let $R = \bigoplus_{g\in G} R_g$ be a finite dimensional strongly G-graded F-algebra.

The group G acts on the set of isomorphism classes of simple R_1-modules. If V be a simple R_1-module, we denote ${}^gV = R_g \otimes_{R_1} V$, and let

$$G_V := \{g \in G \mid R_g \otimes_{R_1} V \simeq V \text{ as } R_1\text{-modules}\}$$

be the stabilizer in G of V.

Theorem 3.2.2. *If M is a simple R-module, then there exists a simple R_1-module V such that V is a direct summand in M. More precisely, ${}_{R_1}M$ is a semisimple R_1-module and has the structure*

$${}_{R_1}M \simeq n \bigoplus_{g\in[G/G_V]} {}^gV, \quad \textit{for some positive integer } n.$$

3.2.3. Let M and V be as above. Because we have a monomorphism $V \stackrel{\iota}{\hookrightarrow} {}_{R_1}M$, there exists the surjective R-homomorphism from $R \otimes_{R_1} V$ to M that takes $r \otimes v$ to $r\iota(v)$.

Definition 3.2.4. We denote by $(R|V)$-mod the full subcategory of R-mod consisting of R-modules M for which there exists an R-epimorphism

$$(R \otimes_{R_1} V)^{(I)} \to M \to 0$$

for some set I. Then $(R|V)$-mod is called the category of *R-modules above V*.

Theorem 3.2.5. *The category* $(R|V)$-mod *is abelian, and coincides with the full subcategory of* R-mod *consisting of R-modules M that viewed as R_1-modules have the structure as in Theorem* (3.2.2).

3.2.6. If we denote $E := \mathrm{End}_R(R \otimes_{R_1} V)^{\mathrm{op}}$ then E is a G-graded algebra, and $R \otimes_{R_1} V$ is a G-graded (R, E)-bimodule. Moreover, we have that $E_g = 0$ for $g \in G\backslash G_V$ (because in this case $V \not\simeq {}^gV$), hence $E = E_{G_V}$ may be regarded as a strongly G_V-graded algebra.

Theorem 3.2.7. *We have the commutative diagram of equivalences of categories*

$$\begin{tikzcd}
(R|V)\text{-mod} \arrow[r, shift left, "\mathrm{Hom}_R(R\otimes_{R_1}V,-)"] \arrow[d, shift left, "(-)_{G_V}"] & E\text{-mod} \arrow[l, shift left, "(R\otimes_{R_1}V)\otimes_E -"] \arrow[d, equal] \\
(R_{G_V}|V)\text{-mod} \arrow[r, shift left, "\mathrm{Hom}_{G_V}(R_{G_V}\otimes_{R_1}V,-)"] \arrow[u, shift left, "R\otimes_{R_{G_V}} -"] & E_{G_V}\text{-mod}. \arrow[l, shift left, "(R_{G_V}\otimes_{R_1}V)\otimes_{E_{G_V}} -"]
\end{tikzcd}$$

3.3 Galois action and Clifford correspondence

We study the relationship between the Clifford theories over F and over K. Consequently we discuss the scalar extension from F to K and the action of the Galois group $\mathrm{Gal}(K/F)$ on $K \otimes_F R$-modules.The general setting is the same as the one for Theorem 1.1.4 with some additional notions introduced bellow. Hence let K/F be an algebraic normal field extension. We denote by $\hat{G}$ the Galois group $\mathrm{Gal}(K/F)$. Then $\hat{G}$ acts on the set of isomorphism classes of simple $K \otimes_F A$-modules. Let W be a simple $K \otimes_F A$-module. Denote by $\hat{G}_W = \{\sigma \in \hat{G} \mid {}^{\sigma}W \simeq W$ as $K \otimes_F A$-modules$\}$ the stabilizer of W.

Let $\hat{W} := \bigoplus_{\sigma\in[\hat{G}/\hat{G}_W]} {}^{\sigma}W$ be the sum of distinct $\hat{G}$-conjugates of W and $R = \bigoplus_{g\in G} R_g$ a finite dimensional strongly G-graded F-algebra, and let $F \leq K$ be an algebraic normal field extension. Denote $KR := K \otimes_F R$, which is a strongly G-graded K-algebra. We suppose that R_1 (and hence R) is defined over a perfect subfield of F.

3.3.1. Let W be a simple KR_1-module and V a simple R_1-module. Denote also

$$\hat{E} := \mathrm{End}_{KR}(KR \otimes_{KR_1} \hat{W})^{\mathrm{op}}.$$

One can see that $\hat{E} = \hat{E}_{G_{\hat{W}}}$, so $\hat{E}$ is strongly $G_{\hat{W}}$-graded.

Notation 3.3.2. We consider the following stabilizers, also called *inertia groups*:

- $I_G(V) := G_V = \{g \in G \mid R_g \otimes_{R_1} V \simeq V$ as R_1-modules$\}$,
- $I_G(W) := G_W = \{g \in G \mid KR_g \otimes_{KR_1} W \simeq W$ as KR_1-modules$\}$,
- $I_{G,F}(W) := \{g \in G \mid$ there exists $\sigma \in \hat{G}$ such that

  $KR_g \otimes_{KR_1} W \simeq {}^{\sigma}W$ as KR_1-modules$\}$,
- $I_G(K \otimes_F V) := \{g \in G \mid KR_g \otimes_{KR_1} K \otimes_F V \simeq K \otimes_F V$ as KR_1-modules$\}$.

Also, denote $T := I_{G,F}(W)$. We obviously have that $I_G(W) \leq I_{G,F}(W) = T \leq G$.

Notation 3.3.3. Apart from the subcategory $(R|V)$-mod introduced in Section 3.2, we consider the following full subcategories:

- $(KR|W)$-mod, consisting of KR-modules M such that there exists an epimorphism of KR-modules

$$(KR \otimes_{KR_1} W)^{(I)} \to M \to 0, \quad \text{for some set } I.$$

- $(KR|W, F)$-mod consisting of KR-modules M such that ${}_{KR_1}M$ is isomorphic to a direct sum of $G \times \hat{G}$-conjugates of W.

- $(KR|K \otimes_F V)$-mod consisting of KR-modules M such that there exists an epimorphism of KR-modules

$$(KR \otimes_{KR_1} K \otimes_F V)^{(I)} \to M \to 0, \quad \text{for some set } I.$$

Theorem 3.3.4. *With the above notation, assume that W is a direct summand of $K \otimes_F V$. Then $I_{G,F}(W) = I_G(K \otimes_F V)$ and the categories $(KR|W, F)$-mod and $(KR|K \otimes_F V)$-mod coincide. Moreover, we have the following commutative diagram of equivalences of categories:*

$$\begin{tikzcd}
(KR|W,F)\text{-mod} \arrow[r, shift left, "\mathrm{Hom}_{KR}(KR\otimes_{KR_1}\hat{W},-)"] \arrow[d, shift left, "\simeq"] & \hat{E}\text{-mod} \arrow[l, shift left, "(KR\otimes_{KR_1}\hat{W})\otimes_{\hat{E}}-"] \arrow[d, equal] \\
(KR_{G_{\hat{W}}}|W,F)\text{-mod} \arrow[u, shift left, "KR\otimes_{KR_{G_{\hat{W}}}}-"] \arrow[r, shift left, "\mathrm{Hom}_{KR_{G_{\hat{W}}}}(KR_{G_{\hat{W}}}\otimes_{KR_1}\hat{W},-)"] & \hat{E}_{G_{\hat{W}}}\text{-mod.} \arrow[l, shift left, "(KR_{G_{\hat{W}}}\otimes_{KR_1}\hat{W})\otimes_{\hat{E}}-"]
\end{tikzcd}$$

3.3.5. We next discuss the relationship between the inertia groups $I_G(V)$ and $T = I_G(K \otimes_F V)$, and between the subcategories $(R|V)$-mod and $(KR|K \otimes_F V)$-mod. We denote $KE := \mathrm{End}_{KR}(KR \otimes_{KR_1} (K \otimes_F V))^{\mathrm{op}}$. As before, $KE = KE_T$ may be regarded as a strongly T-graded K-algebra.

Lemma 3.3.6. *In this setting we have $I_G(V) \le I_G(K \otimes_F V) = T$ and the extension of scalars $K \otimes_F - : R\text{-mod} \to KR\text{-mod}$ induces by restriction a functor*

$$K \otimes_F - : (R|V)\text{-mod} \to (KR|K \otimes_F V)\text{-mod}.$$

Corollary 3.3.7. We have the following commutative diagram of categories and functors:

$$\begin{tikzcd}
(R|V)\text{-mod} \arrow[r, shift left] \arrow[dr, dashed] & E\text{-mod} \arrow[l, shift left] \arrow[dr] \arrow[dl, shift left] & \\
(R_T|V)\text{-mod} \arrow[u, "R\otimes_{R_T}-"] \arrow[ur, shift left] \arrow[dr] & (KR|K\otimes_F V)\text{-mod} \arrow[r, shift left] & KE\text{-mod} \arrow[l, shift left] \arrow[dl, shift left] \\
& (KR_T|K\otimes_F V)\text{-mod.} \arrow[u, "KR\otimes_{KR_T}-"] \arrow[ur, shift left] &
\end{tikzcd}$$

3.4 Quasihomogeneous R-modules

Turull [67], [70] considers the „Clifford theory determined by an R-module" instead of an R_1-module. We discuss here the connections with the point of view of the preceding sections. Let W be a simple KR_1-module as before.

Definition 3.4.1. An R-module M is *W-quasihomogeneous* if $K \otimes_F M \in (KR|W)$-mod.

The first question is whether the Clifford theory depends on the choice of a W-quasihomogeneous module. The next result says that it does not depend.

Theorem 3.4.2. *Assume that M and M' are W-quasihomogeneous R-modules. Then there exists a G-equivariant Morita equivalence between the G-algebras* $\mathrm{End}_{R_1}(M)$ *and* $\mathrm{End}_{R_1}(M')$.

3.4.3. We next consider the following context. Let R and R' be G-graded F-algebras. Let M be a W-quasihomogeneous R-module and M' be a W'-quasihomogeneous R'-module, where W is a KR_1-module and W' a KR'_1-module. Then the question is: when is the Clifford theory determined by M equivalent to the Clifford theory determined by M'?

Corollary 3.4.4. If there exists an isomorphism $\varepsilon : \mathrm{End}_{R_1}(M) \to \mathrm{End}_{R'_1}(M')$, of G-algebras over F, then there exists an equivalence of categories

$$(KR|W, F)\text{-mod} \simeq (KR'|W', F)\text{-mod}.$$

that preserves the gradings of modules and commutes with the action of the $\mathrm{Gal}(K/F)$.

An isomorphism $\mathrm{End}_R(R \otimes_{R_1} M) \simeq \mathrm{End}_R(R \otimes_{R_1} M')$ of G-graded algebras is called an *endoisomorphism* in [70]. But when does an endoisomorphism exist?

Theorem 3.4.5. *Assume that there is a Rickard equivalence between the G-graded F-algebras R and R'. Let M be a simple W-quasihomogeneous R-module and let M' be the corresponding R'-module.*

1) *There exists an isomorphism* $\varepsilon : \mathrm{End}_{R_1}(M) \to \mathrm{End}_{R'_1}(M')$ *of G-algebras over F, induced by Rickard equivalences.*

2) *The simple KR_1-module W also corresponds to a simple KR'_1-module W', and the derived equivalence induces the equivalence* $(KR|W, F)\text{-mod} \simeq (KR'|W', F)\text{-mod}$*of Corollary 3.4.4.*

Chapter 4

Brauer-Clifford classes and character correspondences

The results of [67] can be generalized to the case of group graded algebras. This is the aim of this chapter.

4.1 Motivation

Let G be a group and F a field. We take a strongly G-graded F-algebra R and two R-modules M and M', preferably simple or semisimple. The question we ask is when do these two modules have the same Clifford theory? For this we start by giving some properties of character correspondences and their Brauer-Clifford elements.

4.2 The Brauer-Clifford group

The ingredients to define the Brauer-Clifford group, introduced in the paper [69], are a finite group G and a commutative simple G-ring Z. One of the essential differences from the previous definition consists precisely in this choice of Z, whereas in [67, 68] Z was a commutative simple G-algebra over a field.

If G is a finite group and Z a commutative simple G-ring we say that a central simple G-algebra A over Z is *trivial* if there exists a non-zero G-module M over Z such that $End_Z(M)$ is isomorphic to A as central simple G-algebras over Z. Perhaps the most notable difference between [69] and [67, 68], as far as the definition of the Brauer-Clifford group goes, is that in the latest definition trivial G-algebras are central simple G-algebras over Z whereas under the old definition, trivial G-algebras always where required to have a field as their center.

The definition for the Brauer-Clifford group given in the paper [69] is more general then that in [67, 68], but it is essentially equivalent. We shall now present this refined definition of the notion of Brauer-Clifford group.

Definition 4.2.1. Let G be a finite group, and Z be a commutative simple G-ring. We define the *Brauer-Clifford group* of G over Z to be the set $\mathrm{BrClif}(G, Z)$ together with a binary operation. The elements of $\mathrm{BrClif}(G, Z)$ are equivalence classes of central simple G-algebras of finite rank over Z, under the equivalence given as follows. Suppose A and B are central simple G-algebras of finite rank over Z. Then, we say that A is equivalent to B if and only if there exist trivial central simple G-algebras T_1 and T_2 over Z such that

$$A \otimes_Z T_1 \simeq B \otimes_Z T_2$$

as central G-algebras over Z. The binary operation on $\mathrm{BrClif}(G, Z)$ is that induced by the tensor product over Z of central simple G-algebras over Z.

4.3 The Brauer-Clifford class and the central G-algebra of a character

Let G be a group, Z a commutative simple G-ring, F be a field of characteristic zero, and let $\bar{F}$ be an algebraic closure of F. We assume that all characters take their values in $\bar{F}$. We denote by $\mathrm{Irr}(G)$ the set of all irreducible characters of a finite group G and restriction and induction of a character in the usual way. We wish to study Cifford theory in the special case of strongly graded algebras.

4.3.1. Let G be a finite group and let R be a strongly G-graded semisimple F-algebra. Let ψ be an irreducible character of the algebra $\bar{F}R_H$ where $H \leq G$. Let θ_1 be an irreducible character of $\bar{F}R_1$ which is contained in the restriction of ψ to $\bar{F}R_1$, $\theta_1, \ldots, \theta_r \in \mathrm{Irr}(\bar{F}R_1)$ be the $G \times \mathrm{Gal}(\bar{F}/F)$-orbit of θ_1, and let

$$\bar{\theta} = \theta_1 + \cdots + \theta_r.$$

Let $e_{\theta_1}, \ldots, e_{\theta_r}$ be the corresponding primitive idempotents of $Z(\bar{F}R_1)$, and set $e = e_{\theta_1} + \cdots + e_{\theta_r}$. Then clearly $e \in Z(R_1)$, and if we set $F_0 := e(R_1 \cap Z(R))$ then F_0 is a field.

Remark 4.3.2. If $\Theta := \{\theta_1, \ldots, \theta_r\}$ is any orbit of $G \times \mathrm{Gal}(\bar{F}/F)$ on $\mathrm{Irr}(\bar{F}R_1)$, and if e_Θ denotes the sum of the idempotents corresponding to the irreducible characters in Θ, then e_Θ is a primitive idempotent of $R_1 \cap Z(R)$. The map $\Theta \mapsto e_\Theta$ from the set of orbits of $G \times \mathrm{Gal}(\bar{F}/F)$ on $\mathrm{Irr}(\bar{F}R_1)$ to the set of primitive idempotents of $R_1 \cap Z(R)$ provides a bijection.

Definition 4.3.3. The *central algebra of* ψ with respect to R and F is the G-acted F_0-algebra $eZ(R_1)$. We denote it by $Z(\psi, R, F)$. For each $\theta \in \{\theta_1, \ldots, \theta_r\}$, the central character associated with θ, restricts non-trivially to a map

$$\omega_\theta : Z(\psi, R, F) \to \bar{F}$$

which we also call the *central character associated with* θ.

$Z(\psi, R, F)$ is a commutative central simple G-algebra over the field F_0, uniquely determined by ψ and F and

$$Z(\psi, R, F) \simeq F(\theta_1) \oplus F(\theta_2) \oplus \cdots F(\theta_s),$$

where the G-action on the algebra on the right is obtained from the G-action on Θ. The center algebra holds informations about the character. The next proposition describes how this happens.

Proposition 4.3.4. *Let R and R' be two strongly G-graded semisimple F-algebras, and let ψ_1 and ψ_2 be two irreducible character of the algebra $\bar{F}R_H$ and $\bar{F}R'_H$, respectively, where $H \leq G$. Let $\theta_1 \in \mathrm{Irr}(\bar{F}R_1)$ be contained in $\mathrm{Res}^{\bar{F}R}_{\bar{F}R_1}(\psi_1)$ and $\theta_2 \in \mathrm{Irr}(\bar{F}R'_1)$ be contained in $\mathrm{Res}^{\bar{F}R'}_{\bar{F}R'_1}(\psi_2)$. For $i = 1, 2$ let O_i be the $G \times \mathrm{Gal}(\bar{F}/F)$-orbit of θ_i. We set $Z_1 = Z(\psi_1, R, F)$ and $Z_2 = Z(\psi_2, R', F)$. Then the following are equivalent:*

(1) There exists $\alpha : Z_1 \to Z_2$ an isomorphism of G-algebras over F which sends the central character associated with θ_1 to the central character associated with θ_2.

(2) There is a bijection $\beta : O_1 \to O_2$ which preserves the action of $G \times \mathrm{Gal}(\bar{F}/F)$ and is such that $\beta(\theta_1) = \theta_2$.

Moreover, if α and β exist then they are unique.

4.3.5. Let ψ be an irreducible character of the algebra $\bar{F}R_H$ where $H \leq G$. We let θ_1 be an irreducible character contained in the restriction of ψ to $\bar{F}R_1$, and we let $\theta_1, \ldots, \theta_r \in \mathrm{Irr}(\bar{F}R_1)$ be the $G \times \mathrm{Gal}(\bar{F}/F)$-orbit of θ_1, and we let $\bar{\theta} = \theta_1 + \cdots + \theta_r$. ($\bar{\theta}$ does not depend on the choice of θ_1.)

Recall that the notion of *quasihomogeneous R-module* was introduced in Definition 3.4.1 but we referred to the module as being above a certain other module. Now we give the definition of quasihomogeneous when the module is in a sense "above" a character.

Definition 4.3.6. A R-module M is ψ-*quasihomogeneous* (with respect to R_1) if it is not 0 and the character of $\bar{F} \otimes_F M$, χ, is of the form $m\bar{\theta}_{R_1}$, where m is a positive integer.

We always have such a R-module M over F which is ψ-quasihomogeneous.

Theorem 4.3.7. *Let $Z = Z(\psi, R, F)$ be the center algebra of ψ with respect to F. Suppose that M is any ψ-quasihomogeneous R-module over F. Then $\mathrm{End}_{R_1}(M)$ is a central simple G-algebra over Z and its class in $\mathrm{BrClif}(G, Z)$ does not depend on M.*

Remark 4.3.8. In Theorem 3.4.5 we have that $\mathrm{End}_{R_1}(M)$ is a central simple G-algebra, and can be regarded as representative for the Brauer-Clifford class of W (see [67] and [23] for strongly G-graded algebras). Then Theorem 3.4.2 says that the Brauer-Clifford class of W does not depend on the choice of a W-quasihomogeneous R-module, while Theorem 3.4.5 says that a G-graded derived equivalence over F "preserves Brauer-Clifford classes" (see also [47, Theorem 5.3]).

Definition 4.3.9. Let $Z = Z(\psi, R, F)$ be the center algebra of ψ with respect to R and F. Suppose that M is any ψ-quasihomogeneous R-module over F. We view $\mathrm{End}_{R_1} M$ as a central simple G-algebra over Z, and we denote by

$$[[\psi]] = [[\psi, R, F]] \in \mathrm{BrClif}(G, Z)$$

the element of $\mathrm{BrClif}(G, Z)$ that it defines. We say that this is the element of *the Brauer-Clifford group associated to ψ.*

4.3.10. Let G a finite group, R and R' two strongly G-graded F-algebras that are both semisimple. Let ψ be an irreducible character of the algebra $\bar{F}R_H$ where $H \leq G$ and ψ_1 be an irreducible character of the algebra $\bar{F}R'_{H'}$ where $H' \leq G$. Let $Z = Z(\psi, R, F)$ be the center algebra of ψ with respect to R and F, and let $Z' = Z(\psi_1, R', F)$ be the center algebra of ψ_1 with respect to R' and F. Assume that we have a G-algebra isomorphism $\alpha : Z \to Z'$. We denote by

$$\bar{\alpha} : \mathrm{BrClif}(G, Z) \to \mathrm{BrClif}(G, Z')$$

the isomorphism induced by α. Finally, assume that $\bar{\alpha}([[\psi]]) = [[\psi_1]]$.

Remark 4.3.11. Let $F_0 = Z^G$ and we let $F'_0 = (Z')^G$. Of course, $\alpha(F_0) = F'_0$. Let, for the moment, M be any ψ-quasihomogeneous R-module over F, and let M' be any ψ_1-quasihomogeneous R'-module over F. Then, the center of $\mathrm{End}_{R_1}(M)$ is naturally identified with Z, and the class of $\mathrm{End}_{R'_1}(M')$ in $\mathrm{BrClif}(G, Z)$ is $[[\psi]]$, and the center of $\mathrm{End}_{R'_1}(M')$ is naturally identified with Z', and the class of $\mathrm{End}_{R'_1}(M')$ in $\mathrm{BrClif}(G, Z')$ is $[[\psi_1]]$. Since $\bar{\alpha}[[\psi]] = [[\psi_1]]$, there exist trivial G-algebras T_1 over F_0 and T_2 over F'_0 and an isomorphism of G-algebras over F

$$\beta : \mathrm{End}_{R_1}(M) \otimes_{F_0} T_1 \to \mathrm{End}_{R'_1}(M') \otimes_{F'_0} T_2$$

which restricts to the centers of the algebras (using the identifications) as the map $\alpha : Z \to Z'$. If we tensor with further trivial algebras if necessary, we assume, without loss of generality, that the underlying modules for T_1 and T_2 are free non-zero G-modules and if we tensor these with M and M', respectively we obtain new quasiprimitive modules over F, which we rename M and M' respectively. We obtain the isomorphism β from $A = \mathrm{End}_{R_1}(M)$ to $A' = \mathrm{End}_{R'_1}(M')$. The above yield a one-to-one correspondence from characters of R to characters of R'. For more details see [61].

4.3.12. If H is a subgroup of G then we denote by $\mathrm{Irr}(H, R, \psi)$ the set of irreducible characters ϕ of $\bar{F}R_H$ such that the restriction of ψ to $\bar{F}R_1$ contains at least one irreducible (hence all) characters that are $G \times \mathrm{Gal}(\bar{F}/F)$-conjugate to some irreducible contained in the restriction of ψ to $\bar{F}R_1$. We let $\mathbf{Z}\mathrm{Irr}(H, R, \psi)$ be the set of integer linear combinations of elements of $\mathrm{Irr}(H, R, \psi)$. We do likewise for subgroups of G, R' a strongly G-graded F-algebra and the character ψ_1.

Theorem 4.3.13. *Assume the notations from 4.3.10. Assume furthermore that* $\theta \in \mathrm{Irr}(\bar{F}R_1)$ *is* $G \times \mathrm{Gal}(\bar{F}/F)$*-conjugate to an irreducible character contained in the restriction of* ψ *to* $\bar{F}R_1$*, and, that* $\theta_1 \in \mathrm{Irr}(\bar{F}R'_1)$ *is* $G \times \mathrm{Gal}(\bar{F}/F)$*-conjugate to an irreducible character contained in the restriction of* ψ_1 *to* $\bar{F}R_1$*. Assume that the central character associated with* θ *restricted to* Z *corresponds to the central character associated with* θ_1 *restricted to* Z' *under the isomorphism* α*. Then, there is a bijection* $\phi \mapsto \phi'$ *from the union of* $\mathbb{Z}\mathrm{Irr}(\bar{F}R_S|\psi)$ *to the union of* $\mathbb{Z}\mathrm{Irr}(\bar{F}R'_S|\psi_1)$ *as* S *runs over the subgroups of* G *satisfies the following properties:*

(1) For each H *a subgroup of* G*, restriction of the map* $\phi \mapsto \phi'$ *provides an isomorphism of* $\mathbb{Z}$*-modules from* $\mathbb{Z}\mathrm{Irr}(\bar{F}R_H|\psi)$ *to* $\mathbb{Z}\mathrm{Irr}(\bar{F}R'_H|\psi_1)$*, that preserves the usual inner product, and restriction provides a bijection from* $\mathrm{Irr}(\bar{F}R_H|\psi)$ *to* $\mathrm{Irr}(\bar{F}R'_H, |\psi_1)$*.*

(2) There is some rational constant d*, such that, whenever* H *is a subgroup of* G*,* $\phi \in \mathbb{Z}\mathrm{Irr}(\bar{F}R_H|\psi)$ *and* $\phi \mapsto \phi'$*, then* $\phi'(1) = d\phi(1)$*.*

(3) The map $\phi \mapsto \phi'$ *commutes with induction and restriction of characters, multiplication with characters of* $\bar{F}R_H$*, with any Galois automorphism that fixes* F *and with conjugation by* G*.*

(4) The map $\phi \mapsto \phi'$ *preserves the field of values of irreducible characters and the corresponding elements of Brauer group and in particular the Schur indices.*

(5) If $\phi \mapsto \phi'$ *and* ϕ *is irreducible, then* ϕ *and* ϕ' *have isomorphic center algebras, under an isomorphism that preserves the correspondence. More precisely, suppose that* $\phi \in \mathrm{Irr}(\bar{F}R^S)$*, and* I, H*, are subgroups of* G *with* $I \trianglelefteq H$*. Then, there is a unique* H/I*-algebra isomorphism*

$$\beta : Z(\phi, R_{H/I}, F) \to Z(\phi', R_{H/I}, F)$$

with, for every irreducible character γ contained in $\mathrm{Res}^{R}_{R_I}(\phi)$*, the central character associated to γ corresponds to the central character associated to γ' under β. We let $\bar{\beta}$ denote the group isomorphism between the Brauer-Clifford groups deterined by β and* $[[\phi, R_{H/I}, F]] \in \mathrm{BrClif}(H/I, Z(\phi, R_{H/I}, F))$ *and* $\bar{\beta}([[\phi, R, F]]) = [[\phi', R_{H/I}, F]]$.

4.4 Field extensions

Equality of the Brauer-Clifford classes over a field gives rise to equality over a bigger field. Let F be a field of characteristic zero. We consider that our characters have values in $\bar{F}$. Let K be a subfield of $\bar{F}$ containing F. Let R and R' be two strongly G-graded semisimple F-algebras. Let $\theta \in \mathrm{Irr}(\bar{F}R_1)$, and let $\theta' \in \mathrm{Irr}(\bar{F}R'_1)$, let $Z = Z(\theta, R, F)$ be the center algebra of θ with respect to R and F, and let $Z' = Z(\theta', R', F)$ be the center algebra of θ' with respect to R and F.

Proposition 4.4.1. *Assume there is a G-algebra isomorphism $\alpha : Z \to Z'$, which induces the isomorphism* $\bar{\alpha} : \mathrm{BrClif}(G, Z) \to \mathrm{BrClif}(G, Z')$ *and is such that α sends the central character associated with θ to the central character associated with θ'. Assume, furthermore, that* $\bar{\alpha}([[\theta, R, F]]) = [[\theta', R', F]]$*. Set $S_K = Z(\theta, KR, K)$ and $Z'_K = Z(\theta', KR', K)$. Then, there is a unique G-algebra isomorphism $\beta : Z_K \to Z'_K$, such that β sends the central character associated with θ to the central character associated with θ'. Moreover, if we denote by* $\bar{\beta} : \mathrm{BrClif}(G, Z_K) \to \mathrm{BrClif}(G, Z'_K)$ *the isomorphism induced by β then,* $\bar{\beta}([[\theta, KR, K]]) = [[\theta', KR', K]]$.

4.5 A Morita equivalence over Z

Take M to be a simple R-module, and let ψ be the character of a simple submodule of the $\bar{F}R$-module $\bar{F} \otimes_F M$. Let θ_1 be an irreducible character contained in the restriction of ψ to $\bar{F}A$ and $\theta_1, \ldots, \theta_r$, $\bar{\theta}$, e and F_0 as in Section 4.3, where $A = R_1$. Let $\theta_1, \ldots, \theta_s$ be the representatives for the orbits of the action of $\mathrm{Gal}(\bar{F}/F)$ on $\{\theta_1 \ldots \theta_r\}$.

Then $\mathrm{End}_R(R \otimes_A M)^{\mathrm{op}}$ is a central simple G-graded F_0-algebra over Z, where

$$Z := eZ(A) \simeq F(\theta_1) \oplus F(\theta_2) \oplus \cdots \oplus F(\theta_s)$$

as G-acted F_0-algebras. Instead of M as above we may use another more general R-module: a ψ-quasihomogeneous R-module M'.

Proposition 4.5.1. *If M' is a ψ-quasihomogeneous R-module, then there is a G-graded Morita equivalence over Z between* $\mathrm{End}_R(R \otimes_A M)$ *and* $\mathrm{End}_R(R \otimes_A M')$.

Chapter 5

Modular G-graded algebras and G-algebras of endomorphisms

We wish to study Clifford Theory and field extensions for strongly G-graded algebras. Alexandre Turull in [70], [71] introduced the notion of endoisomorphism showing that there is a natural connection between it and Clifford Theory of finite groups algebras. An endoisomorphism is an isomorphism between G-algebras of endomorphisms, where G is a finite group. We consider here endoisomorphisms between modules over strongly G-graded algebras. An endoisomorphism induces equivalences of categories with some good compatibility properties (see Theorem 5.5.1 and Theorem 5.6.1 below). The results are due to D. Gliţia and can be found in [24].

5.1 General setting

5.1.1. Let p be a prime number. We shall consider the p-modular systems $(\mathcal{K}, \mathcal{O}, k)$ and $(\hat{\mathcal{K}}, \hat{\mathcal{O}}, \hat{k})$ such that $\hat{\mathcal{K}}$ is a finite extension of $\mathcal{K}$ and $\hat{\mathcal{O}}$ is the ring of integers of $\hat{\mathcal{K}}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ and $\hat{\mathfrak{m}}$ be the maximal ideal of $\hat{\mathcal{O}}$.

5.1.2. Let G be a finite group. Let $R = \bigoplus_{g\in G} R_g$ a strongly G-graded $\mathcal{O}$-algebra, free of finite rank over $\mathcal{O}$. We denote $\hat{\mathcal{O}}R = \hat{\mathcal{O}} \otimes_{\mathcal{O}} R$, $\hat{\mathcal{K}}R = \hat{\mathcal{K}} \otimes_{\mathcal{O}} R$ and $\hat{k}R = \hat{k} \otimes_{\mathcal{O}} R$.

We denote by H a subgroup of G. We assume that for any such R and H, the algebra $\mathcal{K}R_H$ is symmetric, $\hat{\mathcal{K}}R_H$ and $\hat{k}R_H$ are split (not necessarily semisimple). Unless otherwise specified, modules are assumed to be finitely generated. We denote by $\mathcal{R}_{\mathcal{K}}(R)$ the *Grothendieck group* of the category of finitely generated $\mathcal{K}R$-modules. If M is an $\mathcal{K}R$-module, we let $[M]$ denote its image in $\mathcal{R}_{\mathcal{K}}(R)$.

Let $\mathrm{Irr}_{\mathcal{K}}(R)$ denote the set of isomorphism classes of simple $\mathcal{K}R$-modules. Then $\mathcal{R}_{\mathcal{K}}(R) = \mathbb{Z}\mathrm{Irr}_{\mathcal{K}}(R)$ is the free abelian group with basis $\mathrm{Irr}_{\mathcal{K}}(R)$. We denote by $\mathrm{IBr}(kR)$ the set of isomorphism classes of simple kR-modules.

5.1.3. The decomposition homomorphism. Let M be an $\mathcal{K}R$-module. An *R-lattice of M* is an R-submodule L of M with the following properties. We take a $\mathcal{K}$-basis in M and let L be the $\mathcal{O}$-submodule of M generated by this basis and by all the products between the elements of this basis and the elements of an $\mathcal{O}$-basis of R. Then L is actually an R-submodule of M with the property that $\mathcal{K} \otimes_{\mathcal{O}} L \simeq M$.

Let L be an R-lattice of M. This yields the kR-module $L/\mathfrak{m}L$, called a *reduction modulo p of M*. This module is not unique even up to isomorphism but all modules have the same composition factors, up to isomorphism, for all choices of L. In this way, the *decomposition map*

$$d : \mathbb{Z}\mathrm{Irr}(\mathcal{K}R) \to \mathbb{Z}\mathrm{IBr}(kR).$$

send the class of M in $\mathcal{R}_{\mathcal{K}}(R)$ to the class of of $L/\mathfrak{m}L$ in $\mathcal{R}_k(R)$.

5.1.4. Recall from [46, Theorem 3.4] that there is an action of $\mathcal{K}G$-modules on $\mathcal{K}R$-modules. Let M be a R-module and N a $\mathcal{K}G$-module. Then we may construct the $\mathcal{K}R$-module $M \otimes_{\mathcal{O}} N$ that is also a module over the $(G \times G)$-graded algebra $R \otimes_{\mathcal{O}} \mathcal{K}G$.

5.2 Categories of modules above a given module

Let R be a strongly G-graded $\mathcal{O}$-algebra, and let M be an R-lattice.

Definition 5.2.1. We denote by $\widehat{M'_{\mathcal{K}}}$ the direct sum of countably infinite number of copies of $\mathcal{K}R$-modules $M \otimes_{\mathcal{O}} \mathcal{K}G$and by $(\mathcal{K}R_H|M)$-mod be the full subcategory of $\mathcal{K}R_H$-mod consisting of $\mathcal{K}R_H$-modules that are quotients of finitely generated $\mathcal{K}R_H$-submodules of $\widehat{M'_{\mathcal{K}}}$. We say that a module from $(\mathcal{K}R_H|M)$ is *above* M.

Definition 5.2.2. The field extension $\mathcal{K}$ of $\mathcal{O}$ is a *good extension for M* if $\mathrm{Res}^{\mathcal{K}R}_{\mathcal{K}R_1}(\mathcal{K}\otimes_{\mathcal{O}}M)$ is a semisimple module and

$$\mathcal{K} \otimes_{\mathcal{O}} \mathrm{End}_{R_1}(M) = \mathrm{End}_{R_1}(\mathcal{K} \otimes_{\mathcal{O}} M).$$

If we assume that $\mathcal{K}R_1$ is semisimple. Then the extension $\mathcal{K}/\mathcal{O}$ is good for M. For the remaining of this section assume that $\mathcal{K}A_1$ is semisimple.

Definition 5.2.3. Let V be a $\mathcal{K}R_1$-module. Denote by $\mathrm{Irr}(\mathcal{K}R|V)$ the set of all isomorphism classes of simple $\mathcal{K}R$-modules whose restriction to R_1 contains a simple summand that is also a summand of V. We say that a module from $\mathrm{Irr}(\mathcal{K}R|V)$ is *above* V.

Proposition 5.2.4. *Let V be the direct sum of all nonisomorphic simple $\hat{\mathcal{K}}R_1$-modules that appear in the decomposition of* $\mathrm{Res}^{R}_{R_1}(\hat{\mathcal{K}} \otimes_{\mathcal{O}} M)$*. Then, a $\hat{\mathcal{K}}R_H$-module W is in* $(\hat{\mathcal{K}}R_H|M)$-mod *if and only if W is an $\mathbb{N}$-linear combination of the elements in* $\mathrm{Irr}(\hat{\mathcal{K}}R_H|V)$.

Definition 5.2.5. Let W be a kR_1-module. We denote by $\mathrm{IBr}(kR|W)$ the set of all isomorphism classes of simple kR_1-modules that restricted to R_1 contain a simple summand that is also a summand of W. We say that a module in $\mathrm{IBr}(kR|W)$ is *above* W.

5.3 Endoisomorphisms and module correspondences

Let R and R' be two strongly G-graded $\mathcal{O}$-algebras. Let M be an R-lattice, and let M' be a R'-lattice. Assume that the extension $\mathcal{K}$ of $\mathcal{O}$ is good for M and M'.

5.3.1. Similar to Section 3.4 we define an *endoisomorphism over* $\mathcal{O}$ from M to M' is an isomorphism of G-algebras over $\mathcal{O}$

$$\epsilon : \mathrm{End}_{R_1}(M) \to \mathrm{End}_{R'_1}(M').$$

We will see that an isomorphism ϵ as above determines a module correspondence κ_ϵ compatible with field extensions and subgroups of G.

Definition 5.3.2. a) A G-algebra Z over $\mathcal{K}$ is called a *center algebra of* $\mathcal{K}R$ if, setting $Z_0 = Z(R_1/J(R_1))$, so Z_0 is a commutative G-algebra over $\mathcal{K}$, then, for some idempotent e of Z_0^G we have $Z = eZ_0$.

b) Let e be the sum of all the primitive idempotents of Z_0^G which act non trivially on $\mathcal{K} \otimes_{\mathcal{O}} M$. Then eZ_0 is called the *center algebra of* R *associated with* $\mathcal{K} \otimes_{\mathcal{O}} M$, and it is denoted by $Z(M, R, \mathcal{K})$.

Theorem 5.3.3. *Let ϵ be an endoisomorphism from M to M'. Then ϵ determines an isomorphism of G-algebras over $\mathcal{K}$ denoted $\bar{\epsilon_{\mathcal{K}}}$ from $Z(M, R, \mathcal{K})$ to $Z(M', R', \mathcal{K})$ and an endoisomorphism*

$$\widehat{\epsilon_{\mathcal{K}}} : \mathrm{End}_{R_1}(\widehat{M_{\mathcal{K}}}) \to \mathrm{End}_{R_1}(\widehat{M'_{\mathcal{K}}}).$$

that in its turn determines a $\mathcal{K}$-linear isomorphism κ_ϵ of categories from $(\mathcal{K}A_H|M)$-mod to $(\mathcal{K}A'_H|M')$-mod.

5.3.4. An isomorphism $\phi : M \to M'$ induces an endoisomorphism ϵ from M to M'. In this case, the isomorphism of categories κ_ϵ takes each module to a module isomorphic to it.

If there is a G-graded Morita equivalence over $\mathcal{O}$ between R and R' such that M corresponds to M', then there is an endoisomorphism from M to M'.

5.4 Endoisomorphisms over $\mathcal{K}$ and over $\mathcal{O}$

Let R and R' be two strongly G-graded $\mathcal{O}$-algebras. Let M be a $\mathcal{K}R$-module, and M' a $\mathcal{K}R'$-module. Starting from an endoisomorphism over the field of fractions $\mathcal{K}$ of the principal ideal domain $\mathcal{O}$, we can obtain the endoisomorphism over $\mathcal{O}$.

Let $E = \operatorname{End}_{\mathcal{K}R_1}(M)$, so E is a finite dimensional $\mathcal{K}$-algebra. Recall that an $\mathcal{O}$-*order* in E is an $\mathcal{O}$-subalgebra B of E that contains a $\mathcal{K}$-basis of E, and such that every element of B is integral over $\mathcal{O}$. One can obtain $\mathcal{O}$-orders in our context by taking first an R-lattice.

Theorem 5.4.1. *Assume that the restriction* $\operatorname{Res}^{\mathcal{K}R}_{\mathcal{K}R_1}(M)$ *is semisimple. Then any* $\mathcal{O}$-*order in* E *is finitely generated as an* $\mathcal{O}$-*module. Moreover,* E *has an* $\mathcal{O}$-*order* B *that is* G-*invariant and maximal among the* G-*invariant orders of* E, *and there is a* G-*invariant lattice* L *such that* $B = \{e \in E \mid e(L) \subseteq L\}$.

Theorem 5.4.2. *Assume that* $\operatorname{Res}^{\mathcal{K}R}_{\mathcal{K}R_1}(M)$ *and* $\operatorname{Res}^{\mathcal{K}R'}_{\mathcal{K}R'_1}(M')$ *are semisimple, and let* ϵ *be an endoisomorphism over* $\mathcal{K}$ *from* M *to* M'. *Then there exists a* G-*invariant* $\mathcal{O}$-*lattice* L *of* M, *a* G-*invariant* $\mathcal{O}$-*lattice* L' *of* M' *and an endoisomorphism* ν *from* $\operatorname{End}_{R_1}(L)$ *to* $\operatorname{End}_{R'_1}(L')$, *such that* $\epsilon = \mathcal{K} \otimes_{\mathcal{O}} \nu$, *and in addition,*

$$\operatorname{End}_{\mathcal{K}R_1}(M) = \mathcal{K} \otimes_{\mathcal{O}} \operatorname{End}_{R_1}(L) \qquad \text{and} \qquad \operatorname{End}_{\mathcal{K}R'_1}(M') = \mathcal{K} \otimes_{\mathcal{O}} \operatorname{End}_{R'_1}(L').$$

5.5 Correspondences in characteristic zero

Let R and R' two strongly G-graded $\mathcal{O}$-algebras, and assume that $\mathcal{K}R_1$ and $\mathcal{K}R'_1$ are semisimple.

Theorem 5.5.1. *Let* M *be a* $\mathcal{K}R$-*module, and* M' *is a* $\mathcal{K}R'$-*module. Let*

$$\epsilon : \operatorname{End}_{\mathcal{K}R_1}(M) \to \operatorname{End}_{\mathcal{K}R'_1}(M')$$

be an endoisomorphism from M *to* M'. *Let* V *be the direct sum of all nonisomorphic simple* $\hat{\mathcal{K}}R_1$-*modules that appear in the decomposition of* $\operatorname{Res}^{\hat{\mathcal{K}}R}_{\hat{\mathcal{K}}R_1}(M)$ *and* V' *be be the direct sum of all nonisomorphic simple* $\hat{\mathcal{K}}R'_1$-*modules that appear in the decomposition of* $\operatorname{Res}^{\hat{\mathcal{K}}R'}_{\hat{\mathcal{K}}R'_1}(M')$. *Then the following statements hold.*

(1) κ_ϵ *induces isomorphisms of* $\mathbb{Z}\operatorname{Irr}(\mathcal{K}H)$-*modules*

$$\kappa_\epsilon : \mathbb{Z}\operatorname{Irr}(\hat{\mathcal{K}} \otimes_{\mathcal{O}} R_H|V) \to \mathbb{Z}\operatorname{Irr}(\hat{\mathcal{K}} \otimes_{\mathcal{O}} R'_H|V').$$

(2) κ_ϵ *sends the simple summands of* $\hat{\mathcal{K}} \otimes_{\mathcal{K}} M$ *to simple summands of* $\hat{\mathcal{K}} \otimes_{\mathcal{K}} M'$ *and commutes with restriction and induction of modules, with the action of* $\operatorname{Gal}(\hat{\mathcal{K}}/\mathcal{K})$ *and with conjugation by* G.

(3) *Let* χ *be the character of a simple module from* $\operatorname{Irr}(\hat{\mathcal{K}} \otimes_O R_H|V)$, *let* $[\chi]$ *denote the element of the Brauer group* $\operatorname{Br}(\mathcal{K}(\chi))$ *associated with it. Then we have* $\mathcal{K}(\kappa_\epsilon(\chi)) = \mathcal{K}(\chi)$ *and* $[\kappa_\epsilon(\chi)] = [\chi]$. *In particular, the Schur indices of the irreducible characters are preserved under* κ_ϵ.

Corollary 5.5.2. Let $W \in \mathrm{Irr}(\mathcal{K}R_1)$ and $W' \in \mathrm{Irr}(\mathcal{K}R'_1)$. Set $Z = Z(W, R, \mathcal{K})$ and $Z' = Z(W', R', \mathcal{K})$ be the respective center algebras, and let $[[W]] \in \mathrm{BrClif}(G, Z)$ and $[[W']] \in \mathrm{BrClif}(G, Z')$ be the respective elements of the Brauer-Clifford group. Suppose that the G-algebra isomorphism $\alpha : Z \to Z'$ sends the central character associated to W to the central character associated with W', and, denoting by $\bar{\alpha}$ the induced group isomorphism between the respective Brauer-Clifford groups we have $\bar{\alpha}([[W]]) = [[W']]$.

Then, there exist M a finitely generated W-quasihomogeneous $\mathcal{K}R$-module, M' a finitely generated W'-quasihomogeneous $\mathcal{K}R'$-module, and an endoisomorphism ϵ over K from M to M' such that:

(1) the module V from Theorem 5.5.1 is the sum of the $G \times \mathrm{Gal}(\hat{\mathcal{K}}/\mathcal{K})$-orbit of W, and V' is the sum of the $G \times \mathrm{Gal}(\hat{\mathcal{K}}/\mathcal{K})$-orbit of W'.

(2) κ_ϵ sends W to W' and induces the isomorphism of $\mathbb{Z}$-modules

$$\kappa_\epsilon : \mathbb{Z}\mathrm{Irr}(\mathcal{K}R_H|W) \to \mathbb{Z}\mathrm{Irr}(\mathcal{K}R'_H|W');$$

5.6 Correspondences in characteristic p

Let R and R' two strongly G-graded $\mathcal{O}$-algebras. Let M be an R-lattice and M' be an R'-lattice. We assume that $\mathcal{K}R_1$ and $\mathcal{K}R'_1$ are semisimple. Assume also that the extension $\hat{k}$ of $\mathcal{O}$ is a good extension for M and M'.

Theorem 5.6.1. *Let W be the direct sum of all the nonisomorphic simple $\hat{k} \otimes_{\mathcal{O}} R_1$-modules that appear in a composition series of $\mathrm{Res}^R_{R_1}(\hat{k} \otimes_{\mathcal{O}} M)$, and let W' be the direct sum of all the nonisomorphic simple $\hat{k} \otimes_{\mathcal{O}} R'_1$-modules that appear in a composition series of $\mathrm{Res}^R_{R_1}(\hat{k} \otimes_{\mathcal{O}} M')$. Let ϵ be an endoisomorphism from M to M'. Then the following statements hold:*

(1) *ϵ induces the endoisomorphism*

$$\mathcal{K} \otimes_{\mathcal{O}} \epsilon : \mathrm{End}_{\mathcal{K}R_1}(\mathcal{K} \otimes_{\mathcal{O}} M) \to \mathrm{End}_{\mathcal{K}R'_1}(\mathcal{K} \otimes_{\mathcal{O}} M')$$

from $\mathcal{K} \otimes_{\mathcal{O}} M$ to $\mathcal{K} \otimes_{\mathcal{O}} M'$, so Theorem 5.5.1 applies.

(2) *κ_ϵ induces the isomorphisms*

$$\kappa_\epsilon : \mathbb{Z}\mathrm{IBr}(\hat{k}R_H|W) \to \mathbb{Z}\mathrm{IBr}(\hat{k}R'_H|W'),$$

of $\mathbb{Z}\mathrm{Irr}(kH)$-modules, sending simple modules to simple modules, and summands of $k \otimes_{\mathcal{O}} M$ to summands of $k \otimes_{\mathcal{O}} M'$.

(3) *κ_ϵ commutes with restriction and induction of modules, and with conjugation by G.*

Chapter 6

Glauberman correspondence and related Morita equivalences

Starting with P-interior algebras, where P is a finite p-group, we prove two theorems establishing certain group graded Morita equivalences. These apply to the case of blocks with normal defect groups, and defect zero blocks of normal subgroups, respectively. The main results are gathered in [26] and are due to D. Gliţia and A. Mărcuş.

6.1 Introduction

This chapter is motivated by several results on the existence of Morita equivalences in the context of the Glauberman-Watanabe correspondence (see [37], [31], [28], [75], [54] and the references given there). In order to explain this, let $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system, let G be a finite group and let A a solvable finite group acting on G such that G and A have coprime order. Let b be an A-invariant block of $\mathcal{O}G$. Under some additional conditions, there exists a Morita equivalence between $b\mathcal{O}G$ and $w(b)\mathcal{O}G^A$ induced by a $(b\mathcal{O}G, w(b)\mathcal{O}G^A)$-bimodule M with the property that regarded as a $G \times G^A$-module, M has a source which is an endopermutation module. Moreover, when we have a splitting p-modular system, this Morita equivalence induces a bijection

$$\pi(G, A) : \mathrm{Irr}_{\mathcal{K}}(G, b) \to \mathrm{Irr}_{\mathcal{K}}(G^A, w(b)),$$

which coincides with the Glauberman correspondence, where the block $w(b)$ of $\mathcal{O}G^A$ is the Watanabe correspondent of b (see [73]).

By induction, this is reduced to the case when we consider blocks lying over a block of a normal p'-subgroup of G, actually of $O_{p'}(G)$. More generaly, instead of a normal p'-subgroup it is useful to consider blocks of defect zero of a normal subgroup (see [29]). It turns out that it is enough to consider a strongly G-graded P-interior $\mathcal{O}$-algebra R, where the p-group P is also a normal subgroup of the group of homogeneous units of

R, and whose identity component R_1 is an $\mathcal{O}$-simple algebra. The identity of R_1 has a defect group $Q \leq P$, and the other algebra is constructed by first considering the Brauer quotient $R_1(Q)$. Theorem 6.6.5 below will generalize the main result of Dade [6] on correspondences above the Glauberman corresponcence (see also Turull [66] and Ladisch [40]), and our approach is more in the spirit of [6] and [8], by systematical use of Clifford extensions.

In fact, our method of proof is inspired from [17], and therefore, our first main result is a generalization of Structure Theorem for blocks with normal defect group due to Külshammer (see also [51, Proposition 14.6], Alperin, Linckelman and Rouquier [2], Fan and Puig [20, Theorem 1.17]). Instead of starting with a block of a group algebra $\mathcal{O}G$, we only consider a separable algebra extension $\mathcal{O}P \to B$, where P is a finite p-group and B has finite $\mathcal{O}$-rank, and then we construct our Morita equivalent algebras from this data.

6.2 Motivation: the Glauberman correspondence

We start off with a presentation of this correspondence and move on to some of the more recent developments that where triggered by a theorem of Atumi Watanabe in 1999 (see [73]), a theorem that motivates the study of Morita equivalences in this context.

Let A and K be finite groups. We assume that A acts on K. We can then construct the semidirect product

$$K \rtimes A = \{(x,a) | x \in K, a \in A\}, \qquad \text{where } (x,a)(y,b) = (x^a y, ab).$$

Moreover, we have the following split exact sequence

$$1 \to K \to K \rtimes A \xrightarrow{\pi} A \to 1$$

There exists a group homomorphism $\lambda : A \to K \rtimes A$ such that $\pi\lambda = 1_A$.

Hypothesis 6.2.1. *Let K be a finite group and A a solvable finite group. We assume that A acts on K and $(|K|, |A|) = 1$.*

6.2.2. Let p be a prime number. We take a p-modular system $(\mathcal{K}, \mathcal{O}, k)$ that is "big enough". In this case $\mathrm{Irr}(K) = \mathrm{Irr}_{\mathcal{K}}(K)$. This happens because any $\mathcal{K}K$ module has a character that determines the isomorphism class. A character χ can be defined either as

$$\chi : K \to \mathcal{K} \quad \text{ or } \chi : K \to \mathbb{C}$$

but $\chi(K) \subseteq \mathcal{K} \cap \mathbb{C}$. We denote $\mathrm{Irr}_{\mathcal{K}}(K)^A = \{\chi \in \mathrm{Irr}_{\mathcal{K}}(K) \mid {}^a\chi = \chi, \forall a \in A\}$.

Theorem 6.2.3 (Glauberman-Isaacs). *Assume Hypothesis 6.2.1. Then there is a bijection*

$$\pi(K, A) : \mathrm{Irr}_{\mathcal{K}}(K)^A \to \mathrm{Irr}_{\mathcal{K}}(K^A)$$

with the following properties:

(1) For all $B \trianglelefteq A$ we have $\pi(K,B)(\mathrm{Irr}_{\mathcal{K}}(K)^A) \subseteq \mathrm{Irr}_{\mathcal{K}}(K^B)^A$. Moreover, in $\mathrm{Irr}(K)^A$ we have

$$\pi(K, A) = \pi(K^B, A/B) \circ \pi(K, B),$$

hence the following diagram commutes

$$\begin{tikzcd} \mathrm{Irr}(K)^A \arrow[r, "\pi(K,A)"] \arrow[d, "\pi(K,B)"'] & \mathrm{Irr}(K^A) \arrow[d, equal] \\ \mathrm{Irr}(K^B)^A \arrow[r, "\pi(K^B,A/B)"'] & \mathrm{Irr}((K^B)^{A/B}). \end{tikzcd}$$

(2) If A is a q-group, where q is a prime number, then $\pi(K,A)(\chi)$ is the unique irreducible character of K^A that is a component of $\mathrm{Res}^K_{K^A}(\chi)$ and has multiplicity prime to q, for all $\chi \in \mathrm{Irr}(K)^A$.

Watanabe's theorem says that the bijection from Theorem 6.2.3 preserves p-blocks.

6.2.4. Permutation and endo-permutation $\mathcal{O}P$-modules.

Definition 6.2.5. A $\mathcal{O}P$-lattice M is called a *permutation lattice* if M has a P-invariant $\mathcal{O}$-basis. The module M is called the *permutation module*.

If M is a permutation module then M is isomorphic to a direct sum of $\mathcal{O}P$-modules induced by subgroups $Q \leq P$, of the form

$$\mathcal{O}P \otimes_{\mathcal{O}Q} V = \mathrm{Ind}_Q^P V,$$

where V is the trivial $\mathcal{O}Q$-module.

Definition 6.2.6. Let M be a $\mathcal{O}P$-module. Then $\mathrm{End}_{\mathcal{O}}(M)$ is a P-algebra free over $\mathcal{O}$, where

$$({}^u f)(m) = f(u^{-1}m).$$

Actually, there exists a homomorphism from P to $\mathrm{Aut}_{\mathcal{O}}(M)$.

We say that M is called the *endo-permutation $\mathcal{O}P$-lattice* if M is a $\mathcal{O}P$ lattice such that $\mathrm{End}_{\mathcal{O}}(M)$ is a permutation $\mathcal{O}P$-module under the conjugation action of P. In other words we require the existence of a P-invariant $\mathcal{O}$-basis (stable basis) of $\mathrm{End}_{\mathcal{O}}(M)$. The module M, in this case, is called an *endo-permutation module*.

6.2.7. The Brauer homomorphism. The following construction works particularly well for modules with a P-stable basis.

6.2.8. Let B be a interior $\mathcal{O}D$-algebra that has a P-stable basis, hence B is a $\mathcal{O}$-algebra, free of finite rank over $\mathcal{O}$ and there exists a $\mathcal{O}$-algebra homomorphism from $\mathcal{O}D$ to B.

The Brauer homomorphism $\mathrm{Br}_D : B^D \to B(D)$, is defined as follows. Let

$$B^D := \{b \in B | {}^u b = b \text{ for all } u \in D\},$$

with ${}^u b = ubu^{-1}$. Let

$$B(D) := k \otimes_{\mathcal{O}} \frac{B^D}{\sum_{Q<D} \mathrm{Tr}_Q^D B^Q},$$

where $B^Q = \{b \in B | {}^u b = b \ \forall u \in Q\}$. Then $\mathrm{Br}_D = \mathrm{Br}_D^B$ is a surjective k-algebra homomorphism.

6.2.9. Categorical equivalences.

Character isometries are often the result of the existence of a Morita or Richard equivalence of blocks. Koshitani and Michler in 2001 in their article [37] and Harris and Linckelmann in 2002 in their article [31] showed that with some additional conditions there exists a Morita equivalence between $b\mathcal{O}K$ and $w(b)\mathcal{O}K^A$, where b is a block of $\mathcal{O}K$ that is A-invariant and $w(b)$ the coresponding block of $\mathcal{O}K^A$.

Theorem 6.2.10. *With the hypothesis of Watanabe's theorem there exists a Morita equivalence between $b\mathcal{O}K \simeq w(b)\mathcal{O}K^A$ induced by a $(b\mathcal{O}K, w(b)\mathcal{O}K^A)$-bimodule M with the property that regarded as a $K \times K^A$-module,M has a source W that is a endo-permutation $\mathcal{O}D$-module. Moreover, this Morita equivalence induces a bijection*

$$\pi(K, A) : \mathrm{Irr}_{\mathcal{K}}(K, b) \to \mathrm{Irr}_{\mathcal{K}}(K^A, w(b))$$

under the condition that either K is p-solvable and $(\mathcal{K}, \mathcal{O}, k)$ is big enough (by [31]) or $D \trianglelefteq K$ and $(\mathcal{K}, \mathcal{O}, k)$ is big enough (by [37]) orb is a nilpotent block and $(\mathcal{K}, \mathcal{O}, k)$ is big enough (by [54]) and some other additional conditions.

By induction, Theorem 6.2.10 is reduced to the case when we have a block of a normal p'-subgroup of K, actually $O_{p'}(K)$. Hence the group algebra $\mathcal{O}O_{p'}(K)$ is semisimple.

6.2.11. More generally, instead of the normal p'-subgroup it is useful to consider blocks of defect zero of a normal subgroup (see [31]). For this one can consider the case of [29], [6], [66], [40] and [75, Paragraph 2.4] when $K \trianglelefteq G$ and b a block with defect zero of $\mathcal{O}K$ where $(\mathcal{K}, \mathcal{O}, k)$ is "big enough" (see the articles of Dade or Harris), $\mathcal{O} = \mathbb{Z}_p$, $\mathcal{K} = \mathbb{Q}_p$ and $k = \mathbb{F}_p$.

6.3 Graded Morita equivalence between algebras

Considering that the theorem (by Glauberman and Isaacs, Theorem 6.2.3) and by Watanabe's theorem refer to solvable groups, by induction their proofs are reduced to a special case. This situation refers to the case when we have a character that is from a p-block with defect zero of a normal subgroup. In this section we present this context following the articles of Dade [6], Turull [66] and Ladisch [40]. Let G be a finite group, $K \trianglelefteq G$ and $K \trianglelefteq M \leq G$ such that M/K is a p-group. Let $\theta \in \mathrm{Irr(K)}$. Then $\theta : K \to \mathbb{C}$ and $\theta(x)$ is an algebraic integer. Let us assume that θ is in a p-block of K having defect 0.

Let $k = \mathbb{F}_p$ the field with p elements, $\mathcal{K} = \mathbb{Q}_p$ the p-adic number field and $\mathcal{O} = \mathbb{Z}_p$ the ring of p-adic integers. Hence, $(\mathcal{K}, \mathcal{O}, k)$ is a p-modular system. Let $\hat{\mathcal{K}} = \mathcal{K}(\theta)$, let $\hat{\mathcal{O}} = \mathcal{O}(\theta)$ and $\hat{k} = \mathbb{Z}_p(\theta)/p\mathbb{Z}_p(\theta)$. Hence, $(\hat{\mathcal{K}}, \hat{\mathcal{O}}, \hat{k})$ is a p-modular system.

6.3.1. The p-block of θ.

The p-block that contains θ can be viewed as follows. There exists a central primitive idempotent $e_\theta \in Z(\hat{\mathcal{O}}K)$ associated to θ. Then $e_\theta\hat{\mathcal{K}}K$ is a central simple $\hat{K}$-algebra. Moreover, we have that $e_\theta\hat{\mathcal{K}}K \simeq M_n(\hat{\mathcal{K}})$ and θ is a (complex) character associated to the unique simple $e_\theta\hat{\mathcal{K}}K$-module $\hat{V}$. If we denote by $\bar{\mathcal{K}}$ the algebraic closure of $\hat{\mathcal{K}}$ then the module θ appears in the decomposition of $\bar{\mathcal{K}} \otimes_{\hat{\mathcal{K}}} \hat{V}$. In this case we have

$$\hat{G} := \mathrm{Gal}(\hat{\mathcal{K}}/\mathcal{K}) \simeq \mathrm{Gal}(\hat{\mathcal{O}}/\mathcal{O}) \simeq \mathrm{Gal}(\hat{k}/k).$$

Moreover, there exists a primitive idempotent $e_{\theta,\mathcal{O}} \in Z(\mathcal{O}K)$ such that $e_{\theta,\mathcal{O}}\mathcal{K}K$ is a simple $\mathcal{K}$-algebra with the center $\hat{\mathcal{K}}$, θ is a component of $V \otimes_{\mathcal{K}} \hat{\mathcal{K}}$ and $e_{\theta,\mathcal{O}}\mathcal{K}K \simeq M_m(\hat{\mathcal{K}})$. Actually, $V \otimes_{\mathcal{K}} \hat{\mathcal{K}}$ is a sum of $\hat{G}$-conjugates of θ.

Remark 6.3.2. $\hat{\mathcal{K}}$ is a splitting field for $e_{\theta,\mathcal{O}}\mathcal{K}K$.

6.3.3. Let $K \trianglelefteq G$, $K \trianglelefteq M \leq G$ and M/K be a p-group. Let $\theta \in \mathrm{Irr}(K)$.

To the character θ corresponds a unique primitive central idempotent $e_{\theta,\mathcal{O}} \in Z(\mathcal{K}K)$ but one can prove that this idempotent is actually in $\mathcal{O}K$. Then we have that $e_{\theta,\mathcal{O}} \in Z(\mathcal{O}K)$ is a primitive idempotent. Moreover,

$$e_{\theta,\mathcal{O}}\mathcal{K}K \simeq M_m(\hat{\mathcal{K}}) \quad \text{and} \quad e_{\theta,\mathcal{O}}\mathcal{O}K \simeq M_n(\hat{\mathcal{O}}).$$

Actually, $e_{\theta,\mathcal{O}}\mathcal{K}K \simeq \mathcal{K} \otimes_{\mathcal{O}} e_{\theta,\mathcal{O}}\mathcal{O}K$. Assume that $e_{\theta,\mathcal{O}}$ is G-invariant, so θ is G-semi-invariant. $e_{\theta,\mathcal{O}}$ has a defect group $P \leq G$ with the property that $Q := P \cap K$ is a defect group of $e_{\theta,\mathcal{O}}$ in K. Because θ has defect group zero we have that $Q = \{1\}$. Denote $M := KP = PK \leq G$.

We consider the Brauer surjective homomorphism

$$\mathrm{Br}_p : (\mathcal{O}K)^p \to kC_K(P)$$

of $\mathcal{O}$-algebras defined as follows

$$\sum_{x\in K}\alpha_x x\longmapsto\sum_{x\in C_K(P)}\bar{\alpha}_x x,$$

where $\alpha\in\mathcal{O}$ is mapped to $\bar{\alpha}\in k=\mathcal{O}/J(\mathcal{O})$ by Br_p. In this case $e_{\theta,\mathcal{O}}$ determines a unique bloc $e_{\varphi,\mathcal{O}}\in Z(\mathcal{O}C_K(P))$ such that $e_{\varphi,\mathcal{O}}\in Z(kC_K(P))$. Hence, the character θ determines a character φ of $C_K(P)$, actually $e_{\varphi,\mathcal{O}}C_K(P)$ is also a central simple $\hat{K}$-algebra having a unique simple module W and φ is a component of $\hat{\mathcal{K}}\otimes_{\mathcal{K}}W$. Also, $e_{\varphi,\mathcal{O}}$ has defect group P in H and defect zero in $C_K(P)=L$ as well. Using the so called Frattini argument we have that $G/K\simeq H/L$. Then we have the G/K-graded algebras $e_{\theta,\mathcal{O}}\mathcal{O}G$ and $e_{\varphi,\mathcal{O}}\mathcal{O}H$.

The next theorem is the main theorem of this section and is due to E.Dade from [6], A.Turull from [66] and F.Ladisch from [40].

Theorem 6.3.4. *There exists a G/K-graded Morita equivalence between the algebras $e_{\theta,\mathcal{O}}\mathcal{O}G$ and $e_{\varphi,\mathcal{O}}\mathcal{O}H$.*

A proof of this theorem in a slightly more general framework considering some central simple $\mathcal{O}$-algebras on which some p-groups act can be given using Section 6.6 .

Remark 6.3.5. If K is a p'-group (hence by Maschke's theorem all blocks of $\mathcal{O}K$ have p-defect zero), then the correspondence $\theta\mapsto\varphi$ described above coincides with the Glauberman correspondence.

Remark 6.3.6. 1) In this case if we denote by $R:=e_{\theta,\mathcal{O}}\mathcal{O}G$ and $S:=e_{\varphi,\mathcal{O}}\mathcal{O}H$, then $R_1=e_{\theta,\mathcal{O}}\mathcal{O}K$ and $S_1=e_{\varphi,\mathcal{O}}\mathcal{O}L$.

2) Because $P\cap K=\{1\}$ we have that $PK/K\simeq P$, hence P can be viewed as a subgroup of G/K.

6.4 Modular group graded algebras

We consider a p-modular system $(\mathcal{K},\mathcal{O},k)$, where k is a perfect field. An important particular case is when $\mathcal{K}=\mathbb{Q}_p$, $\mathcal{O}=\mathbb{Z}_p$ and $k=\mathbb{F}_p$.

Our main objects of study are G-graded crossed products $R=\bigoplus_{g\in G}R_g$, where R is assumed to be free of finite rank over $\mathcal{O}$, so G is a finite group. We have an exact sequence of groups

$$1\to R_1^\times\to \mathrm{hU}(R)\to G\to 1,$$

where $\mathrm{hU}(R)$ denoted the group of homogeneous units of R. We denote $A:=R_1$.

Remark 6.4.1. Let K be a normal subgroup of H, $G=H/K$ and let b a block of $\mathcal{O}K$. Then we take $R=b\mathcal{O}H$ and $R_1=b\mathcal{O}K$. Then H acts on $\mathcal{O}K$ by conjugation, while

H/K acts on $Z(\mathcal{O}K)$. We denote by G_b the stabilizer of G. In this case the algebra $b\mathcal{O}H_b$ is Morita equivalent to $\mathcal{O}Gb\mathcal{O}G$, and $b\mathcal{O}H_b = b\mathcal{O}Hb$, so we can assume without much loss of generality that b is G-invariant. We will no longer work with K and H and refer only to R, G and the defect groups.

6.4.2. We will frequently use the following construction from [52, Chapter 9], which gives a bijection between K-interior H-algebras and G-graded H-interior algebras, where H is a group, K is a normal subgroup of H, $G = H/K$, and $\mathcal{O}H$ is regarded as a G-graded $\mathcal{O}$-algebra in an obvious way (see also [16, Section 2]).

As in [52, 4.2], a K-interior H-algebra is an $\mathcal{O}$-algebra A with group homomorphisms $\varphi : H \to \mathrm{Aut}(A)$ and $\psi : K \to A^\times$ such that, for any $x \in H$, $y \in K$ and $a \in A$, we have $(y \cdot a)^x = y^x \cdot a^x$ and $a^y = y^{-1} \cdot a \cdot y$, where $y \cdot a$ and $a \cdot y$ denote $\psi(y)a$ and $a\psi(y)$ respectively, and $a^x := \varphi(x)^{-1}(a)$. Then A determines a G-graded $\mathcal{O}$-algebra $R := \bigoplus_{g\in G} R_g$ by letting

$$R := A \otimes_{\mathcal{O}K} \mathcal{O}H = \bigoplus_{x\in[H/K]} A \otimes x,$$

and there exists a homomorphism $\psi : \mathcal{O}H \to R$, of G-graded algebras.

Conversely, if $\psi : \mathcal{O}H \to R$ is a homomorphism of G-graded $\mathcal{O}$-algebras, then $A := R_1$ is a K-interior H-algebra, where

$$\varphi : H \to \mathrm{Aut}(A), \qquad \varphi(h)(a) = \psi(h)a\psi(h)^{-1},$$

and $\psi : K \to A^\times$ is the restriction of ψ.

6.4.3. We denote by $J_{\mathrm{gr}}(R)$ the Jacobson radical of the crossed product R, and let $\bar{R} = R/J_{\mathrm{gr}}(R)$. We need a connection between the splittings of the group extension $\mathrm{hU}(R)$ and the splittings of $\mathrm{hU}(\bar{R})$. A generalization of E. Dade, proved in [44, Theorem 3.1.8] states that if the extension $\mathrm{hU}(\bar{R})$ of $\bar{A}^\times$ by G splits and there is $\bar{a} \in \bar{A}$ such that $\mathrm{Tr}_1^G(\bar{a}) = 1$ then there is a bijection between the splittings of $\mathrm{hU}(\bar{R})$ and the $(1 + J(A))$-conjugacy classes of splittings of $\mathrm{hU}(R)$.

6.4.4. Let P be a p-group. Recall that a kP-module M is called *endopermutation* if $\mathrm{End}_k(M)$ has a P-stable basis. By [66, Theorem 3.3], if $\hat{k}/k$ is a field extension and M is an endopermutation $\hat{k}P$-module, then there is an endopermutation kP-module M_0 such that $M \simeq \hat{k} \otimes_k M_0$.

Theorem 6.4.5 (Dade)**.** *Let R be a G-graded crossed product such that $R_1 = kP$, and let M be a G-invariant indecomposable endopermutation kP-module. Let $E = \mathrm{End}_R(R \otimes_{R_1} M)^{\mathrm{op}}$ and $\bar{E} = E/J_{\mathrm{gr}}(R)$. Then the following group extension splits:*

$$1 \to \bar{E}_1^\times \to \mathrm{hU}(\bar{E}) \to G \to 1.$$

6.5 The "normal defect group" situation

The main result of this section is a generalization of the structure theorem for blocks with normal defect group, originally due to Külshammer [38]. There is another approach in [2] using modules, and here we generalize the main result of [17], by establishing a Morita equivalence between strongly graded algebras. We lay down our assumptions in 6.5.1 and 6.5.2, while the algebras in discussion are defined after several steps in 6.5.5 and 6.5.8 below.

6.5.1. Let B be a interior $\mathcal{O}D$-algebra, free of finite rank over $\mathcal{O}$, having a D-stable basis. We suppose that 1 is a primitive idempotent in $Z(B)$, and that B has defect group D. We also assume that ${}_{\mathcal{O}D}B$ and $B_{\mathcal{O}D}$ are projective.

6.5.2. There exists a primitive idempotent $i \in B^D$ such that $B \mid Bi \otimes_{\mathcal{O}D} iB$ as (B, B)-bimodules. Let

$$\gamma = \left\{ aia^{-1} \mid a \in (B^D)^\times \right\}$$

the $(B^D)^\times$-conjugacy class of i. Then the pair $(D, \gamma) = D_\gamma$ is called a *defect pointed group of* B (a notion due to Puig, see [60]). Moreover, $\mathrm{Br}_D(i)$ is a primitive idempotent in $B(D)$, and $\mathrm{Br}_D(\gamma)$ is a point of $B(D)$, because $\mathrm{Br}_D : B^D \to B(D)$ is surjective.

There is a unique maximal ideal of B^D, denoted $\mathfrak{m}_\gamma$, that corresponds to γ such that $\gamma \not\subset \mathfrak{m}_\gamma$, and a unique maximal ideal of $B(D)$, denoted $\mathfrak{m}_{\mathrm{Br}_D(\gamma)}$, that corresponds to $\mathrm{Br}_D(\gamma)$ such that $\mathrm{Br}_D(\gamma) \not\subset \mathfrak{m}_{\mathrm{Br}_D(\gamma)}$. Moreover, we have that

$$B^D/\mathfrak{m}_\gamma \simeq B(D)/\mathfrak{m}_{\mathrm{Br}_D(\gamma)},$$

and we denote by S the simple k-algebra $B(D)/\mathfrak{m}_{\mathrm{Br}_D(\gamma)}$.

Let $\bar{V}$ be the unique (up to a isomorphism) simple S-module. There is a unique central primitive idempotent $e_\gamma \in Z(B(D))$ such that $e_\gamma \mathrm{Br}_\gamma(i) \neq 0$ and the image of $e_\gamma \in B(D)$ via the canonical map $B(D) \to S$ is actually the identity of S.

We will assume that S has Schur index 1, that is, $\hat{k} := \mathrm{End}_S(\bar{V})$ is a field. We have that $\hat{k} = Z(S)$, and $S \simeq \mathrm{End}_{\hat{k}}(\bar{V}) \simeq M_m(\hat{k})$, where $m = \dim_{\hat{k}} \bar{V}$.

6.5.3. Let $C_{B^\times}(D)$ and $N_{B^\times}(D)$ be the centralizer in of D in $B^\times$ and the normalizer of D in $B^\times$, respectively. Then $C_{B^\times}(D)$ is a normal subgroup of $N_{B^\times}(D)$, and we denote

$$G := N_{B^\times}(D)/C_{B^\times}(D) \quad \text{and} \quad \bar{G} := N_{B^\times}(D)/DC_{B^\times}(D).$$

Note that $N_{B^\times}(D)$ acts on B^D and on $B(D)$ as algebra automorphisms. Moreover, we have the following maps that are compatible with the action of $N_{B^\times}(D)$:

$$C_{B^\times}(D) \hookrightarrow B^D \to B(D).$$

In this setting we can construct the G-graded crossed product denoted $B(D) * G$.

6.5.4. It is clear that if $a \in N_{B^\times}(D)$, then aia^{-1} remains a primitive idempotent in B^D. Let

$$N_{B^\times}(D)_\gamma := \{a \in N_{B^\times}(D) \mid a\gamma a^{-1} = \gamma\}$$

be the stabilizer of γ in $N_{B^\times}(D)$. Clearly, $C_{B^\times}(D) \subseteq N_{B^\times}(D)_\gamma$, and let $G_\gamma := N_{B^\times}(D)/C_{B^\times}(D)$ be the stabilizer of γ in G. The stabilizer of e_γ in $N_{B^\times}(D)$ coincides with $N_{B^\times}(D)_\gamma$. We also have that $D \subseteq N_{B^\times}(D)_\gamma$, and we denote

$$\bar{G}_\gamma := N_{B^\times}(D)_\gamma/DC_{B^\times}(D) \quad \text{and} \quad \bar{D} := D/Z(D) \simeq DC_{B^\times}(D)/C_{B^\times}(D).$$

Because the map $D \to B^\times$ is injective, we have that $Z(D) = D \cap C_{B^\times}(D)$, so $\bar{G}_\gamma = G_\gamma/\bar{D}$.

6.5.5. Notice that all the homomorphisms of the diagram

$$\begin{tikzcd} B^D \arrow[r, "\mathrm{Br}_D"] \arrow[d] & B(D) \arrow[d] \\ B^D/\mathfrak{m}_\gamma \arrow[r] & B(D)/\mathfrak{m}_{\mathrm{Br}(\gamma)} = S \end{tikzcd}$$

are $N_{B^\times}(D)_\gamma$-algebra homomorphisms. Observe that $e_\gamma B(D)$ is a $C_{B^\times}(D)$-interior $N_{B^\times}(D)$-acted k-algebra, so as in 6.4.2 we may construct the strongly G_γ-graded crossed product $R := e_\gamma B(D) * G$, where $R_1 := e_\gamma B(D)$.

In addition, since $\hat{k}$ is a perfect field and D is a p-group, there is a group homomorphism $\sigma : D \to S^\times$ such that for all $u \in D$ and all $s \in S$ we have ${}^u s = \sigma(u)s\sigma(u)^{-1}$, hence the D-algebra S is actually a interior D-algebra, Consequenty, S is a $DC_{B^\times}(D)$-interior $N_{B^\times}(D)_\gamma$-acted $\hat{k}$-algebra, and again as in 6.4.2 we can construct the $\bar{G}_\gamma$-graded crossed product k-algebra $\bar{R} := S * \bar{G}_\gamma$.

We will assume that γ is G-invariant, that is, $G = G_\gamma$, because in general, passing from G to G_γ is done by a Morita equivalence.

Remark 6.5.6. When we start with a block B of $\mathcal{O}G$, then there exist the maps

$$\mathcal{O}C_G(D) \hookrightarrow \mathcal{O}G \xrightarrow{\mathrm{Br}_D} kC_G(D),$$

and we have that $e_\gamma B(D) = kC_G(D)e_\gamma$.

6.5.7. The group $\bar{G}$ acts on $\hat{k}$, so we have a group homomorphism

$$\theta : \bar{G} \to \mathrm{Gal}(\hat{k}/k).$$

Let $K \leq \bar{G}$ the kernel of θ. By hypothesis, the extension $\mathcal{O}D \to B$ is separable, hence

$\bar{e}_\gamma \in \mathrm{Tr}_1^{\bar{G}}(\hat{k})$ and $\bar{e}_\gamma \in Z(S) = \hat{k}$. It follows that $\hat{k}$ is a projective module over a group algebra $\hat{k}K$, so by Maschke's theorem $p \nmid |K|$.

The group $N_{B^\times}(D)$ also acts by conjugation on D, and $C_{B^\times}(D)$ acts trivially and we have the group homomorphisms

$$G \to \mathrm{Aut}(D) \to \mathrm{Out}(D), \quad \text{and} \quad \bar{G} \to \mathrm{Out}(D).$$

Notice that this implies that the group $\bar{G}$ (and hence G) is finite. The action of $N_{B^\times}(D)$ on D and on $\hat{k}$ induces the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \bar{D} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \bar{G} \arrow[r] \arrow[d] \arrow[dl, dashed, "\sigma"'] & 1 \\
1 \arrow[r] & \mathrm{Int}(\hat{k}D) \arrow[r] & \mathrm{Aut}_k(\hat{k}D) \arrow[r] & \mathrm{Out}_k(\hat{k}D) \arrow[r] & 1,
\end{tikzcd}$$

By [20, Corollary 3.13], there exists a group homomorphism $\sigma : \bar{G} \to \mathrm{Aut}_k(\hat{k}D)$ that lifts the homomorphism $\bar{G} \to \mathrm{Out}_k(\hat{k}D)$.

6.5.8. Since we have assumed $\bar{V}$ to be $\bar{G}$-invariant, by Clifford theory we have the isomorphism of $\bar{G}$-graded algebras

$$\mathrm{End}_{\bar{R}}(\bar{R} \otimes_S \bar{V})^{\mathrm{op}} \simeq \hat{k}^\theta_\beta \bar{G},$$

where $\hat{k}^\theta_\beta \bar{G}$ is a $\bar{G}$-graded crossed product of $\hat{k}$ and $\bar{G}$ determined by the 2-cocycle $\beta : \bar{G} \times \bar{G} \to \hat{k}^\times$ and the action $\theta : \bar{G} \to \mathrm{Gal}(\hat{k}/k)$. In this case we have a $\bar{G}$-graded Morita equivalence between $\bar{R} = S * \bar{G}$ and $\hat{k}^\theta_\beta \bar{G}$, and moreover the isomorphism $\bar{R} \simeq S \otimes_{\hat{k}} (\hat{k}^\theta_\beta \bar{G})$ of G-graded algebras takes $s\bar{g}$ to $s \otimes_{\hat{k}} g$. By using the homomorphism $\sigma : \bar{G} \to \mathrm{Aut}_k(\hat{k}D)$, we can construct the strongly $\bar{G}$-graded crossed product $(\hat{k}D)^\sigma \bar{G}$, with $\bar{1}$-component $\hat{k}D$.

In fact, $(\hat{k}D)^\sigma_\beta \bar{G}$ can be viewed as a strongly G-graded algebra with the 1-component $\hat{k}Z(D)$, which means that we refine the grading by viewing $\hat{k}D$ as a $\bar{D}$-graded algebra. Let $R' := (\hat{k}D)^\sigma_\beta \bar{G}$, viewed as a G-graded algebra, with 1-component $R'_1 = \hat{k}Z(D)$.

Theorem 6.5.9. *There exists a G-graded Morita equivalence between $R = e_\gamma B(D) * G$ and $R' = (\hat{k}D)^\sigma_\beta \bar{G}$.*

6.6 G-graded $\mathcal{O}P$-interior algebras

6.6.1. Let R be a G-graded $\mathcal{O}$-algebra as in Section 2. The assumptions in this section are as follows:

(1) R is a crossed product of $A := R_1$ and G, hence we have the exact sequence

$$1 \to A^\times \to \mathrm{hU}(R) \to G \to 1.$$

(2) A is a simple k-algebra, and denoting $\hat{k} := Z(A)$, $\hat{k}$ is a Galois extension of k, that is, A has Schur index 1. Denote by V the unique (up to isomorphism) simple A-module.

(3) There exists a finite p-group P and a unital homomorphism $\varphi : \mathcal{O}P \to$ such that ${}_{\mathcal{O}P}R$ and $R_{\mathcal{O}P}$ are projective modules (hence free, so in particular, φ is injective).

(4) $\varphi(P) \cap A^\times = \{1\}$, hence we have the commutative diagram

$$\begin{tikzcd}
 & & P \arrow[r, equal] \arrow[dl, dashed, "\psi"'] \arrow[d] & P \arrow[d] & \\
1 \arrow[r] & A^\times \arrow[r] & \mathrm{hU}(R) \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

Assume that P is a normal subgroup of $\mathrm{hU}(R)$, hence also of G, and we denote $\bar{G} := G/P$.

(5) Let $Q \le P$ be a defect group of kA. We assume that regarded as a central simple $\hat{k}$ algebra, kA is a Dade Q-algebra.

6.6.2. Note that because k is perfect, there exists a unique group homomorphism $\psi : P \to kA^\times$ such that $\det \psi(u) = 1$, and inducing the action of P on A. Obviously ψ extends to an algebra homomorphism $\psi : \mathcal{O}P \to A$, hence we can view A as a interior P-algebra. Then the map $\mathcal{O}Q \to A$ splits as a bimodule map, and the Brauer quotient $A(Q)$ is not zero.

6.6.3. The group $\mathrm{hU}(R)$ acts on the simple algebra kA and on $\hat{k} = Z(kA)$, and moreover, $P(kA)^\times$ acts trivially on $\hat{k}$. So we have a group homomorphism

$$\theta : \bar{G} \to \mathrm{Gal}(\hat{k}/k),$$

and denote by K be the kernel of θ.

6.6.4. Consider the normalizers and centralizers $N_{A^\times}(Q)$, $N_{\mathrm{hU}(R)}(Q)$, $C_{A^\times}(Q)$ and $C_{\mathrm{hU}(R)}(Q)$. Then there exists a group homomorphism $C_{A^\times}(Q) \to A(Q)^\times$, and moreover, this map, and the Brauer homomorphism $A^Q \to A(Q)$ are compatible with the conjugation action of $N_{\mathrm{hU}(R)}(Q)$ on these objects. Denote

$$G' := N_{\mathrm{hU}(R)}(Q)/C_{A^\times}(Q),$$

so G' can be naturally regarded as a subgroup of G. We can now construct, as in 6.4.2, the G'-graded crossed product $R' := A(Q) * G'$, with 1-component $A' := A(Q)$.

Theorem 6.6.5. *Assume that $G' = G$. Then there is a G-graded Morita equivalence over k between kR and R'.*

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