Babeş-Bolyai University
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Contributions in the geometric function theory in $\mathbb{C}^n$

Ph.D. Thesis - Summary

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Contents

Introduction

In this thesis we present general and original results in the geometric function theory of one and several complex variables. One of the main directions in the geometric function theory of one complex variable is the theory of univalent functions. Bieberbach (1916) [6] proved the sharp second coefficient bound for the class S of normalized univalent functions on the unit disc U, and formulated his well known conjecture regarding the coefficient bounds for functions in S. Bieberbach's conjecture was solved by L. de Branges [7] in 1985.

We also mention here main results obtained by the Romanian mathematical school of univalent functions, starting with G. Călugăreanu [8]. He obtained the first necessary and sufficient conditions of univalence for holomorphic functions of one complex variable. On the other hand, S.S. Miller and P.T. Mocanu [80] obtained fundamental results regarding differential subordinations in the complex plane with various applications to univalent functions.

A fundamental result in the theory of univalent functions of one complex variable is the Riemann mapping theorem which provides the conformally equivalence of simply connected domains in $\mathbb{C}$ (see e.g. [44], [57]). However, this result does not hold in $\mathbb{C}^n$, $n \geq 2$ (see e.g. [44]). This shows a basic difference between the one variable theory and that in higher dimensions.

One of the most important tools in the study of univalent functions is the theory of Loewner chains. It is well known that the Loewner chains are basically determined by the Loewner differential equation. We recall that the Loewner differential equation played a main role in the proof of the Bieberbach conjecture, due to L. de Branges [7] (for details, see also [58]). The Loewner theory was also useful in proving several results regarding univalent functions, such as the radius of starlikeness of the class S, analytical characterizations of starlikeness, spirallikeness, convexity, and close-to-convexity in terms of Loewner chains, and univalence criteria on the unit disc U. Pommerenke [93] proved that any function in the class S can be embedded as the first element of a Loewner chain. Also, any function $f \in S$ has parametric representation on U (see [93]). However, these results are not true in higher dimensions. Hence, there exist main differences between the theory of Loewner chains in one and several complex variables (see for example [44, Chapter 8]).

We remark that there exist a number of monographs which refer to various aspects of the theory of univalent functions of one complex variable, including Loewner chains and the Loewner differential equation. We mention here the monographs of Duren [26], Pommerenke [93], Hayman [58], Graham and Kohr [44], Mocanu, Bulboacă and Sălăgean [81]. We also refer to the books of Conway [20], Rosenblum and Rovnyak [103], which contain some classical results related to univalent functions and Loewner chains in the complex plane.

The study of the class $S(B^n)$ of normalized biholomorphic mappings f on the Euclidean unit ball B^n in $\mathbb{C}^n$, $n \geq 2$, was considered by Cartan [9] in 1933. He proved that $S(B^n)$ is not locally uniformly bounded, and thus it is not compact, and there do not exist growth and distortion theorems for the full class $S(B^n)$ (see [9], [29], [44]). Hence there is a basic difference between the theory of univalent functions on U and that on the unit ball in $\mathbb{C}^n$ for $n \geq 2$ (for details, see [44, Chapter 6]). Moreover, the Riemann mapping theorem does not hold in several complex variables. Poincaré [92] proved that the Euclidean unit ball and the unit polydisc in $\mathbb{C}^n$ are not biholomorphically equivalent for $n \geq 2$, although they are homeomorphic.

Cartan [9] found some examples which yield that the Bieberbach conjecture does not have any correspondence to the full class $S(B^n)$, for $n \geq 2$. On the other hand, Cartan [9] conjectured that there are lower and upper estimates regarding $|\det Df(z)|$ for $f \in S(B^n)$, which depend only on $z \in B^n$ (see [9]). Duren and Rudin [29] proved that this conjecture is not true when $n \geq 2$.

Moreover, Cartan [9] recommended the study of starlike and convex mappings in higher dimensions. The first papers related to starlikeness in higher dimensions are due to Matsuno (1955) [79] on the Euclidean unit ball, and Suffridge [108] (1970) on the unit polydisc. Suffridge [109] and Gurganus [49] obtained generalizations of the classical characterization of starlikeness on the unit disc to the unit ball of a complex Banach space. Other necessary and sufficient conditions of starlikeness on bounded balanced pseudoconvex domains in $\mathbb{C}^n$ were obtained by Kikuchi [60], Gong (see [32] and the references therein), Liu [74].

Later contributions were given by Kubicka and Poreda [71], Barnard, FitzGerald and Gong [4], who deduced the sharp growth result for normalized starlike mappings on the unit ball in $\mathbb{C}^n$. Other contributions related to coefficient bounds and distortion results for normalized starlike mappings on the unit ball in $\mathbb{C}^n$ may be found in [44] and the references therein.

Necessary and sufficient conditions of convexity were obtained by Suffridge (1970) [108] on the unit polydisc in $\mathbb{C}^n$, then by Kikuchi [60], Gong, Wang and Yu [36] on the Euclidean unit ball in $\mathbb{C}^n$. Later contributions refer to growth results for normalized convex mappings on B^n, due to Suffridge [111], FitzGerald and Thomas [31], and Liu [74]. On the other hand, sharp coefficient bounds for normalized convex mappings on the unit ball in $\mathbb{C}^n$ were obtained by FitzGerald and Gong [31], Kohr [65] (see also [32] and [44] and the references therein). Distortion results related to estimates of $\|Df(z)\|$ for $f \in K(B^n)$ (the class of normalized convex mappings on the Euclidean unit ball B^n) were obtained by Liczberski and Starkov [72] (see also [32] and the references therein).

Various aspects related to starlikeness, convexity, spirallikeness in $\mathbb{C}^n$ may be found in [32], [36], [38], [44], [60], [88], [96], [101], [102], [110].

Many results in the theory of Loewner chains were generalized to higher dimensions. This subject was initiated by Pfaltzgraff ([88], [89]) who obtained generalizations on the Euclidean unit ball B^n in $\mathbb{C}^n$ of one variable univalence and quasiconformal extension results due to Becker [5]. Poreda ([94], [95]) obtained some applications of parametric representation to growth theorems and coefficient estimates on the unit polydisc in $\mathbb{C}^n$. He also deduced certain generalizations on the unit ball of a complex Banach space (see [96]). The existence and regularity of the theory of Loewner chains in higher dimensions, including geometric and analytical aspects, were considered by Duren, Graham, Hamada and Kohr [27], Graham, Hamada, Kohr [38], Graham, Hamada, Kohr,

Kohr [40], Graham, Kohr and Kohr [46], Hamada [51], Curt and Kohr [24], Poreda ([94], [95]; see also [96]), Arosio [3], Voda [112]. Sufficient conditions of univalence on the unit ball in $\mathbb{C}^n$ were considered by Curt and Pascu [25] (see also [23]; see e.g. [44, Chapter 8] and the references therein), Cristea [21], etc. Various applications related to parametric representation on the unit ball in $\mathbb{C}^n$, including growth, coefficient bounds, embedding results, were obtained in [19], [38], [48], [83], [114], etc.

On the other hand, in higher dimensions there are many differences compared to the one variable Loewner theory (see [44, Chapter 8]). For example, Graham, Hamada and Kohr [38] (see also Poreda [94]) proved that in higher dimensions, there exist normalized biholomorphic mappings f on the unit ball B^n in $\mathbb{C}^n$ which cannot be embedded in Loewner chains $f(z,t)$ such that $f = f(\cdot, 0)$ and $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family. Also, there exist mappings $f \in S(B^n)$ which do not have parametric representation on B^n, $n \geq 2$. On the other hand, in the case of one complex variable, the Loewner differential equation has a unique normalized univalent solution (see [93]). However, in higher dimensions the above result does not hold (see [38]). The study of the form of general solutions to the Loewner differential equation in higher dimensions was considered by Duren, Graham, Hamada and Kohr [27] (see also [3], [51], [112]).

The compactness of the Carathéodory family $\mathcal{M}$ has influenced many results in the theory of Loewner chains in higher dimensions (see [44]). This result was obtained by Graham, Hamada and Kohr [38] in 2002 (see also [55]). Many details and applications of the theory of Loewner chains in several complex variables may be found in the papers [1], [27], [38], [40], [88], [89], [96] (see also [44, Chapter 8] and [23]).

This thesis is divided into five chapters.

- **Chapter 1** gives general results regarding univalent functions of one complex variable. These results are useful in the forthcoming chapters of this thesis. All of these results are presented without proofs. Section 1.1 contains notions and preliminary results regarding holomorphic functions which are used throughout the thesis. Section 1.2 presents the notion of subordination in the complex plane and gives some important properties of holomorphic functions in the unit disc U with positive real part. In Section 1.3 we present some classical and well known properties of univalent functions of one complex variable. We recall the notion of conformal equivalence and we present the Riemann mapping theorem, one of the most significant results in the theory of univalent functions.

  In Section 1.4 we refer to various subclasses of univalent functions on the unit disc. We present the class S of normalized univalent functions on U, the class S^* of normalized starlike functions with respect to the origin, the class K of normalized convex functions. We also refer to the class of close-to-convex functions, the classes of starlike and convex functions of order α, the class of spirallike functions of type γ and the class of almost starlike functions of order α. We shall give coefficient estimates, growth, distortion and covering theorems for these classes. In Section 1.5 we are concerned with some classical results in the theory of Loewner chains and the Loewner differential equation on the unit disc in $\mathbb{C}$. We also present certain applications of the Loewner theory to the study of univalent functions, including univalence criteria and analytical characterizations of important subsets of S in terms of Loewner chains. The results in this section will be useful in the proofs of the main results of this thesis.

- **Chapter 2** gives basic properties of holomorphic functions and holomorphic mappings in $\mathbb{C}^n$. These results are useful in the forthcoming chapters of this thesis and they are presented without proofs. Section 2.1 is devoted to basic properties of holomorphic mappings in the case of several complex variables. Section 2.2 presents the notion of subordination in several complex variables and the generalization of the Carathéodory class $\mathcal{P}$ to $\mathbb{C}^n$, denoted by $\mathcal{M}$. We will give growth and coefficient bounds for the class $\mathcal{M}$ and we include some examples of mappings in the class $\mathcal{M}$. We also present the compactness result related to the class $\mathcal{M}$.

  Section 2.3 is devoted to some subclasses of biholomorphic mappings on the unit ball in $\mathbb{C}^n$. We refer to the classes of normalized starlike, convex, close-to-starlike, and spirallike mappings, respectively. We present analytical characterizations, growth theorems and coefficient bounds, and we give examples of mappings in these classes. We also refer to some subclasses of $S(B^n)$ such as: the class of starlike mappings of order α, the class of ε-starlike mappings, and the class of almost starlike mappings of order α. We shall see that most of these subclasses of $S(B^n)$ have analytical and geometric characterizations. This section does not contain original results, however Definition 2.3.7 is due to Chirilă [12].

  In Section 2.4 we consider generalizations of the Loewner differential equation to higher dimensions. We shall present basic results of Loewner's theory in $\mathbb{C}^n$ and we give various applications, such as univalence criteria and analytical characterizations of certain subclasses of $S(B^n)$ using Loewner chains. Also, we present the class $S_g^0(B^n)$ of mappings with g-parametric representation on the unit ball B^n, where g is a univalent function on the unit disc U that satisfies certain natural assumptions. We shall consider growth and coefficient bounds for mappings in $S_g^0(B^n)$, and we present particular cases of interest. We shall see that all Loewner chains on the unit ball B^n in $\mathbb{C}^n$ are basically determined by the generalized Loewner differential equation (see e.g. [76]). Also, we shall see that each Loewner chain $f(z,t)$ such that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family is generated by its transition mapping (see [41]).

The study of extension operators which take a univalent function f on the unit disc U to a univalent mapping F from the Euclidean unit ball B^n in $\mathbb{C}^n$ into $\mathbb{C}^n$, with the property that $f(z_1) = F(z_1, 0)$, began with the Roper-Suffridge extension operator [101]. This operator was introduced in 1995 in order to construct convex mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$ starting with a convex function on the unit disc. If $f_1, \dots, f_n$ are convex functions on the unit disc U, then $F(z) = (f_1(z_1), \dots, f_n(z_n))$, $z = (z_1, \dots, z_n) \in B^n$, is not necessary a convex mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$, $n \geq 2$. The preservation of convexity under the Roper-Suffridge extension operator was also proved in [43], using a different method. A number of other geometric properties (starlikeness, spirallikeness, starlikeness of a certain order, etc.) have been shown to be preserved by this operator and certain generalizations of it. Graham and Kohr [43] proved that the Roper-Suffridge extension operator preserves the notions of starlikeness and

Bloch mapping, and Graham, Kohr and Kohr [46] proved that it preserves the notion of parametric representation.

Several extension operators that have similar properties to those of the Roper-Suffridge extension operator were studied by Pfaltzgraff and Suffridge [91], Graham, Hamada, Kohr and Suffridge [42] (see also [46]), Muir ([82], [83]), Gong and Liu ([34], [35]), Liu and Liu [77], Xu and Liu [114], etc.

Other recent extension operators that preserve geometric and analytical properties of biholomorphic mappings on the unit ball in $\mathbb{C}^n$ and complex Banach spaces are due to Elin [30] (see also [39]). Further details regarding generalizations of the Roper-Suffridge extension operator may be found in [44, Chapter 11] and [45] (see also, [30], [39], [73], [76], [82], etc.).

Another generalization of the Roper-Suffridge extension operator was given by Pfaltzgraff and Suffridge [91] in 1999. This operator provides a way of extending a locally biholomorphic mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$ to a locally biholomorphic mapping on the Euclidean unit ball B^{n+1} in $\mathbb{C}^{n+1}$. The Pfaltzgraff-Suffridge extension operator was also studied by Graham, Kohr and Pfaltzgraff [48], who obtained a partial answer regarding the preservation of convexity under this operator. They also proved that the Pfaltzgraff-Suffridge extension operator preserves the notion of parametric representation and starlikeness.

- **Chapter 3** is devoted to certain extension operators that preserve certain analytical and geometric properties. This subject began with the Roper-Suffridge extension operator [101], introduced in 1995, in order to construct convex mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$ starting with a convex function on the unit disc. This chapter is based on the original results obtained in [11], [12], [14], [15], [16], [17].

  In Section 3.1 we present the Roper-Suffridge extension operator and its generalizations. We give their main geometric and analytical properties and we show their connection with the theory of Loewner chains. We also discuss the case of the Pfaltzgraff-Suffridge extension operator [91] which provides a way of extending a locally biholomorphic mapping $f \in H(B^n)$ to a locally biholomorphic mapping $F \in H(B^{n+1})$.

  In Section 3.2 we present original results regarding certain generalizations of the Roper-Suffridge extension operator. We prove that these operators preserve the notion of g-Loewner chains, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$ and $\gamma \in (0,1)$. As a consequence, the considered operators preserve certain geometric and analytical properties, such as g-parametric representation, starlikeness of order γ, spirallikeness of type δ and order γ, almost starlikeness of order δ and type γ. This section is based on the original results obtained in [11] and [12]. The main results presented in this section are Theorems 3.2.1, 3.2.7, 3.2.16, Proposition 3.2.11, Corollaries 3.2.2, 3.2.3, 3.2.5, 3.2.6, 3.2.8, 3.2.9, 3.2.13, 3.2.14, 3.2.17, 3.2.18, 3.2.20, 3.2.21.

  In Section 3.3 we present some subordination results associated with a certain extension operator. This section contains original results obtained in [11]. The main results presented in this section are Theorem 3.3.1, Corollaries 3.3.2, 3.3.3, 3.3.4, 3.3.5, 3.3.6, 3.3.7.

  In Section 3.4 we consider some radius problems associated with a certain extension operator. This section contains original results obtained in [11]. The main results are Theorems 3.4.1, 3.4.3, 3.4.4, 3.4.5.

In Section 3.5 we generalize the Pfaltzgraff-Suffridge extension operator and we prove that this generalized operator preserves the notions of parametric representation, starlikeness, spirallikeness of type δ, almost starlikeness of order δ. We consider the preservation of ε-starlikeness under this operator and we obtain a partial answer to the question of whether it preserves convexity. We also obtain a subordination preserving result under the considered operator and we give some particular cases of this result. This section contains original results obtained in [14], [15], [17]. The main results presented in this section are Theorems 3.5.2, 3.5.7, 3.5.12, Corollaries 3.5.3, 3.5.4, 3.5.5, 3.5.6, 3.5.8, 3.5.9, 3.5.10, 3.5.13, 3.5.14.

- **Chapter 4** is devoted to the study of certain subclasses of normalized biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$, generated by using the method of Loewner chains. The theory of Loewner chains provides a powerful tool in the study of various problems related to univalence in one and several complex variables. For example, geometric properties, such as starlikeness, spirallikeness of type α, almost starlikeness of order α, have useful analytical characterizations in terms of Loewner chains. Indeed, if f is a normalized holomorphic mapping on the unit ball B^n, then f is starlike (f is biholomorphic and $f(B^n)$ is a starlike domain with respect to zero) if and only if $f(z,t) = e^t f(z)$ is a Loewner chain. This result is due to Pfaltzgraff and Suffridge [90]. Also, if f is a normalized holomorphic mapping on B^n, then f is spirallike of type α, $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, if and only if $f(z,t) = e^{(1-ia)t} f(e^{iat}z)$ is a Loewner chain, where $a = \tan\alpha$. This result was proved by Hamada and Kohr [54]. Xu and Liu [114] obtained a characterization of almost starlikeness of order α, using Loewner chains, and proved that this notion is preserved by certain extension operators, such as the Roper-Suffridge extension operator.

  Pfaltzgraff and Suffridge [91] proved that if $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$ and if $F(z) = P(z)z$, $z \in B^n$, then F is starlike if and only if

  $$\operatorname{Re}\left[1 + \frac{DP(z)(z)}{P(z)}\right] > 0,\ z \in B^n.$$

  This result was generalized in [38] (compare with [115] in the case of complex Banach spaces) to the case of g-starlikeness, a related notion of starlikeness that will be mentioned in Section 4.1.

  In this chapter we obtain analogous characterizations of certain subclasses of normalized starlike mappings, spirallike mappings of type α, almost starlike mappings of order α, respectively, by using g-Loewner chains, where $g : U \to \mathbb{C}$ is a univalent function which satisfies some natural conditions. Various examples and applications are also obtained. Related results concerning growth, distortion and coefficient bounds of g-starlike mappings on the unit ball in $\mathbb{C}^n$ were obtained in [52], [53], [115].

  This chapter contains original results obtained in [13].

In Section 4.1 we present the classes of g-starlike mappings, g-spirallike mappings of type $\alpha \in (-\pi/2, \pi/2)$, and g-almost starlike mappings of order $\alpha \in [0, 1)$ on B^n. We provide examples of this type of mappings and we obtain their characterization by using g-Loewner chains.

In Section 4.2 we will use these results to prove that, under certain assumptions, the mapping $F : B^n \to \mathbb{C}^n$ given by $F(z) = P(z)z$ is g-starlike, g-spirallike of type $\alpha \in (-\pi/2, \pi/2)$ and g-almost starlike of order $\alpha \in [0, 1)$ on B^n, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that P(0) = 1. More generally, we consider conditions under which F has g-parametric representation on B^n. Various applications of these results are also provided. In this way we obtain concrete examples of mappings which have g-parametric representation on the Euclidean unit ball B^n in $\mathbb{C}^n$. The main definitions and results presented in this chapter are Definitions 4.1.5, 4.1.6, Theorems 4.1.11, 4.1.12, 4.1.13, 4.2.1, Corollaries 4.2.3, 4.2.4, 4.2.11, 4.2.12, 4.2.13, 4.2.14, 4.2.15, 4.2.16.

Extremal problems in one complex variable were intensively studied. A very good treatment concerning extremal problems related to various compact subsets of univalent functions on the unit disc U may be found in the monograph of Hallenbeck and MacGregor [50] and the Ph.D. thesis of Roth [104] (see also [97]).

It is well known that if f is an extreme point for the compact family S of normalized univalent functions on U, or if f is a support point for S, then f maps the unit disc U onto the complement of a continuous arc tending to ∞. Consequently, f cannot be a bounded mapping (see e.g. [26], [50] and [93]). On the other hand, Pell [87] and Kirwan [61] proved that if f is an extreme point (respectively, f is a support point) for the family S of normalized univalent functions on the unit disc U, and if $f(z, t)$ is a Loewner chain such that $f = f(\cdot, 0)$, then $e^{-t}f(\cdot, t)$ is an extreme point of S (respectively, $e^{-t}f(\cdot, t)$ is a support point of S), for all $t \geq 0$.

A generalization of Pell's and Kirwan's results to several complex variables was obtained by Graham, Kohr and Pfaltzgraff [48], in the case of the compact family $\Phi_n(S)$, where Φ_n is the Roper-Suffridge extension operator [101]. Graham, Hamada, Kohr and Kohr [41] and Schleissinger [106] obtained generalizations of the above results to the case of mappings which have parametric representation on B^n. Indeed, the authors in [41] proved that if f is an extreme point for the compact family $S^0(B^n)$ of normalized biholomorphic mappings which have parametric representation on B^n, and if $f(z, t)$ is a Loewner chain such that $f = f(\cdot, 0)$ and $\{e^{-t}f(\cdot, t)\}_{t\geq 0}$ is a normal family on B^n, then $e^{-t}f(\cdot, t)$ is an extreme point of $S^0(B^n)$ for $t \geq 0$. Also, Graham, Hamada, Kohr and Kohr [41] proved that if f is a support point of $S^0(B^n)$ and $f(z, t)$ is a Loewner chain such that $f = f(\cdot, 0)$ and $\{e^{-t}f(\cdot, t)\}_{t\geq 0}$ is a normal family on B^n, then there exists $t_0 > 0$ such that $e^{-t}f(\cdot, t)$ is a support point of $S^0(B^n)$ for $0 \leq t < t_0$. Schleissinger [106] proved recently that this result holds for all $t \in [0, \infty)$.

Muir and Suffridge [84] gave various characterizations of extreme points for convex mappings on B^n. On the other hand, Muir [83] considered extreme points and support points for compact subsets associated with a large family of extension operators. Recently, Voda [112] studied extreme points associated with the Carathéodory family $\mathcal{M}$ and found some differences between the structures of $\mathcal{M}$ in one variable, and $\mathcal{M}$ in higher dimensions.

- **Chapter 5** considers extreme points and support points associated with the compact family $\overline{S_g^0(B^n)}$, where $g : U \to \mathbb{C}$ is a univalent function which satisfies certain natural assumptions. Certain applications and consequences will be obtained. We also consider extreme points and support points associated with extension operators which preserve Loewner chains. In particular, we consider extreme points and support points for the compact family $\Psi_n(\overline{S_g^0(B^n)})$, where Ψ_n is the Pfaltzgraff-Suffridge extension operator [91]. This chapter contains original results due to Chirilă, Hamada and Kohr [18] and Chirilă [17]. These results generalize the work of Pell [87] and Kirwan [61] to several complex variables and continue the work in [41], [48], [106].

  In Section 5.1 we recall the notions of extreme points and support points for compact subsets of $H(B^n)$, where B^n is the Euclidean unit ball in $\mathbb{C}^n$. In Section 5.2 we consider extreme points associated with the compact family $\overline{S_g^0(B^n)}$. The main results presented in this section are Lemma 5.2.1, Theorem 5.2.2, Proposition 5.2.3.

  In Section 5.3 we consider support points associated with the compact family $\overline{S_g^0(B^n)}$. We also obtain a generalization to the n-dimensional case of an extremal principle due to Kirwan and Schober [62] (see also [104]). The main results in this section are Theorems 5.3.1, 5.3.3, Lemma 5.3.4, Corollary 5.3.5.

  In Section 5.4 we consider extreme points and support points for the compact family $\Phi(\overline{S_g^0(B^n)})$, where $\Phi : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(B^{n+1})$ is an extension operator which preserves Loewner chains. In particular, we consider extreme points and support points for the compact family $\Psi_n(\overline{S_g^0(B^n)})$, where Ψ_n is the Pfaltzgraff-Suffridge extension operator. The main results presented in this section are Lemmas 5.4.2, 5.4.4, Theorems 5.4.3, 5.4.5.

The original results presented in this thesis are based on the following papers:

- **T. Chirilă**, *An extension operator associated with certain g-Loewner chains*, Taiwanese J. Math. (**ISI**), **17**, no. 5 (2013), 1819–1837.
- **T. Chirilă**, *Analytic and geometric properties associated with some extension operators*, Complex Var. Elliptic Equ. (**ISI**), to appear, doi.org/10.1080/17476933.2012.746966.
- **T. Chirilă**, *Subclasses of biholomorphic mappings associated with g-Loewner chains on the unit ball in $\mathbb{C}^n$*, Complex Var. Elliptic Equ. (**ISI**), to appear, doi.org/10.1080/17476933.2013.856422.
- **T. Chirilă**, *An extension operator and Loewner chains on the Euclidean unit ball in $\mathbb{C}^n$*, Mathematica (Cluj), **54 (77)** (2012), 116–125.
- **T. Chirilă**, *An extension operator and Loewner chains on some Reinhardt domains in $\mathbb{C}^n$*, Advances in Mathematics: Scientific Journal **1** (2012), 139–145.
- **T. Chirilă**, *Extension operators that preserve geometric and analytic properties of biholomorphic mappings*, in "Topics in Mathematical Analysis and Applications", L. Toth and Th. M. Rassias, Eds., Springer, 2014, to appear.

- **T. Chirilă**, *Extreme points associated with certain extension operators*, in preparation.
- **T. Chirilă**, H. Hamada, G. Kohr, *Extreme points and support points for mappings with g-parametric representation in $\mathbb{C}^n$*, submitted.

The original results presented in this thesis were communicated to the following international conferences:

- 26.08–30.08.2013, 9th International Symposium on Geometric Function Theory and Applications (GFTA 2013), Işik University, Istanbul, Turkey; communication: *Geometric properties of certain extension operators*.
- 27.06–30.06.2013, Joint International Meeting of the AMS and the Romanian Mathematical Society, Alba Iulia, Romania; communication: *A subclass of biholomorphic mappings generated by g-Loewner chains*.
- 27.08–31.08.2012, International Symposium on Geometric Function Theory and Applications (GFTA 2012), Ohrid, R. Macedonia; communication: *Geometric properties associated with generalized Roper-Suffridge extension operators*.
- 26.06–30.06.2012, International Conference on Complex Analysis and Related Topics, the 13th Romanian-Finnish Seminar, Ploieşti, Romania; communication: *Extension operators and g-Loewner chains*.
- 9.02–12.02.2012, 9th Joint Conference on Mathematics and Computer Science, Siófok, Hungary; communication: *An extension operator and Loewner chains on the Euclidean unit ball in $\mathbb{C}^n$*.
- 4.09–8.09.2011, International Symposium on Geometric Function Theory and Applications (GFTA 2011), Cluj-Napoca, Romania; communication: *Extension operators that preserve geometric properties on the unit ball in $\mathbb{C}^n$*.

Keywords

Biholomorphic mapping, convex mapping, extension operator, extreme point, g-almost starlike mapping of order α, g-Loewner chain, g-parametric representation, g-spirallike mapping of type α, g-starlike mapping, Loewner chain, Loewner differential equation, parametric representation, Roper-Suffridge extension operator, spirallike mapping, starlike mapping, subordination, support point.

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Chapter 1

Univalent functions in the complex plane

In this chapter we present general results regarding univalent functions of one complex variable. These results are useful in the forthcoming chapters of this thesis. Next, we refer to the notion of subordination and we recall some basic results regarding the Carathéodory family of holomorphic functions with positive real part on the unit disc. Further, we refer to classical and well known properties of univalent functions of one complex variable. We present various properties of univalent functions. Also, we recall the notion of conformal mapping, and we present the Riemann mapping theorem which provides the conformally equivalence of simply connected domains in $\mathbb{C}$. This is one of the most important results in the theory of univalent functions of one complex variable that does not hold in $\mathbb{C}^n$, $n \geq 2$.

Next, we consider the class S of univalent functions on the unit disc U which are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, for all $f \in S$, and we present main results regarding this class. We are also concerned with some special subclasses of univalent functions on U, such as starlike, convex, spirallike, and close-to-convex functions. These subclasses of univalent functions are based on geometric and analytical characterizations. We also refer to various radius problems associated to the class S and some of its subclasses.

Finally, we present general results from the theory of Loewner chains in the complex plane. This theory provides one of the most important directions in the study of univalent functions. We recall some main results related to Loewner chains and the Loewner differential equation. Finally, several applications of the Loewner theory are provided.

The main bibliographic sources used during the preparation of this chapter are [20], [26], [37], [44], [57], [66], [68], [81], [93].

1.1 General results regarding holomorphic functions

We begin this section with some notions and preliminary results which are used throughout the thesis. For more details, see [57], [66], [68], [99], [105], the main bibliographic sources used during the preparation of this section.

Let $\mathbb{C}$ be the complex plane, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. Let

$$U(z_0; r) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

be the disc of center $z_0 \in \mathbb{C}$ and radius r. Also, let

$$\overline{U}(z_0; r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

be the closed disc of center z_0 and radius r, and let

$$\partial U(z_0; r) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

be the circle of center z_0 and radius r. The disc $U(0, r)$ is denoted by U_r and the unit disc U_1 is denoted by U.

Let $\Omega \subseteq \mathbb{C}$ be an open set. We denote by $H(\Omega)$ the set of holomorphic functions defined on Ω with values in $\mathbb{C}$. Holomorphic functions on the whole complex plane are called *entire functions*.

We next present certain elementary properties of holomorphic functions that will be useful in the next sections (see e.g. [57], [66], [68]). The first result is known as the *open mapping theorem* for holomorphic functions (see e.g. [57], [66]).

Theorem 1.1.1 *Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic function. Then $f(\Omega)$ is a domain in $\mathbb{C}$.*

Another important result in the theory of holomorphic functions is the following *maximum modulus theorem* (see e.g. [57], [66], [68]).

Theorem 1.1.2 *Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If there exists $z_0 \in \Omega$ such that*

$$|f(z_0)| = \max\{|f(z)| : z \in \Omega\},$$

then f is constant on Ω.

An important application of the maximum modulus theorem is the well known Schwarz's lemma (see e.g. [57], [66], [68]).

Corollary 1.1.3 (Schwarz's lemma) *If f is a holomorphic function on U such that $f(0) = 0$ and $|f(z)| < 1$, $z \in U$, then $|f(z)| \leq |z|$, $z \in U$, and $|f'(0)| \leq 1$. If, in addition, there exists some $z_0 \in U \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, or if $|f'(0)| = 1$, then there exists $c \in \mathbb{C}$ such that $|c| = 1$ and $f(z) = cz$, $z \in U$.*

We now recall the notions of locally uniformly bounded families and normal families, and we present the connection between these two notions in the case of families of holomorphic functions (see e.g. [57], [66]).

Definition 1.1.4 Let Ω be an open set in $\mathbb{C}$. Also, let $\mathcal{F} \subseteq H(\Omega)$. We say that $\mathcal{F}$ is *locally uniformly bounded* if for each compact $K \subset \Omega$ there is a constant $M = M(K) > 0$ such that $\|f\|_K \leq M$, for all $f \in \mathcal{F}$, where $\|f\|_K = \max\{|f(z)| : z \in K\}$.

Definition 1.1.5 Let Ω be an open set in $\mathbb{C}$. Also, let $\mathcal{F} \subseteq H(\Omega)$. We say that $\mathcal{F}$ is a *normal family* (*relatively compact family*) if each sequence $\{f_k\}_{k\in\mathbb{N}} \subseteq \mathcal{F}$ contains a subsequence which converges locally uniformly on Ω.

The following well known result due to Montel shows that Definitions 1.1.4 and 1.1.5 are equivalent (see e.g. [57], [66], [68]).

Theorem 1.1.6 (Montel's theorem) *Let $\Omega \subseteq \mathbb{C}$ be an open set and let $\mathcal{F} \subseteq H(\Omega)$. Then $\mathcal{F}$ is a normal family if and only if $\mathcal{F}$ is locally uniformly bounded.*

Note that the above result may be generalized to several complex variables (see [63], [85]). In view of Montel's theorem and the well known characterization of compactness in metric spaces, the following useful result holds (see e.g. [66]). Note that this result also holds in the case of several complex variables (see [63], [85], [98]).

Corollary 1.1.7 *Let Ω be an open set in $\mathbb{C}$. Also, let $\mathcal{F} \subseteq H(\Omega)$. Then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is locally uniformly bounded and closed.*

1.2 Subordination. Holomorphic functions with positive real part

In this section we present the notion of subordination in the complex plane and we give some important properties of holomorphic functions in the unit disc U with positive real part. The main bibliographic sources for this section are [81] and [93].

We first recall the definition of subordination (see e.g. [80], [81]).

Definition 1.2.1 Let $f, g \in H(U)$. We say that f is *subordinate* to g (and write $f \prec g$) if there is a Schwarz function v (i.e. $v \in H(U)$ and $|v(z)| \leq |z|$, $z \in U$) such that $f(z) = g(v(z))$, $z \in U$.

If g is univalent (holomorphic and injective) on U, then the next characterization of subordination holds (see e.g. [81], [93]).

Theorem 1.2.2 *Let $f, g \in H(U)$ be such that g is univalent on U. Then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

Remark 1.2.3 The following result is known as the *subordination principle* (see e.g. [81], [93]): if $f, g \in H(U)$ such that $f(0) = g(0)$, g is univalent on U, and $f(U) \subseteq g(U)$, then $f(U_r) \subseteq g(U_r)$, for all $r \in (0, 1)$.

We next recall the well-known Carathéodory class of holomorphic functions with positive real part on the unit disc (see [44], [81], [93]):

$$\mathcal{P} = \{p \in H(U) : p(0) = 1, \mathrm{Re}\, p(z) > 0, z \in U\}.$$

This class plays an important role in characterizing some special classes of univalent functions on the unit disc, such as starlike, convex, spirallike functions, as well as in the study of Loewner chains and the Loewner differential equation.

We present the following growth and distortion theorem for the class $\mathcal{P}$ (see e.g. [81], [93]).

Theorem 1.2.4 *If $p \in \mathcal{P}$, then the following relations hold*

$$(i)\ \ \frac{1-|z|}{1+|z|} \leq \mathrm{Re}\, p(z) \leq |p(z)| \leq \frac{1+|z|}{1-|z|},\ z \in U,$$

$$(ii)\ \ |p'(z)| \leq \frac{2\mathrm{Re}\, p(z)}{1-|z|^2} \leq \frac{2}{(1-|z|)^2},\ z \in U.$$

These inequalities are sharp and equality holds for $p(z) = \frac{1+\lambda z}{1-\lambda z}$, $z \in U$, for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

We close this section with the following sharp coefficient bounds for the class $\mathcal{P}$ due to Carathéodory (see e.g. [81]).

Theorem 1.2.5 *Let $p \in \mathcal{P}$ be such that $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in U$. Then $|p_n| \leq 2$, $n \geq 1$. This result is sharp and equality holds for $p(z) = \frac{1+\lambda z}{1-\lambda z}$, $z \in U$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$.*

1.3 General results regarding univalent functions

In this section we present some classical and well known properties of univalent functions of one complex variable. Also, we recall the notion of conformal equivalence and we present the Riemann mapping theorem, one of the most significant results in the theory of univalent functions. For more details regarding univalent functions, see e.g. [26], [37], [44], [57], [68], [93], the main bibliographic sources used during the preparation of this section.

We begin with the definition of univalent functions (see e.g. [57], [66], [93]).

Definition 1.3.1 Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$. The function f is *univalent* on Ω if f is holomorphic and injective on Ω.
We denote by $H_u(\Omega)$ the set of univalent functions on Ω.

The following result gives a necessary condition of univalence (see e.g. [57], [66]). However, Theorem 1.3.2 does not provide a sufficient condition of univalence.

Theorem 1.3.2 *Let Ω be a domain in $\mathbb{C}$ and let $f \in H_u(\Omega)$. Then $f'(z) \neq 0$, $z \in \Omega$.*

The following result, due to Alexander, Noshiro, Warschawski and Wolff (see e.g. [44], [81]), yields a sufficient condition of univalence for holomorphic functions on convex domains in $\mathbb{C}$.

Theorem 1.3.3 *Let $\Omega \subseteq \mathbb{C}$ be a convex domain and let $f \in H(\Omega)$. If $\operatorname{Re} f'(z) > 0$, $z \in \Omega$, then f is univalent on Ω.*

The following generalization of Theorem 1.3.3 was obtained by Ozaki and Kaplan [59]. This result plays a key role in the definition of close-to-convexity, as we shall see in a next section.

Theorem 1.3.4 *Let $\Omega \subseteq \mathbb{C}$ be a domain. Also, let $f, g \in H(\Omega)$ be such that $g \in H_u(\Omega)$ and $g(\Omega)$ is a convex domain in $\mathbb{C}$. If*

$$\operatorname{Re}\left[\frac{f'(z)}{g'(z)}\right] > 0, \quad z \in \Omega,$$

then $f \in H_u(\Omega)$.

We next present some important examples of univalent functions (see e.g. [66], [81], [93]).

Example 1.3.5 Let $f : U \to \mathbb{C}$ be given by $f(z) = \dfrac{z}{(1-z)^2}$, $z \in U$. Then f is univalent on U and maps the unit disc U onto $\mathbb{C} \setminus \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \leq -1/4,\ \operatorname{Im}\zeta = 0\}$. Note that f is called the *Koebe function* (see e.g. [26], [44], [93]). As we shall see in the next sections, this function plays a special role in many extremal results in the theory of univalent functions.

Moreover, let $f_\theta : U \to \mathbb{C}$ be given by $f_\theta(z) = \dfrac{z}{(1-e^{i\theta}z)^2}$, where $\theta \in \mathbb{R}$. We note that f_θ is a rotation of angle $-\theta$ of the function f, since $f_\theta(z) = e^{-i\theta} f(e^{i\theta}z)$, $z \in U$. Therefore f_θ is univalent on U and maps the unit disc U onto the complex plane except for a radial slit to ∞ which starts from the point $(-1/4)e^{-i\theta}$ (see e.g. [26], [44], [93]).

The following result, known as Hurwitz's theorem, is very useful in various applications concerning univalent functions (see e.g. [26], [57], [66], [93]).

Theorem 1.3.6 (*e.g.* **[57]**) *Let Ω be a domain in $\mathbb{C}$ and let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of univalent functions on Ω such that $f_k \to f$ locally uniformly on Ω. Then f is either constant or univalent on Ω.*

We next present the notion of conformal equivalence of domains in $\mathbb{C}$, as well as a fundamental result regarding this notion, namely the Riemann mapping theorem (see for details, [2], [26], [57], [66], [93], [99]).

Definition 1.3.7 Let Ω_1 and Ω_2 be domains in $\mathbb{C}$ and let $f : \Omega_1 \to \Omega_2$. We say that f is a *conformal mapping* if f is univalent on Ω_1 and $f(\Omega_1) = \Omega_2$. In this case, the domains Ω_1 and Ω_2 are said to be *conformally equivalent* (see e.g. [2], [57]).

A fundamental result in the theory of univalent functions is given by the *Riemann mapping theorem*. For more details and applications, see [2], [86], [99], [105].

Theorem 1.3.8 (*e.g.* **[44]**, **[57]**) *Let Ω be a simply connected domain in $\mathbb{C}$ such that $\Omega \neq \mathbb{C}$. Then Ω and the unit disc U are conformally equivalent. Moreover, if $z_0 \in \Omega$ is a given point, then there exists a unique conformal mapping f of Ω onto U such that $f(z_0) = 0$ and $f'(z_0) > 0$.*

The next property of conformal equivalence of simply connected domains in $\mathbb{C}$ follows directly from Theorem 1.3.8 (see e.g. [57], [66]).

Corollary 1.3.9 *Any two simply connected domains in $\mathbb{C}$ and different from $\mathbb{C}$ are conformally equivalent.*

1.4 Families of normalized univalent functions on the unit disc

In this section we refer to various subclasses of univalent functions on the unit disc. We present the class S of normalized univalent functions on U, the class S^* of normalized starlike functions with respect to the origin, the class K of normalized convex functions. We also refer to the class of close-to-convex functions, the classes of starlike and convex functions of order α, the class of spirallike functions of type γ and the class of almost starlike functions of order α. We shall see that most of these subclasses of S have analytical and geometric characterizations.

The main bibliographic sources used in this section are [26], [44], [81], [93].

1.4.1 The class S

First, we consider the class S of normalized univalent functions on the unit disc. The choice of the unit disc is justified by the Riemann mapping theorem. Hence, the study of univalent functions on simply connected domains in $\mathbb{C}$ may be reduced to the unit disc.

Definition 1.4.1 (**[26], [93]**) Let S be the class of univalent functions f on the unit disc U, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus,

$$S = \{f \in H_u(U) : f(0) = f'(0) - 1 = 0\}.$$

Any function $f \in S$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ z \in U.$$

Next, we recall some well known properties of functions in the class S.

Theorem 1.4.2 provides the sharp second coefficient bound for the class S. This result is due to L. Bieberbach [6].

Theorem 1.4.2 *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S$, then $|a_2| \leq 2$. The equality $|a_2| = 2$ holds if and only if f is a rotation of the Koebe function.*

Starting with the inequality $|a_2| \leq 2$ for functions in the class S, Bieberbach formulated the following conjecture in 1916 [6], which was solved by L. de Branges in 1985 [7]:

Conjecture 1.4.1 (Bieberbach's conjecture) *If $f \in S$ has the Taylor series expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then $|a_k| \leq k$, $k \geq 2$. The equality $|a_k| = k$, $k \geq 2$, holds if and only if f is a rotation of the Koebe function.*

An application of Theorem 1.4.2 is the well known *Koebe* $1/4$-*covering theorem* for the class S (see e.g. [26], [93]).

Theorem 1.4.3 *Let $f \in S$. Then $f(U) \supseteq U_{1/4}$.*

We next present the growth and distortion theorem for the class S (see e.g. [26], [37, I p. 65], [93]). This result is another application of Theorem 1.4.2.

Theorem 1.4.4 ([6]) *If $f \in S$, then the following sharp inequalities hold:*

$$(i)\quad \frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2},\ z \in U,$$

$$(ii)\quad \frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3},\ z \in U,$$

$$(iii)\quad \frac{1-|z|}{1+|z|} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+|z|}{1-|z|},\ z \in U.$$

Equality in each of the above relations holds if and only if f is a rotation of the Koebe function.

Using Theorem 1.4.4 and Corollary 1.3.6, as well as the characterization of compact subsets of $H(U)$, we deduce that S is compact (see e.g. [81]).

Corollary 1.4.5 *S is a compact subset of $H(U)$.*

We remark that the relation (i) from Theorem 1.4.4 gives a necessary, but not a sufficient condition for univalence. Necessary and sufficient conditions of univalence may be found in [28], [62], [70] (see also [44] and the references therein).

1.4.2 The class $S(M)$

1.4.3 Starlike functions

We next consider the class S^* of normalized starlike functions on the unit disc. We will also present the class of starlike functions of order α. Various properties related to starlike functions may be found in [26], [37], [44], [81], [93].

Definition 1.4.6 (**[81]**) Let $f \in H(U)$ be such that $f(0) = 0$. The function f is said to be *starlike* if f is univalent on U and $f(U)$ is a starlike domain with respect to the origin.

We next present the analytical characterization of starlikeness (see e.g. [26], [44], [81], [93]).

Theorem 1.4.7 (**[26]**, **[93]**) *Let $f \in H(U)$ be such that $f(0) = 0$. Then f is starlike if and only if $f'(0) \neq 0$ and*

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0,\ z \in U.$$

Let S^* be the class of normalized starlike functions on U.

Remark 1.4.8 The growth and distortion theorem for the class S (Theorem 1.4.4) also holds for the class S^* (see e.g. [78], [81]). Moreover, Bieberbach's conjecture also holds for the class S^* (see e.g. [78], [81]).

Starlike functions of order α

We next give a brief presentation of the class S^*_α of starlike functions of order α. For more details and applications, see [37], [100].

Definition 1.4.9 (**[100]**) Let $\alpha \in [0, 1)$. The class of *starlike functions of order α* is defined by

$$S^*_\alpha = \left\{ f \in H(U) :\ f(0) = 0,\ f'(0) = 1, \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha,\ z \in U \right\}.$$

Note that $S^*_\alpha \subseteq S^*$, $\alpha \in [0, 1)$, and $S^*_0 = S^*$.

There exists the following connection between the classes S^*_α and S^* (see e.g. [81]).

Theorem 1.4.10 *Let $\alpha \in [0, 1)$. The function $f \in S^*_\alpha$ if and only if $g \in S^*$, where $g(z) = z\left[\frac{f(z)}{z}\right]^{\frac{1}{1-\alpha}}$. We choose the branch of the power function such that $\left.\left[\frac{f(z)}{z}\right]^{\frac{1}{1-\alpha}}\right|_{z=0} = 1$.*

From Theorem 1.4.10 we obtain the growth theorem for the class S^*_α, $\alpha \in [0, 1)$ (see e.g. [37, I p. 140], [44], [81]).

Theorem 1.4.11 *If $f \in S^*_\alpha$, $\alpha \in [0, 1)$, then*

$$\frac{|z|}{(1 + |z|)^{2(1-\alpha)}} \le |f(z)| \le \frac{|z|}{(1 - |z|)^{2(1-\alpha)}},\ z \in U.$$

These inequalities are sharp.

1.4.4 Convex and close-to-convex functions

In this section we are concerned with the class K of normalized convex functions on the unit disc. In the second part of this section, we will also present the class of convex functions of order α and the class of close-to-convex functions, respectively.

Other details regarding convex and close-to-convex functions on the unit disc may be found in [26], [37], [44], [81], [93].

Convex functions

Definition 1.4.12 (**[81]**) Let $f : U \to \mathbb{C}$ be a holomorphic function. The function f is said to be *convex* if f is univalent on U and $f(U)$ is a convex domain.

We next present the well known analytical characterization of convexity on the unit disc (see e.g. [26], [81], [93]).

Theorem 1.4.13 (**[26]**, **[93]**) *If $f \in H(U)$, then f is convex if and only if $f'(0) \neq 0$ and*

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0,\ z \in U. \tag{1.4.1}$$

We denote by K the class of normalized convex functions on U. Hence, $K \subset S^* \subset S$.

We next present the growth and distortion theorem for the class K (see e.g. [78], [81]).

Theorem 1.4.14 *If $f \in K$, then the following relations hold*

(i) $\dfrac{|z|}{1+|z|} \leq |f(z)| \leq \dfrac{|z|}{1-|z|},\ z \in U,$

(ii) $\dfrac{1}{(1+|z|)^2} \leq |f'(z)| \leq \dfrac{1}{(1-|z|)^2},\ z \in U.$

These inequalities are sharp and equality holds in a point different from 0 for $f(z) = \dfrac{z}{1-\lambda z}$, $\lambda \in \mathbb{C},\ |\lambda| = 1$.

From Theorems 1.4.7 and 1.4.13, we obtain Alexander's duality theorem, which shows the connection between the classes S^* and K (see e.g. [81]). We remark that this result cannot be generalized to the case of normalized convex mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$ (see [110]; see also, e.g. [44] and [101]).

Theorem 1.4.15 *Let $f \in H(U)$ be such that $f(0) = 0$. Then the function f is convex on U if and only if the function $F(z) = zf'(z)$ is starlike on U.*

We next refer to sharp estimates related to the class K.

Theorem 1.4.16 (**[78]**) *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to the class K. Then $|a_n| \leq 1$, $n \geq 2$. These estimates are sharp and the equality $|a_n| = 1$, $n \geq 2$, holds if and only if $f(z) = \dfrac{z}{1-\lambda z}$, $\lambda \in \mathbb{C},\ |\lambda| = 1$.*

The next result provides the connection between the classes K and $S^*_{1/2}$. This result was given by A. Marx and E. Strohhäcker (see e.g. [44], [81]).

Theorem 1.4.17 *If $f \in K$, then $f \in S^*_{1/2}$. This result is sharp.*

A generalization of the above result to higher dimensions was obtained in [22] and [64].

Convex functions of order α

We next give a brief presentation of the notion of convexity of order α [100].

Definition 1.4.18 (**[100]**) Let $f : U \to \mathbb{C}$ be a holomorphic function. We say that f is *convex of order* $\alpha \in [0,1)$ if $f'(0) \neq 0$ and

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha,\ z \in U.$$

We denote by $K(\alpha)$ the class of normalized convex functions of order α on U. Clearly, $K(\alpha) \subseteq K$ for $\alpha \in [0,1)$, $K(0) = K$.

The following generalization of Theorem 1.4.17 is due to Jack (see e.g. [44]). This result is not sharp. For a sharp result, see e.g. [80]. A generalization of this result to $\mathbb{C}^n$ was obtained in [22] and [64].

Theorem 1.4.19 *If $f \in K(\alpha)$, $\alpha \in [0,1)$, then $f \in S^*_\beta$, where*

$$\beta = \beta(\alpha) = \frac{2\alpha - 1 + \sqrt{(2\alpha-1)^2 + 8}}{4}. \tag{1.4.2}$$

The following connection between the classes $K(\alpha)$, $\alpha \in [0,1)$, and S^* is useful in applications (see e.g. [44], [81]).

Theorem 1.4.20 *$f \in K(\alpha)$ if and only if $g \in S^*$, where $g(z) = z(f'(z))^{\frac{1}{1-\alpha}}$, $z \in U$. We choose the branch of the power function such that $(f'(z))^{\frac{1}{1-\alpha}}|_{z=0} = 1$.*

Close-to-convex functions

We next present another important subclass of univalent functions on the unit disc U, namely the class of close-to-convex functions. Further details regarding close-to-convexity may be found in [26], [44], [81], [93].

The following definition is due to Kaplan [59].

Definition 1.4.21 The function $f \in H(U)$ is said to be *close-to-convex* if there exists a convex function g on U such that

$$\mathrm{Re}\left[\frac{f'(z)}{g'(z)}\right] > 0,\ z \in U.$$

We note that any close-to-convex function is univalent on U, by Theorem 1.3.4.

We denote by C the class of normalized close-to-convex functions on U. The following inclusion relation holds:

$$K \subset S^* \subset C \subset S.$$

We now present a very important geometric characterization of close-to-convex functions, due to Kaplan [59] (see also [26], [93]).

Theorem 1.4.22 (**[59]**) *Let $f : U \to \mathbb{C}$ be a normalized locally univalent function. Then f is close-to-convex if and only if*

$$\int_{\tau_1}^{\tau_2} \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] d\tau > -\pi,\ z = re^{i\tau},$$

for each $r \in (0,1)$ and for each $\tau_1, \tau_2 \in \mathbb{R}$ such that $0 \leq \tau_2 - \tau_1 \leq 2\pi$.

Reade proved that the coefficients of normalized close-to-convex functions satisfy the same bounds as in the Bieberbach conjecture (see e.g. [44], [81]).

1.4.5 Spirallike functions

The notion of spirallikeness was defined by L. Špaček [107] in 1932, and is a generalization of the notion of starlikeness. For more details regarding spirallike functions, see e.g. [26], [44], [81], [93].

First, we recall the notion of spirallikeness of type γ ([107]; see also [26], [37, I p. 148], [81]).

Definition 1.4.23 (**[107]**) A *logarithmic γ-spiral* (or *γ-spiral*), $\gamma \in (-\pi/2, \pi/2)$, is a curve in the complex plane given by

$$w(t) = w_0 e^{-(\cos\gamma - i\sin\gamma)t},\ t \in \mathbb{R},$$

where $w_0 \in \mathbb{C}^*$.

A domain $\Omega \subset \mathbb{C}$ which contains the origin is said to be *spirallike of type* γ, $\gamma \in (-\pi/2, \pi/2)$, if for each $w_0 \in \Omega \setminus \{0\}$, the arc of γ-spiral connecting w_0 with the origin is contained in Ω.

Definition 1.4.24 (**[107]**)

(i) Let $f \in H(U)$ be such that $f(0) = 0$ and let $\gamma \in (-\pi/2, \pi/2)$. The function f is said to be *spirallike of type* γ on the unit disc U if f is univalent on U and $f(U)$ is a spirallike domain of type γ.

(ii) Let $f \in H(U)$ be such that $f(0) = 0$. The function f is said to be *spirallike* if there exists $\gamma \in (-\pi/2, \pi/2)$ such that f is spirallike of type γ.

It is clear that spirallike functions of type 0 are starlike. We denote by $\hat{S}_\gamma$ the class of normalized spirallike functions of type γ on the unit disc, $\gamma \in (-\pi/2, \pi/2)$. Thus, $\hat{S}_\gamma \subset S$.

We next present the analytical characterization of spirallike functions ([107]; see also [44], [81], [93]).

Theorem 1.4.25 (**[107]**) *Let $f \in H(U)$ be such that $f(0) = 0$, $f'(0) \neq 0$, and let $\gamma \in (-\pi/2, \pi/2)$. Then f is spirallike of type γ if and only if*

$$\mathrm{Re}\left[e^{i\gamma}\frac{zf'(z)}{f(z)}\right] > 0,\ z \in U.$$

The following duality theorem between the classes S^* and $\hat{S}_\gamma$ provides many examples of spirallike functions on the unit disc (see e.g. [44], [81]).

Theorem 1.4.26 *Let $\gamma \in (-\pi/2, \pi/2)$ and let $\delta = e^{-i\gamma}\cos\gamma$. Then $f \in \hat{S}_\gamma$ if and only if there exists $g \in S^*$ such that $f(z) = z\left[\frac{g(z)}{z}\right]^\delta$, $z \in U$. We choose the branch of the power function such that $\left[\frac{g(z)}{z}\right]^\delta\Big|_{z=0} = 1$.*

1.4.6 Radius problems for subclasses of S

We next consider some radius problems associated with the class S and some of its subclasses. First, we recall the concept of the radius for a certain property in a certain set (see e.g. [37, II p. 84]; see also e.g. [44], [81]).

Definition 1.4.27 Given $\mathcal{F}$ a nonempty subset of S and a property $\mathcal{P}$ which the functions in $\mathcal{F}$ may or may not have in a disc U_r, the radius for the property $\mathcal{P}$ in the set $\mathcal{F}$ is denoted by $R_{\mathcal{P}}(\mathcal{F})$ and is the largest R such that every function in the set $\mathcal{F}$ has the property $\mathcal{P}$ in each disc U_r for every $r < R$.

We let $R_{S^*}(\mathcal{F})$ be the radius of starlikeness of $\mathcal{F}$, $R_K(\mathcal{F})$ the radius of convexity, $R_{S^*_\alpha}(\mathcal{F})$ the radius of starlikeness of order α and $R_{\hat{S}_\gamma}(\mathcal{F})$ the radius of spirallikeness of type γ of $\mathcal{F}$.

The radius of convexity of S was determined by Nevanlinna (see e.g. [37]).

Theorem 1.4.28 $R_K(S) = 2 - \sqrt{3}$.

Moreover, $R_K(S^*) = 2 - \sqrt{3}$ (see e.g. [37, II p. 86]).

The radius of starlikeness of S was obtained by Grunsky (see e.g. [37]).

Theorem 1.4.29 $R_{S^*}(S) = \tanh\frac{\pi}{4} = \frac{e^{\pi/2} - 1}{e^{\pi/2} + 1} \approx 0.66.$

1.5 The theory of Loewner chains in the complex plane

In this section we are concerned with some classical results in the theory of Loewner chains and the Loewner differential equation on the unit disc in $\mathbb{C}$. We also present certain applications of the Loewner theory to the study of univalent functions, including univalence criteria and analytical characterizations of important subsets of S in terms of Loewner chains. Generalizations of these results to the case of several complex variables will be discussed in the next chapter. The results in this section will be useful in the proofs of the main results of this thesis.

The main bibliographic sources used in this section are [20], [26], [44], [81], [93], [103].

1.5.1 General results regarding Loewner chains

We begin this section with the definition of a univalent subordination chain (see [93]; see also [44], [81]).

Definition 1.5.1 (**[93]**) The function $f : U \times [0,\infty) \to \mathbb{C}$ is said to be a *Loewner chain* (normalized univalent subordination chain) if $f(\cdot,t)$ is univalent on U, $f(0,t) = 0$, $f'(0,t) = e^t$ for $t \geq 0$, and $f(\cdot,s) \prec f(\cdot,t)$, whenever $0 \leq s \leq t < \infty$.

Here $f'(0,t) = \frac{\partial f}{\partial z}(0,t)$.

The above subordination condition is equivalent to the fact that there exists a unique univalent Schwarz function $v = v(z,s,t)$, called the *transition function* associated to $f(z,t)$, such that

$$f(z,s) = f(v(z,s,t),t), \quad z \in U, \quad t \geq s \geq 0.$$

We note that if $f(z,t)$ is a Loewner chain, then $f(\cdot,0) \in S$. The following theorem shows that any function in the class S can be embedded as the first element of a Loewner chain. This result is due to Pommerenke [93].

Theorem 1.5.2 *For each $f \in S$, there exists a Loewner chain $f(z,t)$ such that $f(z) = f(z,0)$, $z \in U$.*

From now on, if f is a function which is holomorphic with respect to $z \in U$ and is also a function of other real variable, we denote by $f'(z,\cdot)$ the derivative of f with respect to z, and thus $f'(z,\cdot) = \frac{\partial f}{\partial z}(z,\cdot)$.

We now recall some main results regarding the Loewner differential equation on the unit disc. The following theorem due to Pommerenke [93] shows that the solution of the initial value problem (1.5.1) generates a Loewner chain (see also, e.g. [44], [81], [103]).

Theorem 1.5.3 (**[93]**) *Let $p = p(z,t) : U \times [0,\infty) \to \mathbb{C}$ satisfy the following conditions:*

(i) $p(\cdot,t) \in \mathcal{P}$, for each $t \geq 0$;

(ii) $p(z,\cdot)$ is a measurable function on $[0,\infty)$, for each $z \in U$.

Then for each $z \in U$ and $s \geq 0$, the initial value problem

$$\frac{\partial v}{\partial t} = -vp(v,t), \text{ a.e. } t \geq s, \; v(z,s,s) = z, \tag{1.5.1}$$

has a unique solution $v(t) = v(z,s,t) = e^{s-t}z + \ldots$. Moreover, for fixed $s \geq 0$ and $z \in U$, $v(z,s,\cdot)$ is Lipschitz continuous on $[s,\infty)$ locally uniformly with respect to z. Also, $v(\cdot,s,t)$ is a univalent Schwarz function for $t \in [s,\infty)$. In addition, for each $s \geq 0$, the limit

$$f(z,s) = \lim_{t\to\infty} e^t v(z,s,t) \tag{1.5.2}$$

exists locally uniformly on U, and $f(z,s)$ is a Loewner chain, which satisfies the differential equation

$$\frac{\partial f}{\partial t}(z,t) = zp(z,t)f'(z,t), \quad \text{a.e.} \quad t \geq 0,$$

for all $z \in U$.

The following theorem is due to Pommerenke (see [93]). Theorem 1.5.4 provides a necessary and sufficient condition for a function $f(z,t)$ to be a Loewner chain. This result is very useful in applications and shows that Loewner chains are related to the Loewner differential equation. For more details, see [44], [93] (see also [103]).

Theorem 1.5.4 (**[93]**) *The function $f : U \times [0,\infty) \to \mathbb{C}$ such that $f(0,t) = 0,\ f'(0,t) = e^t,\ t \geq 0$, is a Loewner chain if and only if the following relations hold:*

(i) *There exist $r \in (0,1)$ and a constant $M > 0$ such that $f(\cdot,t)$ is holomorphic on U_r for each $t \geq 0$, $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in U_r$, and*

$$|f(z,t)| \leq Me^t,\ z \in U_r,\ t \geq 0.$$

(ii) *There exists a function $p(z,t)$ such that $p(\cdot,t) \in \mathcal{P}$ for each $t \geq 0$, $p(z,\cdot)$ is measurable on $[0,\infty)$ for each $z \in U$, and for almost all $t \geq 0$,*

$$\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t),\ z \in U_r.$$

1.5.2 Loewner chains and univalent functions on the unit disc

In this section we present characterizations of some subclasses of univalent functions by using Loewner chains (see e.g. [44]). We also present some well known univalence criteria on the unit disc, which were obtained by the method of Loewner chains.

Loewner chains and subclasses of univalent functions

The following theorem presents the characterization of starlikeness in terms of Loewner chains (see [93]).

Theorem 1.5.5 *Let f be a normalized holomorphic function on U. Then f is starlike if and only if*

$$f(z,t) = e^t f(z),\ z \in U,\ t \geq 0$$

is a Loewner chain.

A generalization of Theorem 1.5.5 is given in the following result, which represents the characterization of spirallikeness of type γ by using Loewner chains (see [93]).

Theorem 1.5.6 *Let f be a normalized holomorphic function on U and let $\gamma \in (-\pi/2, \pi/2)$. Also, let $a = \tan\gamma$. Then f is spirallike of type γ if and only if*

$$f(z,t) = e^{(1-ia)t} f(e^{iat}z),\ z \in U,\ t \geq 0$$

is a Loewner chain.

The next theorem provides the characterization of convexity by using Loewner chains.

Theorem 1.5.7 ([44], [93]) *Let f be a normalized holomorphic function on U. Then $f \in K$ if and only if*

$$f(z,t) = f(z) + (e^t - 1)zf'(z),\ z \in U,\ t \geq 0$$

is a Loewner chain.

Loewner chains and univalence criteria on the unit disc

We conclude this section with some applications of the theory of Loewner chains to univalence criteria on the unit disc. Generalizations of some of these results to higher dimensions will be discussed in the next chapter. The following result was obtained by Becker [5].

Theorem 1.5.8 *Let $f : U \to \mathbb{C}$ be a normalized holomorphic function. If*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,\ z \in U, \tag{1.5.3}$$

then f is univalent on U.

The following criterion for univalence is due to Ahlfors and Becker (see [5]; see also e.g. [44]).

Theorem 1.5.9 *Let $f : U \to \mathbb{C}$ be a normalized holomorphic function. Also, let $c \in \mathbb{C}$ be such that $|c| \leq 1$, $c \neq -1$. If*

$$\left|(1 - |z|^2)\frac{zf''(z)}{f'(z)} + c|z|^2\right| \leq 1,\ z \in U,$$

then f is univalent on U.

Remark 1.5.10 If we take $c = 0$ in Theorem 1.5.9, we obtain the sufficient condition for univalence due to Becker [5], given in Theorem 1.5.8.

Chapter 2

Biholomorphic mappings in several complex variables

In this chapter we present basic properties of holomorphic functions and holomorphic mappings in $\mathbb{C}^n$. We recall a well known result in the theory of holomorphic mappings in higher dimensions, which yields that the Euclidean unit ball and the unit polydisc in $\mathbb{C}^n$ are not biholomorphically equivalent for $n \geq 2$, and thus the Riemann mapping theorem does not hold in several complex variables. We present some known results related to holomorphic mappings in $\mathbb{C}^n$ that are useful in the next sections of this thesis. We also present basic ideas related to subordination, and the generalization to higher dimensions of the Carathéodory class of functions with positive real part on the unit disc. We present growth and distortion theorems, as well as coefficient estimates for this class, due to Graham, Hamada and Kohr [38], Pfaltzgraff [88], and Poreda [94]. One of the most important results in this direction is the compactness of the Carathéodory family $\mathcal{M}$. This result was obtained in 2002 by Graham, Hamada and Kohr [38], and has influenced many results in the theory of Loewner chains in higher dimensions (see [44]).

We also study several subclasses of biholomorphic mappings on the Euclidean unit ball and the unit polydisc in $\mathbb{C}^n$, such as starlike, convex, spirallike, close-to-starlike mappings, and we give analytical and geometric properties of these classes.

Further, we recall the notions of Loewner chains and the associated transition mappings in higher dimensions. On the other hand, we present the generalized Loewner differential equation in several complex variables, and the connection with Loewner chains. In fact, as in the case of one complex variable, all Loewner chains are basically determined by the generalized Loewner differential equation. However, in higher dimensions there are many differences compared to the one variable theory (see [44, Chapter 8]). We present applications of the theory of Loewner chains in characterizing certain subclasses of biholomorphic mappings. We also recall some univalence criteria on the unit ball in $\mathbb{C}^n$, based on the theory of Loewner chains in $\mathbb{C}^n$. Finally, we consider the class of mappings which have parametric representation on the Euclidean unit ball B^n in $\mathbb{C}^n$. This class is the analogous of the class S to higher dimensions. Many details and applications of the theory of Loewner chains in several complex variables may be found in the main papers of

Pfaltzgraff ([88] and [89]) and Poreda [96], and the monographs of Graham and Kohr [44], and Curt [23].

2.1 Preliminary results

This section is devoted to basic properties of holomorphic mappings in the case of several complex variables. The results in this section are classical and may be found in [10], [63], [69], [98], main bibliographic sources used during the preparation of this section. These results will be useful in the next chapters of this thesis.

2.1.1 Holomorphic functions in $\mathbb{C}^n$

We begin this section by recalling some well known results related to holomorphic functions of several complex variables.

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Let $a \in \mathbb{C}^n$ and let $r > 0$. Let

$$B^n(a, r) = \{z \in \mathbb{C}^n : \|z - a\| < r\}$$

be the open ball of center a and radius r. The closure of $B^n(a, r)$ is denoted by $\overline{B}^n(a, r)$ and the boundary by $\partial B^n(a, r)$. We write B^n_r instead of $B^n(0, r)$. The open unit ball B^n_1 is denoted by B^n and is called the Euclidean unit ball in $\mathbb{C}^n$.

The open polydisc $P^n(a, R)$ of center $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ and multiradius $R = (r_1, \dots, r_n) \in \mathbb{R}^n_+$ is defined by

$$P^n(a, R) = U(a_1, r_1) \times \cdots \times U(a_n, r_n).$$

If $r_1 = \ldots = r_n = r$, we denote this polydisc by $P^n(a, r)$. The unit polydisc $P^n(0, 1)$ is denoted by P^n. Clearly, P^n is the unit ball in $\mathbb{C}^n$ with respect to the maximum norm, $\|z\|_\infty = \max_{1\leq j\leq n} |z_j|$, and thus

$$P^n = \{z \in \mathbb{C}^n : \|z\|_\infty < 1\}.$$

We recall that $\Omega \subseteq \mathbb{C}^n$ is a *Reinhardt domain* if, whenever $(z_1, \dots, z_n) \in \Omega$ and $\theta_j \in \mathbb{R}$, $j = 1, \dots, n$, we have $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in \Omega$ (see e.g. [44]).

Let Ω be an open subset of $\mathbb{C}^n$. We next give the definition of holomorphic functions on Ω (see e.g. [44], [69], [85]).

Definition 2.1.1 Let $f : \Omega \to \mathbb{C}$ be a function. We say that f is *holomorphic* if f is continuous on Ω and holomorphic in each variable separately, i.e. for each $w = (w_1, \dots, w_n) \in \Omega$ and $j = 1, \dots, n$, the function of one complex variable

$$f(w_1, \dots, w_{j-1}, \cdot, w_{j+1}, \dots, w_n)$$

is holomorphic on the open set

$$\{\zeta \in \mathbb{C} : (w_1, \dots, w_{j-1}, \zeta, w_{j+1}, \dots, w_n) \in \Omega\}.$$

Hartogs showed that the assumption of continuity in Definition 2.1.1 can be omitted, and thus any holomorphic function in each variable separately (partially holomorphic function) is holomorphic (see e.g. [10], [69]). We denote by $H(\Omega, \mathbb{C})$ the set of holomorphic functions from an open set $\Omega \subseteq \mathbb{C}^n$ into $\mathbb{C}$. Holomorphic functions on the whole space $\mathbb{C}^n$ are called *entire functions*.

We next present basic properties of holomorphic functions that will be useful in the next sections.

The following result is known as the *open mapping theorem* and shows that any nonconstant holomorphic function is open (see e.g. [63], [85]).

Theorem 2.1.2 *Let $f : \Omega \to \mathbb{C}$ be a nonconstant holomorphic function, where Ω is a domain (open and connected set) in $\mathbb{C}^n$. Then $f(\Omega)$ is a domain in $\mathbb{C}$.*

Note that the above result is not true in the case of holomorphic mappings $f : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$, where $m > 1$ (see e.g. [98]). Still, a generalization of this result holds in the case of locally biholomorphic mappings from domains in $\mathbb{C}^n$ into $\mathbb{C}^n$ (see [63]).

The next result given in Theorem 2.1.3 is an analogous of Montel's theorem to the case of several complex variables (see e.g. [63], [85], [98]).

Theorem 2.1.3 (Montel's theorem) *Let $\Omega \subseteq \mathbb{C}^n$ be an open set and let $\mathcal{F} \subseteq H(\Omega, \mathbb{C})$. Then $\mathcal{F}$ is a normal family if and only if $\mathcal{F}$ is locally uniformly bounded.*

As in the case of one complex variable, we have the following characterization of compact subsets in $H(\Omega, \mathbb{C})$, where $\Omega \subset \mathbb{C}^n$ is an open set (see e.g. [63], [85], [98]). This result will be useful in the forthcoming sections.

Corollary 2.1.4 *Let Ω be an open set in $\mathbb{C}^n$ and let $\mathcal{F} \subseteq H(\Omega, \mathbb{C})$. Then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is locally uniformly bounded and closed.*

2.1.2 Holomorphic mappings in $\mathbb{C}^n$

We next discuss the case of holomorphic mappings from open subsets of $\mathbb{C}^n$ into $\mathbb{C}^m$, where $m > 1$.

First, we recall the well known notion of a holomorphic mapping from an open set in $\mathbb{C}^n$ into $\mathbb{C}^m$ (see e.g. [44], [63], [69]).

Definition 2.1.5 Let Ω be an open subset of $\mathbb{C}^n$ and let $f : \Omega \to \mathbb{C}^m$ be a mapping, where $m > 1$. We say that f is *holomorphic* if each component f_k of f is a holomorphic function from Ω into $\mathbb{C}$, for $k = 1, \dots, m$.

We denote by $H(\Omega)$ the set of holomorphic mappings from Ω into $\mathbb{C}^n$.

We next give some basic results in the theory of holomorphic mappings in $\mathbb{C}^n$. The following result is a generalization of the maximum principle to holomorphic mappings (see e.g. [63], [85]). We consider the space $\mathbb{C}^n$ with respect to an arbitrary norm $\|\cdot\|$.

Theorem 2.1.6 *Let Ω be a domain in $\mathbb{C}^n$ and let $f : \Omega \to \mathbb{C}^m$ be a holomorphic mapping. If there exists $z_0 \in \Omega$ such that*

$$\|f(z_0)\| = \max\{\|f(z)\| : z \in \Omega\},$$

then $\|f(z)\|$ is constant on Ω.

A consequence of Theorem 2.1.6 is the following generalization of Schwarz's lemma to holomorphic mappings defined on the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm (see e.g. [63], [85]).

Corollary 2.1.7 (Schwarz's lemma) *If $f : B^n \subset \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic mapping such that $f(0) = 0$ and $\|f(z)\| < 1$, $z \in B^n$, then $\|f(z)\| \leq \|z\|$, $z \in B^n$, and $\|Df(0)\| \leq 1$. In addition, if there exists $z_0 \in B^n \setminus \{0\}$ such that $\|f(z_0)\| = \|z_0\|$, then $\|f(\zeta z_0)\| = \|\zeta z_0\|$, for all $\zeta \in \mathbb{C}$ such that $|\zeta| < 1/\|z_0\|$.*

We next present the notions of biholomorphy and univalence in several complex variables (see e.g. [44], [63], [69], [85]). Also, we give some examples of biholomorphic mappings on the unit ball B^n in $\mathbb{C}^n$.

Definition 2.1.8 Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $f : \Omega \to \mathbb{C}^n$.

(i) The mapping f is a *biholomorphic mapping* if f is a holomorphic mapping of Ω onto a domain $\Omega' \subseteq \mathbb{C}^n$ and f has an inverse f^{-1} which is holomorphic on Ω'. In this case, the domains Ω and Ω' are called *biholomorphically equivalent*.

(ii) We say that f is a *univalent mapping* if f is holomorphic and injective on Ω.

As in the case of one complex variable, the notions of biholomorphy and univalence are equivalent (see e.g. [85], [98]). We note that Theorem 2.1.9 does not hold in infinite dimensional complex Banach spaces (see [110]).

Theorem 2.1.9 (**[85], [98]**) *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $f : \Omega \to \mathbb{C}^n$ be a mapping. Then f is univalent on Ω if and only if f is biholomorphic from Ω onto $f(\Omega)$.*

The following result due to Poincaré [92] is well known in several complex variables and shows that in higher dimensions the Euclidean unit ball and the unit polydisc are not biholomorphically equivalent, although they are homeomorphic (see e.g. [85], [98]). Hence, the Riemann mapping theorem does not hold in several complex variables (see [98]).

Theorem 2.1.10 *The Euclidean unit ball B^n and the unit polydisc P^n are not biholomorphically equivalent for $n \geq 2$.*

We next give the definition of a locally biholomorphic mapping (see e.g. [44], [63], [85]).

Definition 2.1.11 Let Ω be a domain in $\mathbb{C}^n$ and let $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping. We say that f is *locally biholomorphic* on Ω if for each $z \in \Omega$, there exists an open and connected neighborhood $V \subset \Omega$ of z such that the restriction $f|_V : V \to f(V)$ is biholomorphic.

Remark 2.1.12 It is well known that if $\Omega \subseteq \mathbb{C}^n$ is a domain and $f \in H(\Omega)$, then f is locally biholomorphic on Ω if and only if $J_f(z) \neq 0$, $z \in \Omega$, where $J_f(z) = \det Df(z)$.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm,

$$\|A\| = \sup\{\|A(z)\| : \|z\| = 1\},$$

and let I_n be the identity of $L(\mathbb{C}^n, \mathbb{C}^n)$. If $\Omega \subseteq \mathbb{C}^n$ is an open set which contains the origin, and if $f \in H(\Omega)$, then f is called *normalized* if $f(0) = 0$ and $Df(0) = I_n$.

We denote by $S(\Omega)$ the set of normalized biholomorphic mappings on a domain $\Omega \subset \mathbb{C}^n$. In particular, we denote by $S(B^n)$ the set of normalized biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$. Obviously, if $n = 1$, then $S(B^1) = S$ is the usual set of normalized univalent functions on the unit disc U.

We also denote by $\mathcal{L}S_n(\Omega)$ the set of normalized locally biholomorphic mappings on a domain $\Omega \subseteq \mathbb{C}^n$. In particular, we denote by $\mathcal{L}S_n(B^n)$ the set of normalized locally biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$. In the case of one complex variable, we denote $\mathcal{L}S_1(B^1) = \mathcal{L}S$.

2.2 Subordination. The Carathéodory class in $\mathbb{C}^n$

We next present the notion of subordination in several complex variables and the generalization of the Carathéodory class $\mathcal{P}$ to $\mathbb{C}^n$, denoted by $\mathcal{M}$.

The main bibliographic sources used in this section are [44], [38] and [88].

We first recall the definition of subordination (see e.g. [44]).

Definition 2.2.1 Let $f, g \in H(B^n)$. We say that f is *subordinate* to g (and write $f \prec g$) if there is a Schwarz mapping v (i.e. $v \in H(B^n)$ and $\|v(z)\| \leq \|z\|$, $z \in B^n$) such that $f(z) = g(v(z))$, $z \in B^n$.

As in the case $n = 1$, if g is biholomorphic on B^n, the following characterization of subordination holds (see e.g. [44]).

Theorem 2.2.2 *Let $f, g \in H(B^n)$ be such that g is biholomorphic on B^n. Then $f \prec g$ if and only if $f(0) = g(0)$ and $f(B^n) \subseteq g(B^n)$.*

The following class of normalized holomorphic mappings on B^n plays the role of the Carathéodory class in $\mathbb{C}^n$ (see [49], [88], [110]; see also e.g. [44], [63]):

$$\mathcal{M} = \{h \in H(B^n) : h(0) = 0, Dh(0) = I_n, \operatorname{Re}\langle h(z), z\rangle > 0, z \in B^n \setminus \{0\}\}.$$

In the case $n = 1$, $h \in \mathcal{M}$ if and only if $p \in \mathcal{P}$, where $h(\zeta) = \zeta p(\zeta)$, for $\zeta \in U$. The class $\mathcal{M}$ plays an important role in the study of Loewner chains and the Loewner differential equation in several complex variables, as well as in characterizing certain classes of biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$ (for details, see [44]).

The following theorem is due to Pfaltzgraff [88] (see also [49], in the case of complex Banach spaces).

Theorem 2.2.3 *If $h : B^n \to \mathbb{C}^n$ is a mapping in the class $\mathcal{M}$, then*

$$\|z\|^2 \frac{1 - \|z\|}{1 + \|z\|} \leq \operatorname{Re} \langle h(z), z \rangle \leq \|z\|^2 \frac{1 + \|z\|}{1 - \|z\|}, \ z \in B^n. \tag{2.2.1}$$

These estimates are sharp.

Note that the following result holds for mappings in the class $\mathcal{M}$. The lower bound in (2.2.2) is a direct consequence of the relation (2.2.1) in Theorem 2.2.3. The upper bound in (2.2.2) is stronger than the upper bound in (2.2.1) and was obtained by Graham, Hamada and Kohr [38].

Theorem 2.2.4 *If $h : B^n \to \mathbb{C}^n$ belongs to the class $\mathcal{M}$, then*

$$r \frac{1 - r}{1 + r} \leq \|h(z)\| \leq \frac{4r}{(1 - r)^2}, \ \|z\| = r < 1. \tag{2.2.2}$$

Theorem 2.2.4 and the fact that the class $\mathcal{M}$ is closed lead to the following conclusion. This result was obtained by Graham, Hamada and Kohr [38] (see also [55]).

Corollary 2.2.5 (**[38]**, **[55]**) *The class $\mathcal{M}$ is compact in $H(B^n)$.*

2.3 Subclasses of biholomorphic mappings on B^n

In this section we present some subclasses of biholomorphic mappings on the unit ball in $\mathbb{C}^n$. We refer to the classes of normalized starlike, convex, close-to-starlike, and spirallike mappings, respectively. We also refer to some subclasses of $S(B^n)$ such as: the class of starlike mappings of order α, the class of ε-starlike mappings, and the class of almost starlike mappings of order α. We shall see that most of these subclasses of $S(B^n)$ have analytical and geometric characterizations.

The main bibliographic sources used in this section are [44], [63], [110] (see also [32]).

This section does not contain original results, however Definition 2.3.7 is due to Chirilă [12].

2.3.1 Starlike mappings

Next we discuss the case of starlike mappings on the unit ball of $\mathbb{C}^n$. The following results are generalizations to higher dimensions of the properties of starlike functions on the unit disc. We consider $\mathbb{C}^n$ with respect to the usual Euclidean inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$, $z \in \mathbb{C}^n$. However, the following results remain true with respect to an arbitrary norm in $\mathbb{C}^n$ (see e.g. [32], [44]).

We first give the definition of starlikeness on the Euclidean unit ball B^n (see e.g. [44], [63]).

Definition 2.3.1 Let $f : B^n \to \mathbb{C}^n$ be a holomorphic mapping. We say that f is *starlike* if f is biholomorphic on B^n, $f(0) = 0$ and $f(B^n)$ is a starlike domain with respect to the origin.

The following analytical characterization of starlikeness on the Euclidean unit ball B^n is due to Matsuno [79]. Suffridge generalized this result to the unit polydisc in $\mathbb{C}^n$ (see [108]). Gurganus [49] and Suffridge [109] generalized this result on the unit ball of a complex Banach space.

Theorem 2.3.2 ([79], [109]) *Let $f : B^n \to \mathbb{C}^n$ be a locally biholomorphic mapping such that $f(0) = 0$. Then f is starlike if and only if*

$$\operatorname{Re}\langle [Df(z)]^{-1} f(z), z\rangle > 0,\ z \in B^n \setminus \{0\}. \tag{2.3.1}$$

We denote by $S^*(B^n)$ the class of normalized starlike mappings on B^n. If $n = 1$, this class reduces to the class S^* of normalized starlike functions on the unit disc U.

We next present the growth theorem for mappings in $S^*(B^n)$ due Kubicka and Poreda [71], and Barnard, FitzGerald and Gong [4].

Theorem 2.3.3 ([4], [71]) *If $f : B^n \to \mathbb{C}^n$ is a normalized starlike mapping, then*

$$\frac{\|z\|}{(1+\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^2},\ z \in B^n.$$

This result is sharp. Also, $f(B^n) \supseteq B^n_{1/4}$.

Starlike mappings of order α

We next present the notion of starlikeness of order α on the Euclidean unit ball B^n in $\mathbb{C}^n$, due to Curt [22] and Kohr [64].

Definition 2.3.4 Let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping and let $\alpha \in [0, 1)$. The mapping f is said to be *starlike of order* α if

$$\operatorname{Re}\left[\frac{\|z\|^2}{\langle [Df(z)]^{-1} f(z), z\rangle}\right] > \alpha,\ z \in B^n \setminus \{0\}.$$

Let $S^*_\alpha(B^n)$ denote the set of starlike mappings of order α on B^n.

We present the growth theorem for starlike mappings of order $\alpha \in [0, 1)$ on B^n (see [22], [38]).

Theorem 2.3.5 *Let $f \in S^*_\alpha(B^n)$, $\alpha \in [0, 1)$. Then*

$$\frac{\|z\|}{(1+\|z\|)^{2(1-\alpha)}} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^{2(1-\alpha)}},\ z \in B^n.$$

These estimates are sharp.

Almost starlike mappings of order α

We next present the class of almost starlike mappings of order α on the Euclidean unit ball in $\mathbb{C}^n$ (see [114]).

Definition 2.3.6 Let $0 \leq \alpha < 1$. A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is *almost starlike of order* α if

$$\mathrm{Re}\,\langle [Df(z)]^{-1} f(z), z\rangle > \alpha \|z\|^2,\ z \in B^n \setminus \{0\}.$$

A closely related notion to almost starlikeness of order α is that of almost starlikeness of order α and type γ, where $\alpha \in [0,1)$ and $\gamma \in [0,1)$. This notion was introduced by Chirilă [12].

Definition 2.3.7 Let $\alpha \in [0,1)$ and $\gamma \in [0,1)$. A mapping $f \in \mathcal{L}S_n(B^n)$ is almost starlike of order α and type γ if

$$\mathrm{Re}\left[\frac{1}{\frac{1}{(1-\alpha)\|z\|^2}\langle [Df(z)]^{-1} f(z), z\rangle - \frac{\alpha}{1-\alpha}}\right] > \gamma,\ z \in B^n \setminus \{0\}. \tag{2.3.2}$$

Remark 2.3.8 (see e.g. **[11]**) It is easily seen that if f satisfies (2.3.2), then f is also almost starlike of order α. Hence $f \in S(B^n)$ (see [114]). In fact, f is also starlike on B^n.

2.3.2 Convex and close-to-starlike mappings

In the following, we present the case of convex mappings on the Euclidean unit ball and the unit polydisc in $\mathbb{C}^n$.

In the second part of this section, we also present the class of close-to-starlike mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$, as well as the class of ε-starlike mappings.

The main bibliographic sources used in this section are [32], [44], [63] (see also [33], [110]).

First, we give the definition of convex mappings on the unit ball B^n in $\mathbb{C}^n$ with respect to an arbitrary norm $\|\cdot\|$ (see e.g. [44], [63]).

Definition 2.3.9 A holomorphic mapping $f : B^n \to \mathbb{C}^n$ is *convex* if f is biholomorphic and $f(B^n)$ is a convex domain.

We denote by $K(B^n)$ the set of normalized convex mappings on the unit ball B^n. If $n = 1$, $K(B^n)$ reduces to the class K of normalized convex functions on the unit disc U.

Convex mappings on the unit polydisc P^n in $\mathbb{C}^n$

The following characterization of convexity on the unit polydisc P^n in $\mathbb{C}^n$ was obtained by Suffridge [108]. In this result, we consider $\mathbb{C}^n$ with respect to the maximum norm $\|\cdot\|_\infty$.

Theorem 2.3.10 *Let $f : P^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping. Then f is convex if and only if there exist $f_j \in K$, $j = 1, \ldots, n$, such that*

$$f(z) = (f_1(z_1), \ldots, f_n(z_n)),\ z = (z_1, \ldots, z_n) \in P^n.$$

We next present the growth theorem for mappings in the class $K(P^n)$ [74] (cf. [44]).

Theorem 2.3.11 (**[74]**) *If $f : P^n \to \mathbb{C}^n$ is a mapping in $K(P^n)$, then*

$$\frac{\|z\|_\infty}{1 + \|z\|_\infty} \leq \|f(z)\|_\infty \leq \frac{\|z\|_\infty}{1 - \|z\|_\infty},\ z \in P^n.$$

These estimates are sharp.

Convex mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$

The next theorem presents the analytical characterization of convexity on the Euclidean unit ball B^n in $\mathbb{C}^n$. This result is due to Kikuchi [60] and Gong, Wang and Yu [36].

Theorem 2.3.12 (**[36], [60]**) *If $f : B^n \to \mathbb{C}^n$ is a locally biholomorphic mapping, then f is convex if and only if*

$$1 - \operatorname{Re} \langle [Df(z)]^{-1} D^2 f(z)(u, u), z\rangle > 0, \tag{2.3.3}$$

for all $z \in B^n$ and $u \in \mathbb{C}^n$ such that $\|u\| = 1$ and $\operatorname{Re}\langle z, u\rangle = 0$.

We now give some examples of convex mappings (see e.g. [44], [63]).

Example 2.3.13 (i) The mapping $f : P^n \to \mathbb{C}^n$ given by

$$f(z) = \left(\frac{z_1}{1 - z_1}, \ldots, \frac{z_n}{1 - z_n}\right),\ z = (z_1, \ldots, z_n) \in P^n,$$

is convex on the unit polydisc in $\mathbb{C}^n$. However, the restriction of this mapping to the Euclidean unit ball B^n in $\mathbb{C}^n$ is not convex, for $n \geq 2$ (see [36]; see also [32]).

(ii) The mapping $f : B^n \to \mathbb{C}^n$ given by

$$f(z) = \left(\frac{z_1}{1 - z_1}, \ldots, \frac{z_n}{1 - z_1}\right),\ z = (z_1, \ldots, z_n) \in B^n,$$

is convex on the Euclidean unit ball B^n in $\mathbb{C}^n$ (see [101] and [102]).

We now give the growth result for mappings in $K(B^n)$, where B^n is the Euclidean unit ball in $\mathbb{C}^n$. This result was obtained by Suffridge [111], FitzGerald and Thomas [31], Liu [74].

Theorem 2.3.14 *If $f : B^n \to \mathbb{C}^n$ is a mapping in $K(B^n)$, then*

$$\frac{\|z\|}{1+\|z\|} \leq \|f(z)\| \leq \frac{\|z\|}{1-\|z\|},\ z \in B^n.$$

These estimates are sharp.

In one dimension, Marx-Strohhäcker theorem states that a normalized convex function is starlike of order $1/2$ (see Theorem 1.4.17). This result was generalized to several complex variables by Curt [22] and Kohr [64].

Theorem 2.3.15 *If $f \in K(B^n)$, then $f \in S^*_{1/2}(B^n)$. This result is sharp.*

Close-to-starlike mappings

Next, we consider the notion of close-to-starlikeness on the Euclidean unit ball B^n in $\mathbb{C}^n$. The extension to higher dimensions of the class of close-to-convex functions was considered by Pfaltzgraff and Suffridge [90].

Definition 2.3.16 (**[90]**) A normalized holomorphic mapping $f : B^n \to \mathbb{C}^n$ is *close-to-starlike* if there exist $h \in \mathcal{M}$ and a mapping $g \in S^*(B^n)$ such that

$$Df(z)h(z) = g(z),\ z \in B^n. \tag{2.3.4}$$

It is obvious that any mapping $f \in S^*(B^n)$ is also close-to-starlike (with respect to itself).

If $n = 1$, Definition 2.3.16 reduces to the definition of close-to-convex functions on the unit disc U (Definition 1.4.21), by using Alexander's theorem.

ε-starlike mappings

We next present the notion of ε-starlikeness introduced by Gong and Liu [34]. This notion interpolates between starlikeness and convexity as ε ranges from 0 to 1. The notion of ε-starlikeness will be useful in the next chapter.

Definition 2.3.17 (**[34]**) Let Ω be a domain in $\mathbb{C}^n$ which contains the origin and let $f : \Omega \to \mathbb{C}^n$ be a biholomorphic mapping such that $f(0) = 0$. We say that f is *ε-starlike*, $0 \leq \varepsilon \leq 1$, if $f(\Omega)$ is starlike with respect to each point in $\varepsilon f(\Omega)$, i.e.

$$(1-\lambda)f(z) + \lambda\varepsilon f(w) \in f(\Omega),\ \lambda \in [0,1],\ z, w \in \Omega.$$

When $\varepsilon = 0$ we obtain the family of starlike mappings on Ω, and when $\varepsilon = 1$ we obtain the family of convex mappings on Ω.

Certain results regarding ε-starlike mappings may be found in [34], [35].

2.3.3 Spirallike mappings

We next present the class of spirallike mappings in several complex variables. The notion of spirallikeness with respect to a normal linear operator whose eigenvalues have positive real part was given by Gurganus [49]. Suffridge [110] generalized the notion of spirallikeness to complex Banach spaces.

Definition 2.3.18 (**[110]**; e.g. **[44]**) Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$, where

$$m(A) = \min\{\mathrm{Re}\,\langle A(z), z\rangle : \|z\| = 1\}.$$

A normalized biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is *spirallike relative to A* if $e^{-tA} f(B^n) \subseteq f(B^n)$ for all $t \geq 0$, where

$$e^{-tA} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k A^k.$$

The following result is due to Suffridge [110] and represents a generalization to higher dimensions of Theorem 1.4.25. The case when A is normal was considered by Gurganus [49].

Theorem 2.3.19 (**[110]**, **[49]**) *Let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping and let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$. Then f is spirallike relative to A if and only if*

$$\mathrm{Re}\,\langle [Df(z)]^{-1} A f(z), z\rangle > 0, \quad z \in B^n \setminus \{0\}. \tag{2.3.5}$$

Remark 2.3.20 If $A = e^{-i\alpha} I_n$, $\alpha \in (-\pi/2, \pi/2)$, we obtain the class of *spirallike mappings of type* α, studied by Hamada and Kohr [54]. In this case the condition (2.3.5) becomes

$$\mathrm{Re}\,[e^{-i\alpha} \langle [Df(z)]^{-1} f(z), z\rangle] > 0, \quad z \in B^n \setminus \{0\}. \tag{2.3.6}$$

This class has a similar behaviour as the class of spirallike functions of type α on the unit disc U.

A closely related notion to spirallikeness of type α is that of spirallikeness of type α and order γ, where $\alpha \in (-\pi/2, \pi/2)$ and $\gamma \in [0, 1)$. This notion was studied by Liu and Liu [77] and Chirilă [11] (cf. [54]).

Definition 2.3.21 (**[77]**; cf. **[11]**) Let $f \in \mathcal{L}S_n(B^n)$, $\alpha \in (-\pi/2, \pi/2)$ and $\gamma \in [0, 1)$. We say that f is spirallike of type α and order γ if

$$\mathrm{Re}\left[\frac{1}{(1 - i\tan\alpha)\frac{1}{\|z\|^2}\langle [Df(z)]^{-1} f(z), z\rangle + i\tan\alpha}\right] > \gamma, \ z \in B^n \setminus \{0\}. \tag{2.3.7}$$

Remark 2.3.22 (see e.g. **[11]**) From (2.3.7) we deduce that if f is spirallike of type α and order γ, then f is also spirallike of type α. Hence $f \in S(B^n)$ (see [54]). In fact, f has parametric representation, as we shall see in the next section (cf. [38]).

2.4 The theory of Loewner chains in several complex variables

In this section we shall present basic results of Loewner's theory in $\mathbb{C}^n$ and we give various applications, such as univalence criteria and analytical characterizations of certain subclasses of $S(B^n)$ using Loewner chains. Also, we present the class $S_g^0(B^n)$ of mappings with g-parametric representation on the unit ball B^n, where g is a univalent function on the unit disc U that satisfies certain natural assumptions.

The main bibliographic sources used in this section are [44], [23], [38] and [47] (see also [88] and [96]).

2.4.1 General results regarding Loewner chains in $\mathbb{C}^n$

We shall present main results regarding Loewner chains and the Loewner differential equation in several complex variables.

First, we present the definition of Loewner chains in several complex variables (see e.g. [23], [44], [88], [96]).

Definition 2.4.1 (**[88]**, **[44]**) A mapping $f : B^n \times [0,\infty) \to \mathbb{C}^n$ is said to be a *Loewner chain* (normalized univalent subordination chain) if $f(\cdot,t)$ is biholomorphic on B^n, $f(0,t)=0$, $Df(0,t)=e^tI_n$, for $t\geq 0$, and $f(\cdot,s)\prec f(\cdot,t)$, whenever $0\leq s\leq t<\infty$.

The above subordination condition is equivalent to the fact that there is a unique biholomorphic Schwarz mapping $v=v(z,s,t)$, called the *transition mapping* associated to $f(z,t)$, such that (see [88], [44])

$$f(z,s)=f(v(z,s,t),t),\ z\in B^n,\ t\geq s\geq 0.$$

We next present main results regarding the Loewner differential equation in several complex variables. The first result is due to Pfaltzgraff [88, Theorem 2.1]. Poreda [96] studied the initial value problem (2.4.1) on the unit ball of a complex Banach space.

Theorem 2.4.2 (**[88]**) *Let $h : B^n\times[0,\infty)\to\mathbb{C}^n$ satisfy the following conditions:*

(*i*) $h(\cdot,t)\in\mathcal{M}$ *for* $t\geq 0$*;*

(*ii*) $h(z,\cdot)$ *is measurable on* $[0,\infty)$ *for* $z\in B^n$.

Then for each $s\geq 0$ and $z\in B^n$, the initial value problem

$$\frac{\partial v}{\partial t}=-h(v,t),\ \text{a.e. } t\geq s,\ v(s)=z, \tag{2.4.1}$$

has a unique locally absolutely continuous solution $v(t)=v(z,s,t)=e^{s-t}z+\dots$. Moreover, for fixed s and t, $0\leq s\leq t<\infty$, $v_{s,t}(z)=v(z,s,t)$ is a univalent Schwarz mapping, and for fixed $s\geq 0$ and $z\in B^n$ it is a Lipschitz function of $t\geq s$ locally uniformly with respect to z.

From now on, if $f = f(z, \cdot)$ is a mapping which is holomorphic with respect to $z \in B^n$ and also depends of other real variable, we write $Df(z, \cdot)$ for the differential of f in the z variable.

The following result shows that if $v(t) = v(z, s, t)$ is the unique solution of the initial value problem (2.4.1), then $\lim_{t\to\infty} e^t v(z, s, t)$ exists and gives rise to a Loewner chain (cf. [96]; see also e.g. [44]).

Theorem 2.4.3 (*cf.* **[96]**; *e.g.* **[44]**) *Let $h : B^n \times [0, \infty) \to \mathbb{C}^n$ satisfy the conditions (i)-(ii) of Theorem* 2.4.2. *Let $v(t) = v(z, s, t)$ be the solution of the initial value problem* (2.4.1). *Then for each $s \geq 0$, the limit*

$$\lim_{t\to\infty} e^t v(z, s, t) = f(z, s),\ s \geq 0, \tag{2.4.2}$$

exists locally uniformly on B^n. In addition, $f(\cdot, s)$ is univalent on B^n and $f(z, s) = f(v(z, s, t), t)$ for all $z \in B^n$ and $0 \leq s \leq t < \infty$. Then $f(z, t)$ is a Loewner chain which has the property that $\{e^{-t} f(\cdot, t)\}_{t\geq 0}$ is a normal family on B^n. Moreover, $f(z, \cdot)$ is a locally Lipschitz function on $[0, \infty)$ locally uniformly with respect to $z \in B^n$ and for a.e. $t \geq 0$ the following relation holds

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t),\ \forall z \in B^n.$$

The following theorem due to Pfaltzgraff [88] represents a main result in the theory of Loewner chains in several complex variables, and generalizes Theorem 1.5.4. Poreda obtained an analogous result for complex Banach spaces (see e.g. [96]). We will use this result in various applications of the Loewner theory in the forthcoming chapters.

Theorem 2.4.4 ([88]) *Let $h = h(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ satisfy the conditions (i)-(ii) of Theorem* 2.4.2.

Let $f = f(z, t) : B^n \times [0, \infty) \to \mathbb{C}^n$ be a mapping such that $f(\cdot, t) \in H(B^n)$, $f(0, t) = 0$, $Df(0, t) = e^t I_n$, for $t \geq 0$, and $f(z, \cdot)$ is locally absolutely continuous on $[0, \infty)$ locally uniformly with respect to $z \in B^n$. Assume that

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t),\ \text{a.e.}\ t \geq 0,\ \forall\, z \in B^n. \tag{2.4.3}$$

Further, assume that there exists an increasing sequence $\{t_m\}_{m\in\mathbb{N}}$ such that $t_m > 0$, $t_m \to \infty$ and $\lim_{m\to\infty} e^{-t_m} f(z, t_m) = F(z)$ locally uniformly on B^n. Then $f(z, t)$ is a Loewner chain and

$$\lim_{t\to\infty} e^t v(z, s, t) = f(z, s)$$

locally uniformly on B^n for each $s \geq 0$, where $v(t) = v(z, s, t)$ is the solution of the initial value problem (2.4.1), *for all $z \in B^n$.*

In higher dimensions, Graham, Kohr and Kohr [47] (see also [44]) proved that if $f(z, t)$ is a Loewner chain on B^n, then $f(z, \cdot)$ is locally Lipschitz on $[0, \infty)$ locally uniformly with respect to $z \in B^n$. Also, Graham, Hamada and Kohr [38], Curt and Kohr [24] proved that any Loewner chain on the Euclidean unit ball B^n in $\mathbb{C}^n$ satisfies the generalized Loewner differential equation (2.4.4).

Theorem 2.4.5 ([47], [24]) *Let $f(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain. Then there exists a mapping $h = h(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ such that $h(\cdot,t) \in \mathcal{M}$ for each $t \geq 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$ for each $z \in B^n$ and the following relation holds*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \text{ a.e. } t \geq 0, \; \forall z \in B^n. \tag{2.4.4}$$

2.4.2 Loewner chains and biholomorphic mappings on the unit ball B^n

We next give characterizations of certain subclasses of biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$ by using Loewner chains.

Loewner chains and subclasses of biholomorphic mappings

The characterization of starlike mappings by using Loewner chains was obtained by Pfaltzgraff and Suffridge [90].

Theorem 2.4.6 *Let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping. Then f is starlike if and only if $f(z,t) = e^t f(z)$ is a Loewner chain.*

The following theorem due to Hamada and Kohr [54] generalizes Theorem 1.5.6 to n-dimensions, and shows that spirallike mappings of type α can be characterized by using Loewner chains.

Theorem 2.4.7 *Let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping and let $\alpha \in \mathbb{R}$, $|\alpha| < \pi/2$. Then f is a spirallike mapping of type α if and only if $f(z,t) = e^{(1-ia)t} f(e^{iat}z)$, $z \in B^n$, $t \geq 0$, is a Loewner chain, where $a = \tan\alpha$.*

The following result gives the characterization of almost starlike mappings of order α by using Loewner chains. This result was obtained by Xu and Liu [114] in the context of complex Banach spaces.

Theorem 2.4.8 *Let f be a normalized locally biholomorphic mapping on B^n and let $\alpha \in [0,1)$. Then f is almost starlike of order α if and only if $f(z,t) = e^{\frac{1}{1-\alpha}t} f(e^{\frac{\alpha}{\alpha-1}t}z)$, $z \in B^n$, $t \geq 0$, is a Loewner chain.*

Loewner chains and univalence criteria on B^n

We next give some applications of the theory of Loewner chains to univalence criteria in several complex variables. The following result represents a generalization to n-dimensions of Becker's criterion (see Theorem 1.5.8). This generalization was obtained by Pfaltzgraff in 1974 [88].

Theorem 2.4.9 *Let $f : B^n \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping which satisfies*

$$(1 - \|z\|^2)\|[Df(z)]^{-1}D^2f(z)(z, \cdot)\| \le 1,\ z \in B^n.$$

Then f is univalent on B^n.

For other univalence criteria on the unit ball, based on the method of Loewner chains, see [21], [25].

2.4.3 Parametric representation on B^n

We next present the class of mappings which admit parametric representation on the Euclidean unit ball B^n in $\mathbb{C}^n$.

We first give the definition of parametric representation (see [38], [47]; cf. [94], [95]).

Definition 2.4.10 A normalized mapping $f \in H(B^n)$ has *parametric representation* if there exists a Loewner chain $f(z, t)$ such that $\{e^{-t}f(z, t)\}_{t\ge0}$ is a normal family on B^n (locally uniformly bounded family) and $f = f(\cdot, 0)$.

We denote by $S^0(B^n)$ the set of mappings which have parametric representation.

When $n = 1$, we know that $S^0(U) = S$ [93] (see Theorem 1.5.2). However, if $n \ge 2$, then $S^0(B^n)$ is a proper subset of $S(B^n)$ (see [38]; cf. [94], [95]). Note that many important subsets of $S(B^n)$, such as $S^*(B^n)$ and $\hat{S}_\alpha(B^n)$, are also subsets of $S^0(B^n)$ (see [38]).

The following growth result holds for mappings in $S^0(B^n)$. This result was obtained by Poreda [94] on the unit polydisc in $\mathbb{C}^n$, and by Graham, Hamada and Kohr [38] on the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm.

Theorem 2.4.11 (**[38]**, **[94]**) *If $f \in S^0(B^n)$ then*

$$\frac{\|z\|}{(1 + \|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1 - \|z\|)^2},\ z \in B^n.$$

These estimates are sharp. Hence $f(B^n) \supseteq B^n_{1/4}$.

Note that mappings in $S(B^n)$, $n \ge 2$, need not satisfy the above result (see e.g. [44]). Hence $S(B^n)$ is larger then the class $S^0(B^n)$ for $n \ge 2$. Moreover, the class $S^0(B^n)$ is compact, as proven by Graham, Kohr and Kohr [47].

Corollary 2.4.12 *$S^0(B^n)$ is a compact set in the topology of $H(B^n)$.*

Mappings with g-parametric representation on B^n

It is natural to introduce the notion of g-parametric representation. We begin with the following assumption (see [38]).

Definition 2.4.13 Let $g \in H(U)$ be a univalent function such that $g(0) = 1$, $g(\bar{\zeta}) = \overline{g(\zeta)}$ for $\zeta \in U$ (i.e. g has real coefficients), Re $g(\zeta) > 0$ on U, and assume that g satisfies the following conditions for $r \in (0,1)$:

$$\begin{cases} \min_{|\zeta|=r} \operatorname{Re} g(\zeta) = \min\{g(r), g(-r)\} \\ \max_{|\zeta|=r} \operatorname{Re} g(\zeta) = \max\{g(r), g(-r)\} \end{cases} \tag{2.4.5}$$

We define the class $\mathcal{M}_g$, where g satisfies the assumptions of Definition 2.4.13. This class was introduced by Graham, Hamada and Kohr [38].

Definition 2.4.14 The class $\mathcal{M}_g$ is given by

$$\mathcal{M}_g = \Big\{h : B^n \to \mathbb{C}^n : h \in H(B^n),\, h(0) = 0,\, Dh(0) = I_n, \Big\langle h(z), \frac{z}{\|z\|^2}\Big\rangle \in g(U),\, z \in B^n\Big\}.$$

Note that $\langle h(z), \frac{z}{\|z\|^2}\rangle$ is understood to have the value 1 (its limiting value) when $z = 0$. Clearly, $\mathcal{M}_g \subseteq \mathcal{M}$ and if $g(\zeta) \equiv \frac{1-\zeta}{1+\zeta}$, then $\mathcal{M}_g \equiv \mathcal{M}$. Note that $\mathcal{M}_g \neq \emptyset$, since $\mathrm{id}_{B^n} \in \mathcal{M}_g$.

We next give the definition of g-Loewner chains. This notion is due to Graham, Hamada and Kohr [38] (compare with [47] and [94] for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$).

Definition 2.4.15 We say that a mapping $f = f(z,t) : B^n \times [0,\infty) \to \mathbb{C}^n$ is a *g-Loewner chain* if $f(z,t)$ is a Loewner chain such that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on B^n (locally uniformly bounded family) and the mapping $h = h(z,t)$ which occurs in the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e. } t \geq 0, \quad \forall\, z \in B^n, \tag{2.4.6}$$

satisfies the condition $h(\cdot,t) \in \mathcal{M}_g$ for a.e. $t \geq 0$.

We now give the definition of g-parametric representation, due to Graham, Hamada and Kohr [38] (compare with [47] and [94] for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$).

Definition 2.4.16 A normalized mapping $f \in H(B^n)$ has g-parametric representation if there exists a g-Loewner chain $f(z,t)$ such that $f = f(\cdot,0)$.

Let $S_g^0(B^n)$ be the set of mappings which have g-parametric representation.

When $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in U$, the set $S_g^0(B^n)$ reduces to the set $S^0(B^n)$ of mappings which have usual parametric representation (see [38]).

Graham, Hamada and Kohr [38] proved that $K(B^n) \subset S_g^0(B^n)$, where $g(\zeta) \equiv 1 - \zeta$. This inclusion represents one of the motivations for the study of g-parametric representation and g-Loewner chains in higher dimensions.

Note that the growth theorem for the class $S_g^0(B^n)$ was obtained by Graham, Hamada and Kohr [38], and implies that $S_g^0(B^n)$ is locally uniformly bounded (see also [44]).

Chapter 3

Extension operators that preserve geometric and analytical properties

In this chapter we are concerned with certain extension operators which take a univalent function f on the unit disc U to a univalent mapping F from the Euclidean unit ball B^n in $\mathbb{C}^n$ into $\mathbb{C}^n$, with the property that $f(z_1) = F(z_1, 0)$. This subject began with the Roper-Suffridge extension operator [101], introduced in 1995, which has the property that if f is a convex function of U then F is a convex mapping of B^n. A number of other geometric properties (starlikeness, spirallikeness, starlikeness of a certain order, etc.) have been shown to be preserved by this operator and certain generalizations of it. Graham and Kohr [43] showed that the Roper-Suffridge extension operator preserves the notions of starlikeness and Bloch mapping, and Graham, Kohr and Kohr [46] showed that it preserves the notions of parametric representation and spirallikeness of type δ.

In the first part of this chapter we present the Roper-Suffridge extension operator and its generalizations. We give their main geometric and analytical properties and we show their connection with the theory of Loewner chains. We also discuss the case of the Pfaltzgraff-Suffridge extension operator [91] which provides a way of extending a locally biholomorphic mapping $f \in H(B^n)$ to a locally biholomorphic mapping $F \in H(B^{n+1})$.

In the second part of this chapter we present original results regarding certain generalizations of the Roper-Suffridge extension operator. We prove that these operators preserve the notion of g-Loewner chains, where $g(\zeta) = (1-\zeta)/(1+(1-2\gamma)\zeta)$, $|\zeta| < 1$ and $\gamma \in (0,1)$. As a consequence, the considered operators preserve certain geometric and analytical properties, such as g-parametric representation, starlikeness of order γ, spirallikeness of type δ and order γ, almost starlikeness of order δ and type γ. We generalize the Pfaltzgraff-Suffridge extension operator and we prove that this generalized operator preserves the notions of parametric representation, starlikeness, spirallikeness of type δ, almost starlikeness of order δ. We consider the preservation of ε-starlikeness under this operator and we obtain a partial answer to the question of whether it preserves convexity. We also study subordination preserving results under the above mentioned operators and we consider radius problems associated with them.

The main bibliographic sources used throughout this chapter are [34], [35], [38], [42], [43],

[44], [45], [46], [48], [56], [67], [75], [77], [82], [91], [101], [102].

The original results presented in this chapter can be found in [11], [12], [14], [15], [16].

3.1 General results regarding extension operators

3.1.1 The Roper-Suffridge extension operator

For $n \geq 2$, let $\tilde{z} = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$ such that $z = (z_1, \tilde{z}) \in \mathbb{C}^n$.

The Roper-Suffridge extension operator provides a way of extending a locally univalent function on the unit disc U to a locally biholomorphic mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$. This operator was introduced by Roper and Suffridge in 1995 [101] in order to construct convex mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$ starting with a convex function on the unit disc. If $f_1, \ldots, f_n$ are convex functions on the unit disc U, then $F(z) = (f_1(z_1), \ldots, f_n(z_n))$, $z = (z_1, \ldots, z_n) \in B^n$, is not necessary a convex mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$ (see Example 2.3.13 (i)).

The Roper-Suffridge extension operator $\Phi_n : \mathcal{L}S \to \mathcal{L}S_n(B^n)$ is defined by [101]

$$\Phi_n(f)(z) = (f(z_1), \tilde{z}\sqrt{f'(z_1)}),\ z = (z_1, \tilde{z}) \in B^n. \tag{3.1.1}$$

We choose the branch of the power function such that $\sqrt{f'(z_1)}\big|_{z_1=0} = 1$.

Roper and Suffridge [101] proved the following result:

Theorem 3.1.1 *If f is a convex function on the unit disc U, then $F = \Phi_n(f)$ is a convex mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$. Hence $\Phi_n(K) \subseteq K(B^n)$.*

A different proof of Theorem 3.1.1 was given by Graham and Kohr in 2000 (see [43]). Graham and Kohr [43] also proved the following result, which shows that the Roper-Suffridge extension operator preserves the notion of starlikeness.

Theorem 3.1.2 *If $f \in S^*$, then $F = \Phi_n(f) \in S^*(B^n)$. Hence $\Phi_n(S^*) \subseteq S^*(B^n)$.*

Hamada, Kohr and Kohr [56] showed that the operator Φ_n also preserves starlikeness of order $1/2$. This property is related to convexity, since $K(B^n) \subset S^*_{1/2}(B^n)$ (see Theorem 2.3.15).

Theorem 3.1.3 *If $f \in S^*_{1/2}$, then $F = \Phi_n(f) \in S^*_{1/2}(B^n)$. Hence $\Phi_n(S^*_{1/2}) \subseteq S^*_{1/2}(B^n)$.*

Graham, Kohr and Kohr [46] showed that the Roper-Suffridge extension operator preserves the notion of spirallikeness of type $\delta \in (-\pi/2, \pi/2)$.

Theorem 3.1.4 *If $f \in \hat{S}_\delta$, $\delta \in (-\pi/2, \pi/2)$, then $F = \Phi_n(f) \in \hat{S}_\delta(B^n)$. Hence $\Phi_n(\hat{S}_\delta) \subseteq \hat{S}_\delta(B^n)$.*

We now give the connection between the Roper-Suffridge extension operator and the Loewner theory. Graham, Kohr and Kohr (see [46]; see also [44]) obtained the following result, which shows that the operator Φ_n preserves the notion of parametric representation.

Theorem 3.1.5 *If $f \in S$ and $F = \Phi_n(f)$, then $F \in S^0(B^n)$. Hence $\Phi_n(S) \subseteq S^0(B^n)$.*

Graham, Kohr and Kohr (see [46]; see also [42]) obtained the following result, which provides the radius of starlikeness associated with $\Phi_n(S)$.

Theorem 3.1.6 $R_{S^*}(\Phi_n(S)) = \tanh(\pi/4)$.

The radius of convexity associated with $\Phi_n(S)$ and $\Phi_n(S^*)$ was obtained by Graham, Kohr and Kohr (see [46]; see also [42]).

Theorem 3.1.7 $R_K(\Phi_n(S)) = R_K(\Phi_n(S^*)) = 2 - \sqrt{3}$.

3.1.2 Generalizations of the Roper-Suffridge extension operator

In this section we are interested in other extension operators that have similar properties to those of the Roper-Suffridge extension operator. For several generalizations of the Roper-Suffridge extension operator, see [34], [35], [42], [43], [44], [45], [46], [67], [76], [82]. Other recent extension operators that preserve geometric and analytical properties of biholomorphic mappings on the unit ball in $\mathbb{C}^n$ and complex Banach spaces are due to Elin [30] (via semigroups theory), and Graham, Hamada and Kohr [39] (via univalent subordination chains).

Graham, Kohr and Kohr [46] considered the following operator

$$\Phi_{n,\alpha}(f)(z) = F(z) = (f(z_1), \tilde{z}(f'(z_1))^\alpha),\ z = (z_1, \tilde{z}) \in B^n, \tag{3.1.2}$$

where $\alpha \in [0, 1/2]$, and f is a locally univalent function on U, normalized by $f(0) = f'(0) - 1 = 0$. We choose the branch of the power function such that $(f'(z_1))^\alpha|_{z_1=0} = 1$. When $\alpha = 1/2$, this operator reduces to the Roper-Suffridge extension operator.

Graham, Kohr and Kohr [46] obtained a number of extension results related to the operator $\Phi_{n,\alpha}$, $\alpha \in [0, 1/2]$.

Theorem 3.1.8 ([46]) *Let $f \in \mathcal{L}S$, $\alpha \in [0, 1/2]$.*

(i) If $f \in S$, then $\Phi_{n,\alpha}(f)$ can be embedded in a Loewner chain and moreover $\Phi_{n,\alpha}(f) \in S^0(B^n)$. Hence $\Phi_{n,\alpha}(S) \subseteq S^0(B^n)$.

(ii) If $f \in S^*$, then $\Phi_{n,\alpha}(f) \in S^*(B^n)$. Hence $\Phi_{n,\alpha}(S^*) \subseteq S^*(B^n)$.

(iii) If $f \in \hat{S}_\delta$, $\delta \in (-\pi/2, \pi/2)$, then $\Phi_{n,\alpha}(f) \in \hat{S}_\delta(B^n)$. Hence $\Phi_{n,\alpha}(\hat{S}_\delta) \subseteq \hat{S}_\delta(B^n)$.

Graham, Kohr and Kohr [46] also proved that convexity is preserved under the operator $\Phi_{n,\alpha}$ only if $\alpha = 1/2$, i.e. only in the case of the Roper-Suffridge extension operator.

Graham, Kohr and Kohr [46] obtained the following radius of starlikeness for the class $\Phi_{n,\alpha}(S)$.

Theorem 3.1.9 $R_{S^*}(\Phi_{n,\alpha}(S)) = \tanh(\pi/4)$, *for all* $\alpha \in [0, 1/2]$.

Graham, Hamada, Kohr and Suffridge [42] considered the operator $\Phi_{n,\alpha,\beta}$ given by

$$\Phi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), \tilde{z}\left(\frac{f(z_1)}{z_1}\right)^{\alpha}(f'(z_1))^{\beta}\right),\ z=(z_1,\tilde{z})\in B^n, \tag{3.1.3}$$

where $\alpha \geq 0$, $\beta \geq 0$, and f is a locally univalent function on U, normalized by $f(0)=f'(0)-1=0$, and such that $f(z_1)\neq 0$ for $z_1\in U\backslash\{0\}$. The branches of the power functions are chosen such that

$$\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\Bigg|_{z_1=0} = 1 \text{ and } (f'(z_1))^{\beta}|_{z_1=0}=1.$$

Note that the operator $\Phi_{n,0,1/2}$ reduces to the Roper-Suffridge extension operator.

Graham, Hamada, Kohr and Suffridge [42] obtained certain extension results related to the operator $\Phi_{n,\alpha,\beta}$, where $\alpha\in[0,1]$, $\beta\in[0,1/2]$, and $\alpha+\beta\leq 1$.

Theorem 3.1.10 ([42]) *Let $f\in\mathcal{L}S$, $\alpha\in[0,1]$, $\beta\in[0,1/2]$, $\alpha+\beta\leq 1$.*

(i) *If $f\in S$, then $\Phi_{n,\alpha,\beta}(f)$ can be embedded in a Loewner chain and moreover $\Phi_{n,\alpha,\beta}(f)\in S^0(B^n)$. Hence $\Phi_{n,\alpha,\beta}(S)\subseteq S^0(B^n)$.*

(ii) *If $f\in S^*$, then $\Phi_{n,\alpha,\beta}(f)\in S^*(B^n)$. Hence $\Phi_{n,\alpha,\beta}(S^*)\subseteq S^*(B^n)$.*

Moreover, Graham, Hamada, Kohr and Suffridge [42] showed that $\Phi_{n,\alpha,\beta}(K)\subset K(B^n)$ only if $(\alpha,\beta)=(0,1/2)$, i.e. only in the case of the Roper-Suffridge extension operator.

The radius of starlikeness for the class $\Phi_{n,\alpha,\beta}(S)$ was obtained by Graham, Hamada, Kohr and Suffridge [42].

Theorem 3.1.11 $R_{S^*}(\Phi_{n,\alpha,\beta}(S))=\tanh(\pi/4)$, *for $\alpha\in[0,1]$, $\beta\in[0,1/2]$, such that $\alpha+\beta\leq 1$.*

The following extension operator was introduced by Muir [82]. The purpose of this operator was to provide examples of extreme points of $K(B^n)$, starting with extreme points of K (see [84]).

Definition 3.1.12 (**[82]**) Let $Q:\mathbb{C}^{n-1}\to\mathbb{C}$ be a homogeneous polynomial of degree 2. The Muir operator $\Phi_{n,Q}:\mathcal{L}S\to\mathcal{L}S_n(B^n)$ is defined by

$$\Phi_{n,Q}(f)(z)=(f(z_1)+Q(\tilde{z})f'(z_1), \tilde{z}\sqrt{f'(z_1)}),\ z=(z_1,\tilde{z})\in B^n.$$

We choose the branch of the power function such that $\sqrt{f'(z_1)}|_{z_1=0}=1$.

In the case $Q\equiv 0$, the Muir operator reduces to the Roper-Suffridge extension operator.

Muir [82] proved the following result. Note that (ii) was also obtained by Kohr [67].

Theorem 3.1.13 ([82]) (i) *$\Phi_{n,Q}(K)\subseteq K(B^n)$ if and only if $\|Q\|\leq 1/2$.*

(ii) *$\Phi_{n,Q}(S^*)\subseteq S^*(B^n)$ if and only if $\|Q\|\leq 1/4$.*

Kohr [67] proved the following result regarding the Muir operator:

Theorem 3.1.14 $\Phi_{n,Q}(S) \subseteq S^0(B^n)$*, whenever* $\|Q\| \leq 1/4$.

Extension operators which preserve certain geometric properties (e.g. starlikeness and convexity) on more general Reinhardt domains in $\mathbb{C}^n$ were considered by Gong and Liu (see [34], [35]), Liu and Liu [76]. Further details regarding generalizations of the Roper-Suffridge extension operator may be found in [44, Chapter 11] and [45] (see also, [30], [39], [73], [76], [82], etc.)

3.1.3 The Pfaltzgraff-Suffridge extension operator

Another generalization of the Roper-Suffridge extension operator was given by Pfaltzgraff and Suffridge [91] in 1999. This operator provides a way of extending a locally biholomorphic mapping on the Euclidean unit ball B^n in $\mathbb{C}^n$ to a locally biholomorphic mapping on the Euclidean unit ball B^{n+1} in $\mathbb{C}^{n+1}$.

For $n \geq 1$, let $z' = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$.

The Pfaltzgraff-Suffridge extension operator $\Psi_n : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(B^{n+1})$ is defined by (see [91])

$$\Psi_n(f)(z) = F(z) = \Big(f(z'), z_{n+1}[J_f(z')]^{\frac{1}{n+1}}\Big), \; z = (z', z_{n+1}) \in B^{n+1}, \tag{3.1.4}$$

where $J_f(z') = \det Df(z')$, $z' \in B^n$. We choose the branch of the power function such that $[J_f(z')]^{\frac{1}{n+1}}\big|_{z'=0} = 1$. Then $F = \Psi_n(f) \in \mathcal{L}S_{n+1}(B^{n+1})$ whenever $f \in \mathcal{L}S_n(B^n)$. Moreover, if $f \in S(B^n)$ then $F \in S(B^{n+1})$. Note that if $n = 1$, then Ψ_1 reduces to the Roper-Suffridge extension operator Φ_2.

Pfaltzgraff and Suffridge [91] proposed the following conjecture regarding the preservation of convexity under the operator Ψ_n.

Conjecture 3.1.1 *If* $f \in K(B^n)$ *then* $\Psi_n(f) \in K(B^{n+1})$.

The operator Ψ_n was also studied by Graham, Kohr and Pfaltzgraff [48]. They obtained a partial answer to Conjecture 3.1.1 (see [48]).

For $a \in (0, 1]$, let

$$\Omega_{a,n} = \{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : \; |z_{n+1}|^2 < a^{\frac{2n}{n+1}}(1 - \|z'\|^2)\}.$$

Then $\Omega_{a,n} \subseteq B^{n+1}$ and $\Omega_{1,n} = B^{n+1}$. Graham, Kohr and Pfaltzgraff [48] proved the following convexity result.

Theorem 3.1.15 *Let* $f \in K(B^n)$, $a_1, a_2 > 0$ *be such that* $a_1 + a_2 \leq 1$ *and let* $F = \Psi_n(f)$. *Then*

$$(1 - \lambda)F(z) + \lambda F(w) \in F(\Omega_{a_1+a_2,n}), \; z \in \Omega_{a_1,n}, \; w \in \Omega_{a_2,n}, \; \lambda \in [0, 1].$$

Graham, Kohr and Pfaltzgraff [48] proved the following results, which show that the Pfaltzgraff-Suffridge extension operator preserves the notions of parametric representation and starlikeness.

Theorem 3.1.16 (i) *If* $f \in S^0(B^n)$, *then* $F = \Psi_n(f) \in S^0(B^{n+1})$. *Hence* $\Psi_n(S^0(B^n)) \subseteq S^0(B^{n+1})$.

(ii) *If* $f \in S^*(B^n)$, *then* $F = \Psi_n(f) \in S^*(B^{n+1})$. *Hence* $\Psi_n(S^*(B^n)) \subseteq S^*(B^{n+1})$.

Hamada, Kohr and Kohr [56] obtained a generalization of the Pfaltzgraff-Suffridge extension operator on some Reinhardt domains in $\mathbb{C}^n$.

The notions of continuous extension operator $\Phi : \mathcal{L}S \to \mathcal{L}S_n(B^n)$ and Loewner chain preserving extension operator were recently introduced by Muir [83]. The generalization of these notions for continuous operators $\Phi : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(B^{n+1})$ was recently given by Graham, Hamada, Kohr and Kohr [41].

Definition 3.1.17 **([41])** The mapping $\Phi : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(B^{n+1})$ is called an *extension operator* if Φ is continuous (with respect to the compact open topologies of $H(B^n)$ and $H(B^{n+1})$) and

$$\Phi(f)(z',0) = (f(z'),0), \quad \forall\, f \in \mathcal{L}S_n(B^n), \quad z' \in B^n.$$

If Φ is an extension operator, we say that Φ *preserves Loewner chains* (cf. [83], for $n = 1$) provided that, whenever $f = f(z',t) : B^n \times [0,\infty) \to \mathbb{C}^n$ is a Loewner chain, the mapping $F = F(z,t) : B^{n+1} \times [0,\infty) \to \mathbb{C}^{n+1}$ given by

$$F(\cdot,t) = e^t\Phi(e^{-t}f(\cdot,t)), \quad t \geq 0, \tag{3.1.5}$$

is also a Loewner chain on $B^{n+1} \times [0,\infty)$.

We remark that all the extension operators considered in this section are concrete examples of Loewner chain preserving extension operators.

3.2 g-Loewner chains associated with generalized Roper-Suffridge extension operators

In this section we are concerned with the extension operators $\Phi_{n,\alpha}$, $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ that provide a way of extending a locally univalent function f on the unit disc U to a locally biholomorphic mapping $F \in H(B^n)$, where B^n is the Euclidean unit ball in $\mathbb{C}^n$. By using the method of Loewner chains, we prove that if f can be embedded as the first element of a g-Loewner chain on the unit disc, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$ for $|\zeta| < 1$ and $\gamma \in (0,1)$, then $F = \Phi_{n,\alpha}(f)$ can also be embedded as the first element of a g-Loewner chain on B^n, whenever $\alpha \in [0,\frac{1}{2}]$. In particular, if f is starlike of order γ on U (resp. f is spirallike of type δ and order γ on U, where $\delta \in (-\pi/2,\pi/2)$), then $F = \Phi_{n,\alpha}(f)$ is also starlike of order γ on B^n (resp. $F = \Phi_{n,\alpha}(f)$ is spirallike of type δ and order γ on B^n). Also, if f is almost starlike of order δ and type γ on U, where $\delta \in [0,1)$, then $F = \Phi_{n,\alpha}(f)$ is almost starlike of order δ and type γ on B^n. Similar ideas are applied in the case of the Muir extension operator $\Phi_{n,Q}$, where Q is a homogeneous polynomial of degree 2 on $\mathbb{C}^{n-1}$ such that $\|Q\| \leq \frac{1-|2\gamma-1|}{8\gamma}$, $\gamma \in (0,1)$, and in the case of the extension operator $\Phi_{n,\alpha,\beta}$.

Throughout this section we consider g-Loewner chains with $g \in H(U)$ given by

$$g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}, \quad |\zeta| < 1,$$

where $\gamma \in (0,1)$. Then g maps the unit disc onto the open disc of center $1/(2\gamma)$ and radius $1/(2\gamma)$. Hence, in this case the class $\mathcal{M}_g$ is given by (see Definition 2.4.14)

$$\mathcal{M}_g = \Big\{ h \in H(B^n) : h(0) = 0, Dh(0) = I_n, \Big| \frac{1}{\|z\|^2} \langle h(z), z \rangle - \frac{1}{2\gamma} \Big| < \frac{1}{2\gamma},\ z \in B^n \backslash \{0\} \Big\}.$$

We remark that this section is based on the original results obtained in [11] and [12].

3.2.1 The operator $\Phi_{n,\alpha}$ and g-Loewner chains

The main result of this section is given in Theorem 3.2.1 below due to Chirilă [12]. This result yields that the operator $\Phi_{n,\alpha}$ given by (3.1.2) preserves the notion of g-Loewner chain for $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, where $\gamma \in (0,1)$. In the case $\gamma = 0$, see [46] (see also Theorem 3.1.8).

Theorem 3.2.1 ([12]) *Assume $f \in S$ can be embedded as the first element of a g-Loewner chain, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0,1)$. Then $F = \Phi_{n,\alpha}(f)$ can be embedded as the first element of a g-Loewner chain on B^n, for $\alpha \in [0,1/2]$.*

In view of Theorem 3.2.1, Chirilă [12] obtained the following particular cases.

Corollary 3.2.2 ([12]) *If $f : U \to \mathbb{C}$ has g-parametric representation and $\alpha \in [0,1/2]$, then $F = \Phi_{n,\alpha}(f) \in S_g^0(B^n)$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$, and $\gamma \in (0,1)$.*

The following result was obtained by Hamada, Kohr and Kohr [56], in the case $\alpha = \gamma = 1/2$, and by Liu [75], in the case $\gamma \in (0,1)$ and $\alpha \in [0,1/2]$. Chirilă [12] gave a different proof of this result using g-Loewner chains.

Corollary 3.2.3 *If $f \in S_\gamma^*$, $\gamma \in (0,1)$ and $\alpha \in [0,1/2]$, then $F = \Phi_{n,\alpha}(f) \in S_\gamma^*(B^n)$. In particular, the Roper-Suffridge extension operator preserves the notion of starlikeness of order γ.*

The following remark follows from Corollary 3.2.3 (see [12]).

Remark 3.2.4 Since $K \subset S_{1/2}^*$ (see Theorem 1.4.17), it follows in view of Corollary 3.2.3 that $\Phi_{n,\alpha}(K) \subset S_{1/2}^*(B^n)$ for $\alpha \in [0, \frac{1}{2}]$. However, $\Phi_{n,\alpha}(K) \nsubseteq K(B^n)$ for $\alpha \neq 1/2$ (see [46]).

The following result is due to Liu and Liu [77] (see also [75]). Chirilă [12] obtained this result by using g-Loewner chains.

Corollary 3.2.5 *Let $\alpha \in [0, 1/2]$, $\delta \in (-\pi/2, \pi/2)$ and $\gamma \in (0, 1)$. Also, let $f : U \to \mathbb{C}$ be a spirallike function of type δ and order γ on U, and let $F = \Phi_{n,\alpha}(f)$. Then F is also spirallike of type δ and order γ on B^n.*

Xu and Liu [114] proved that certain extension operators preserve the notion of almost starlikeness of order δ. Chirilă [12] proved the following preservation result of almost starlikeness of order δ and type γ in the case of the operator $\Phi_{n,\alpha}$.

Corollary 3.2.6 ([12]) *Let $\alpha \in [0, 1/2]$, $\delta \in [0, 1)$ and $\gamma \in (0, 1)$. Also, let $f : U \to \mathbb{C}$ be an almost starlike function of order δ and type γ. Then $F = \Phi_{n,\alpha}(f)$ is almost starlike of order δ and type γ on B^n.*

3.2.2 The Muir extension operator and g-Loewner chains

Chirilă [12] proved that the Muir extension operator $\Phi_{n,Q}$ given by Definition 3.1.12 preserves the notion of g-Loewner chains, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$ for $\zeta \in U$, and $\gamma \in (0, 1)$. In the case $\gamma = 0$, see [67] (see Theorem 3.1.14). Theorem 3.2.7 represents the main result of this section.

Theorem 3.2.7 ([12]) *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \le \frac{1-|2\gamma-1|}{8\gamma}$, where $\gamma \in (0, 1)$. Assume $f \in S$ can be embedded as the first element of a g-Loewner chain. Then $F = \Phi_{n,Q}(f)$ can be embedded as the first element of a g-Loewner chain on B^n, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$.*

In view of the above result, Chirilă [12] deduced that the operator $\Phi_{n,Q}$ preserves the notion of g-parametric representation and starlikeness of order γ, whenever $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0, 1)$, and $\|Q\| \le \frac{1-|2\gamma-1|}{8\gamma}$.

Corollary 3.2.8 ([12]) *Let $\gamma \in (0, 1)$ and let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \le \frac{1-|2\gamma-1|}{8\gamma}$. If $f : U \to \mathbb{C}$ has g-parametric representation, then $F = \Phi_{n,Q}(f) \in S_g^0(B^n)$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$.*

The following result was obtained by Wang and Liu [113]. Chirilă [12] obtained this result by using the method of g-Loewner chains.

Corollary 3.2.9 *Let $\gamma \in (0, 1)$ and let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \le \frac{1-|2\gamma-1|}{8\gamma}$. If $f \in S_\gamma^*$, then $F = \Phi_{n,Q}(f) \in S_\gamma^*(B^n)$.*

Remark 3.2.10 Let $f(z_1, t)$ be a Loewner chain such that $f(\cdot, t)$ is a convex function on U for $t \ge 0$. Then it is well known that the following relation holds (see e.g. [44], [81] and [93]):

$$\left| \frac{1-|z_1|^2}{2} \cdot \frac{f''(z_1,t)}{f'(z_1,t)} - \overline{z}_1 \right| \le 1, \quad |z_1| < 1, \quad t \ge 0. \tag{3.2.1}$$

Taking into account Remark 3.2.10, Chirilă [12] obtained the following improvement of Theorem 3.2.7 in the case of g-Loewner chains $f(z_1,t)$ such that $f(\cdot,t)$ is convex on U for $t \geq 0$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0,1)$ (cf. [67] and [82]).

Proposition 3.2.11 ([12]) *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \leq \frac{1-|2\gamma-1|}{4\gamma}$, where $\gamma \in (0,1)$. Assume $f \in S$ can be embedded as the first element of a g-Loewner chain $f(z_1,t)$ such that $f(\cdot,t)$ is convex on U for $t \geq 0$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$. Then $F = \Phi_{n,Q}(f)$ can be embedded as the first element of a g-Loewner chain on B^n for $t \geq 0$.*

Remark 3.2.12 Let $f \in K(\gamma)$. Then $f \in S^*_\beta$, where $\beta = \beta(\gamma)$ is given by (1.4.2). Since $\beta(\gamma) \in (\gamma, 1)$, it follows that $f \in S^*_\gamma$ too, and thus $f(z_1,t) = e^t f(z_1)$ is a g-Loewner chain on U, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$. Obviously, $f(\cdot,t)$ is convex for $t \geq 0$, since f is convex on U.

In view of Proposition 3.2.11 and Remark 3.2.12, Chirilă [12] obtained the following particular cases (cf. [67], [82]).

Corollary 3.2.13 ([12]) *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \leq \frac{1-|2\gamma-1|}{4\gamma}$, where $\gamma \in (0,1)$. If $f \in K(\gamma)$, then $F = \Phi_{n,Q}(f)$ can be embedded as the first element of a g-Loewner chain on B^n, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$.*

Corollary 3.2.14 ([12]) *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2 such that $\|Q\| \leq \frac{1}{2}$, and let $f \in K$. Then $F = \Phi_{n,Q}(f)$ can be embedded as the first element of a g-Loewner chain on B^n, where $g(\zeta) = 1 - \zeta$, $|\zeta| < 1$.*

Using the previous corollaries, Chirilă [12] showed that the following remark holds.

Remark 3.2.15
(i) If $f \in K(\gamma)$, then $\Phi_{n,Q}(f) \in S^*_\gamma(B^n)$, whenever $\gamma \in (0,1)$ and $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2 such that $\|Q\| \leq \frac{1-|2\gamma-1|}{4\gamma}$ (cf. [82]).

(ii) If $f \in K$, then $\Phi_{n,Q}(f) \in S^*_{1/2}(B^n)$, whenever $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2 such that $\|Q\| \leq 1/2$ (cf. [82]).

Chirilă [12] showed that the bound $\|Q\| \leq \frac{1}{2}$ is the best possible for Corollary 3.2.14 (cf. [82]).

3.2.3 The operator $\Phi_{n,\alpha,\beta}$ and g-Loewner chains

The main result of this section is given in Theorem 3.2.16 below due to Chirilă [11]. This result yields that the operator $\Phi_{n,\alpha,\beta}$ given by (3.1.3) preserves the notion of g-Loewner chains for $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, where $\gamma \in (0,1)$. This result was obtained by Graham, Hamada, Kohr and Suffridge [42], in the case $\gamma = 0$ (see Theorem 3.1.10). In the case $\alpha = 0$ and $\gamma \in (0,1)$, Theorem 3.2.16 was obtained by Chirilă [12] (see Theorem 3.2.1).

Theorem 3.2.16 ([11]) *Assume $f \in S$ can be embedded as the first element of a g-Loewner chain, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, and $\gamma \in (0,1)$. Then $F = \Phi_{n,\alpha,\beta}(f)$ can be embedded as the first element of a g-Loewner chain on B^n for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$.*

In view of Theorem 3.2.16, Chirilă [11] obtained the following particular cases. Corollary 3.2.17 was obtained by Graham, Hamada, Kohr and Suffridge [42], in the case $\gamma = 0$. Also, Corollary 3.2.17 was obtained by Chirilă [12], in the case $\alpha = 0$ (see Corollary 3.2.2).

Corollary 3.2.17 ([11]) *If $f : U \to \mathbb{C}$ has g-parametric representation and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$, then $F = \Phi_{n,\alpha,\beta}(f) \in S_g^0(B^n)$, where $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in U$, and $\gamma \in (0,1)$.*

The following result was obtained by Hamada, Kohr and Kohr [56], in the case $\alpha = 0$, $\beta = \gamma = 1/2$, and by Liu [75], in the case $\gamma \in (0,1)$ and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$. If $\gamma = 0$, the result below was obtained by Graham, Hamada, Kohr and Suffridge [42]. Chirilă [11] proved this result by using the method of g-Loewner chains.

Corollary 3.2.18 *If $f \in S_\gamma^*$ and $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$, then $F = \Phi_{n,\alpha,\beta}(f) \in S_\gamma^*(B^n)$, where $\gamma \in (0,1)$. In particular, the Roper-Suffridge extension operator preserves the notion of starlikeness of order γ.*

The following remark follows from Corollary 3.2.18 (see [11]).

Remark 3.2.19 Since $K \subset S_{1/2}^*$ (see Theorem 1.4.17), it follows in view of Corollary 3.2.18 that $\Phi_{n,\alpha,\beta}(K) \subset S_{1/2}^*(B^n)$ for $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$. However, $\Phi_{n,\alpha,\beta}(K) \nsubseteq K(B^n)$ for $(\alpha, \beta) \neq (0, 1/2)$ (see [42]).

The following result is due to Liu and Liu [77] (see also [75]). Chirilă [11] obtained a different proof by using the method of g-Loewner chains.

Corollary 3.2.20 *Let $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$, $\delta \in (-\pi/2, \pi/2)$ and $\gamma \in (0,1)$. Also, let $f : U \to \mathbb{C}$ be a spirallike function of type δ and order γ on U, and let $F = \Phi_{n,\alpha,\beta}(f)$. Then F is also spirallike of type δ and order γ on B^n.*

Chirilă (cf. [12]) obtained the following preservation result of almost starlikeness of order δ and type γ in the case of the operator $\Phi_{n,\alpha,\beta}$. Corollary 3.2.21 was obtained by Chirilă [12], in the case $\alpha = 0$ (see Corollary 3.2.6).

Corollary 3.2.21 *Let $\alpha \in [0,1]$, $\beta \in [0,1/2]$, $\alpha + \beta \leq 1$, $\delta \in [0,1)$ and $\gamma \in (0,1)$. Also, let $f : U \to \mathbb{C}$ be an almost starlike function of order δ and type γ. Then $F = \Phi_{n,\alpha,\beta}(f)$ is almost starlike of order δ and type γ on B^n.*

3.3 Subordination associated with the operator $\Phi_{n,\alpha,\beta}$

In this section we present some subordination results associated with the operator $\Phi_{n,\alpha,\beta}$. Chirilă [11] obtained a subordination preserving result under the operator $\Phi_{n,\alpha,\beta}$. More precisely, the following theorem holds (see [56], in the case $\alpha = 0$ and $\beta = 1/2$):

Theorem 3.3.1 ([11]) *Let $f, g : U \to \mathbb{C}$ be two locally univalent functions such that $f(0) = g(0) = 0$, $f'(0) = a$ and $g'(0) = b$, where $0 < a \leq b$. Assume that $f(z_1) \neq 0$ and $g(z_1) \neq 0$ for $0 < |z_1| < 1$. If $\alpha \geq 0$, $\beta \in [0, 1/2]$ and $f \prec g$, then $\Phi_{n,\alpha,\beta}(f) \prec \Phi_{n,\alpha,\beta}(g)$. We choose the branches of the power functions such that*

$$[f'(z_1)]^\beta|_{z_1=0} = a^\beta,\ \left[\frac{f(z_1)}{z_1}\right]^\alpha\Bigg|_{z_1=0} = a^\alpha,$$

$$[g'(z_1)]^\beta|_{z_1=0} = b^\beta,\ \left[\frac{g(z_1)}{z_1}\right]^\alpha\Bigg|_{z_1=0} = b^\alpha.$$

Chirilă [11] obtained certain consequences of the above result. These results were obtained in [56], for $\alpha = 0$ and $\beta = 1/2$.

Corollary 3.3.2 ([11]) *Let $f \in \mathcal{L}S$ and $M \geq 1$ be such that $|f(z_1)| \leq M$, $z_1 \in U$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Then $\|\Phi_{n,\alpha,\beta}(f)(z)\| \leq M$, $z \in B^n$, whenever $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$.*

Corollary 3.3.3 ([11]) *Let $f \in \mathcal{L}S$ and $M \geq 1$ be such that $|f(z_1)| \leq M$, $z_1 \in U$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Then $\Phi_{n,\alpha,\beta}(f) \in S^0(B^n_r)$, where $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha+\beta \leq 1$, and $r = 1/(M + \sqrt{M^2 - 1})$.*

Corollary 3.3.4 ([11]) *Let $f : U \to \mathbb{C}$ be a locally univalent function on U such that $f(0) = 0$ and $f'(0) = a$, where $a \in (0, 1]$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Also let $g \in S$ and assume that $f \prec g$. Then $\|\Phi_{n,\alpha,\beta}(f)(z)\| \leq \|z\|/(1 - \|z\|)^2$, $z \in B^n$, whenever $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$.*

Corollary 3.3.5 ([11]) *Let f be a locally univalent function on the unit disc U with $f(0) = 0$ and $f'(0) = a \in (0, 1]$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Also let $g \in S^*_\gamma$, $\gamma \in (0, 1)$, and assume that $f \prec g$. Then $\|\Phi_{n,\alpha,\beta}(f)(z)\| \leq \|z\|/(1 - \|z\|)^{2(1-\gamma)}$, $z \in B^n$, whenever $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$.*

In view of Corollary 3.3.5, Chirilă [11] obtained the following consequence.

Corollary 3.3.6 ([11]) *Let f be a locally univalent function on the unit disc U with $f(0) = 0$ and $f'(0) = a \in (0, 1]$. Assume that $f(z_1) \neq 0$ for $0 < |z_1| < 1$. Also let $g \in K$ and assume that $f \prec g$. Then $\|\Phi_{n,\alpha,\beta}(f)(z)\| \leq \|z\|/(1 - \|z\|)$, $z \in B^n$, whenever $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$.*

We now present another consequence of Theorem 3.3.1 due to Chirilă [11]. This result was obtained by Hamada, Kohr and Kohr [56] for $\alpha = 0$ and $\beta = 1/2$.

Corollary 3.3.7 ([11]) *Let $F = \Phi_{n,\alpha,\beta}(f)$ and $G = \Phi_{n,\alpha,\beta}(g)$ where f is a locally univalent function on the unit disc such that $f(0) = 0$, $f'(0) = a \in (0,1]$, $f(z_1) \neq 0$ for $0 < |z_1| < 1$, $g \in K$, $\alpha \geq 0$, $\beta \in [0,1/2]$. Assume $DF(z)(z) \prec DG(z)(z)$, $z \in B^n$. Then $F(z) \prec G(z)$, $z \in B^n$.*

3.4 Radius problems and the operator $\Phi_{n,\alpha,\beta}$

We next consider certain radius problems associated with the operator $\Phi_{n,\alpha,\beta}$. These results are due to Chirilă [11].

Graham, Kohr and Kohr [46] obtained the radius of starlikeness and the radius of convexity associated with $\Phi_n(S)$ (see Theorems 3.1.6 and 3.1.7). Also, Graham, Hamada, Kohr and Suffridge [42] obtained the radius of starlikeness associated with $\Phi_{n,\alpha,\beta}(S)$ (see Theorem 3.1.11). In this section, we shall be concerned with other radius problems for some subsets of $S(B^n)$ associated with the operator $\Phi_{n,\alpha,\beta}$.

Chirilă [11] obtained the following result regarding the radius of spirallikeness of type δ for the set $\Phi_{n,\alpha,\beta}(S)$.

Theorem 3.4.1 ([11]) $R_{\hat{S}_\delta}(\Phi_{n,\alpha,\beta}(S)) = \tanh\left[\frac{\pi}{4} - \frac{|\delta|}{2}\right]$*, for $\alpha \in [0,1]$, $\beta \in [0,1/2]$ such that $\alpha + \beta \leq 1$ and $\delta \in (-\pi/2, \pi/2)$.*

Remark 3.4.2 If we take $\delta = 0$, $\alpha \in [0,1]$, $\beta \in [0,1/2]$ with $\alpha + \beta \leq 1$, then from Theorem 3.4.1 we obtain that $R_{S^*}(\Phi_{n,\alpha,\beta}(S)) = \tanh(\pi/4)$. This result was obtained by Graham, Hamada, Kohr and Suffridge [42].

Chirilă [11] proved the following result regarding the radius of starlikeness of order γ for the class $\Phi_{n,\alpha,\beta}(S)$.

Theorem 3.4.3 ([11]) $R_{S^*_\gamma}(\Phi_{n,\alpha,\beta}(S)) = r$*, where r is the unique root of the equation*

$$\left(\frac{1-r}{1+r}\right)^{\cos x} \cos x - \gamma = 0,$$

for $\gamma \in (0, 1/e)$, in which $x = x(r)$, $0 < x < \pi$ is uniquely determined by the equation

$$\sin x \ln\left(\frac{1+r}{1-r}\right) - x = 0$$

and $r = \frac{1-\gamma}{1+\gamma}$, for $\gamma \in [1/e, 1)$.

Using the fact that $R_K(S^*_{1/2}) = \sqrt{2\sqrt{3}-3}$ (see e.g. [37, II p. 87]), and the fact that the Roper-Suffridge extension operator preserves convexity (see Theorem 3.1.1), we may obtain the following result due to Chirilă [11].

Theorem 3.4.4 ([11]) $R_K(\Phi_n(S^*_{1/2})) = \sqrt{2\sqrt{3}-3}$.

Similarly, using the results regarding radii of univalence given in [37, Chapter 13] and the fact that the operator $\Phi_{n,\alpha,\beta}$ preserves the notions of starlikeness (see Theorem 3.1.10), starlikeness of order $\gamma \in (0,1)$ (see Corollary 3.2.18) and spirallikeness of type $\delta \in (-\pi/2, \pi/2)$ (see e.g. [75]), Chirilă [11] obtained the following results.

Theorem 3.4.5 ([11]) *If $\alpha \in [0,1]$, $\beta \in [0,1/2]$ such that $\alpha+\beta \leq 1$, then the following relations hold:*

(i) $R_{\hat{S}_\delta}(\Phi_{n,\alpha,\beta}(S^*_\gamma))$ *is the smallest positive root of*

$$((1-2\gamma)\cos\delta)x^2 - 2(1-\gamma)x + \cos\delta = 0,\ \delta \in (-\pi/2, \pi/2),\ \gamma \in (0,1).$$

(ii) $R_{S^*_\gamma}(\Phi_{n,\alpha,\beta}(K)) = \sin(\gamma\pi/2)$*,* $\gamma \in (0,1)$.

(iii) $R_{\hat{S}_\delta}(\Phi_{n,\alpha,\beta}(K)) = \cos\delta$*,* $0 \leq \delta < 1$.

(iv) $R_{S^*}(\Phi_{n,\alpha,\beta}(\hat{S}_\delta)) = 1/(\cos\delta + |\sin\delta|)$*,* $\delta \in (-\pi/2, \pi/2)$.

3.5 A generalization of the Pfaltzgraff-Suffridge extension operator

In this section we are concerned with an extension operator $\Psi_{n,\alpha}$, $\alpha \geq 0$, that provides a way of extending a locally biholomorphic mapping $f \in H(B^n)$ to a locally biholomorphic mapping $F \in H(B^{n+1})$. In the case $\alpha = 1/(n+1)$, this operator reduces to the Pfaltzgraff-Suffridge extension operator (3.1.4). By using the method of Loewner chains, we prove that if $f \in S^0(B^n)$, then $\Psi_{n,\alpha}(f) \in S^0(B^{n+1})$, whenever $\alpha \in [0, 1/(n+1)]$. In particular, if $f \in S^*(B^n)$, then $\Psi_{n,\alpha}(f) \in S^*(B^{n+1})$, and if f is spirallike of type $\delta \in (-\pi/2, \pi/2)$ on B^n, then $\Psi_{n,\alpha}(f)$ is also spirallike of type δ on B^{n+1}. We also prove that if f is almost starlike of order $\delta \in [0,1)$ on B^n, then $\Psi_{n,\alpha}(f)$ is almost starlike of order δ on B^{n+1}. Finally we prove that if $f \in K(B^n)$ and $1/(n+1) \leq \alpha \leq 1/n$, then the image of $F = \Psi_{n,\alpha}(f)$ contains the convex hull of the image of some egg domain contained in B^{n+1}. An extension of this result to the case of ε-starlike mappings will be also considered. We will also obtain a subordination preserving result under the above operator.

This section contains original results obtained in [14], [15]. Most of the results in this section are generalizations of certain results due to Graham, Kohr and Pfaltzgraff (see [48]), in the case $\alpha = \frac{1}{n+1}$.

3.5.1 Loewner chains and the operator $\Psi_{n,\alpha}$

We start by introducing the operator $\Psi_{n,\alpha}$. This operator was considered by Chirilă [14].

For $n \geq 1$, set $z' = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$.

Definition 3.5.1 Let $\alpha \geq 0$. The extension operator $\Psi_{n,\alpha} : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(B^{n+1})$ is defined by [14]

(3.5.1) $$\Psi_{n,\alpha}(f)(z) = F(z) = \Big(f(z'), z_{n+1}[J_f(z')]^\alpha\Big),\ z = (z', z_{n+1}) \in B^{n+1}.$$

We choose the branch of the power function such that $[J_f(z')]^\alpha\big|_{z'=0} = 1$. Then $F = \Psi_{n,\alpha}(f) \in \mathcal{L}S_{n+1}(B^{n+1})$ whenever $f \in \mathcal{L}S_n(B^n)$. Also, if $f \in S(B^n)$ then $F \in S(B^{n+1})$.

If $\alpha = 1/(n+1)$, the operator $\Psi_{n,1/(n+1)}$ reduces to the Pfaltzgraff-Suffridge extension operator Ψ_n given by (3.1.4) [91]. If $n = 1$ and $\alpha = 1/2$, then $\Psi_{1,1/2}$ reduces to the well-known Roper-Suffridge extension operator Φ_2 (see [101]). Note that the operator $\Psi_{1,\alpha}$, $\alpha \in [0, \frac{1}{2}]$, was considered by Graham, Kohr and Kohr [46].

The main result of this section is given in Theorem 3.5.2 due to Chirilă [14]. This result states that the operator $\Psi_{n,\alpha}$ preserves the notion of Loewner chains. In the case $\alpha = \frac{1}{n+1}$, see [48] (see Theorem 3.1.16).

Theorem 3.5.2 ([14]) *Assume $f \in S(B^n)$ can be imbedded as the first element of a Loewner chain $f(z', t)$. Then $F = \Psi_{n,\alpha}(f)$ can also be imbedded as the first element of a Loewner chain $F(z, t)$, for $\alpha \in \left[0, \frac{1}{n+1}\right]$.*

Taking into account Theorem 3.5.2, we obtain that the operator $\Psi_{n,\alpha}$ preserves the notions of parametric representation, starlikeness, spirallikeness of type δ, and almost starlikeness of order δ. These results are due to Chirilă [14]. Note that Corollaries 3.5.3 and 3.5.4 were obtained by Graham, Kohr and Pfaltzgraff [48] in the case $\alpha = \frac{1}{n+1}$.

Corollary 3.5.3 ([14]) *Assume $f \in S^0(B^n)$. Then $F = \Psi_{n,\alpha}(f) \in S^0(B^{n+1})$, for $\alpha \in \left[0, \frac{1}{n+1}\right]$.*

Corollary 3.5.4 ([14]) *Assume $f \in S^*(B^n)$. Then $F = \Psi_{n,\alpha}(f) \in S^*(B^{n+1})$, for $\alpha \in \left[0, \frac{1}{n+1}\right]$.*

Corollary 3.5.5 ([14]) *Assume $f \in \hat{S}_\delta(B^n)$, where $\delta \in (-\pi/2, \pi/2)$. Then $F = \Psi_{n,\alpha}(f) \in \hat{S}_\delta(B^{n+1})$, for $\alpha \in \left[0, \frac{1}{n+1}\right]$.*

The following result due to Chirilă [14] yields that the operator $\Psi_{n,\alpha}$ preserves the notion of almost starlikeness of order $\delta \in [0, 1)$ (compare with [114]).

Corollary 3.5.6 ([14]) *Assume f is an almost starlike mapping of order δ on B^n, where $\delta \in [0, 1)$. Then $F = \Psi_{n,\alpha}(f)$ is an almost starlike mapping of order δ on B^{n+1}, where $\alpha \in \left[0, \frac{1}{n+1}\right]$.*

3.5.2 ε-starlikeness and the operator $\Psi_{n,\alpha}$

We next discuss the case of ε-starlike mappings associated with the operator $\Psi_{n,\alpha}$, for $\alpha \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$.

For $a \in (0,1]$, let

$$\Omega_{a,n,\alpha} = \{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : |z_{n+1}|^2 < a^{2n\alpha}(1 - \|z'\|^2)^{(n+1)\alpha}\}.$$

Then $\Omega_{a,n,\alpha} \subseteq B^{n+1}$. For $a = 1$ and $\alpha = \frac{1}{n+1}$, we obtain that $\Omega_{1,n,\frac{1}{n+1}} = B^{n+1}$.

The following theorem is due to Chirilă [14]. When $\varepsilon = 1$, this result gives a partial answer to the question of whether $\Psi_{n,\alpha}$ preserves convexity. We remark that Theorem 3.5.7 was obtained in [48], in the case $\alpha = \frac{1}{n+1}$ and $\varepsilon = 1$ (compare with [35]).

Theorem 3.5.7 ([14]) *Let $\varepsilon \in [0,1]$ and let $f : B^n \to \mathbb{C}^n$ be a normalized ε-starlike mapping. Also let $F = \Psi_{n,\alpha}(f)$, for $\alpha \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$, and let $a_1, a_2 > 0$ be such that $a_1 + a_2 \leq 1$. Then*

$$(1-\lambda)F(z) + \lambda\varepsilon F(w) \in F(\Omega_{a_1+a_2,n,\alpha}),\ z \in \Omega_{a_1,n,\alpha},\ w \in \Omega_{a_2,n,\alpha},\ \lambda \in [0,1].$$

Taking $\varepsilon = 1$ in Theorem 3.5.7, Chirilă [14] obtained the following convexity result for the operator $\Psi_{n,\alpha}$. In the case $\alpha = \frac{1}{n+1}$, see [48] (see Theorem 3.1.15).

Corollary 3.5.8 ([14]) *If $f \in K(B^n)$ and $F = \Psi_{n,\alpha}(f)$, then $(1-\lambda)F(z) + \lambda F(w) \in F(\Omega_{a_1+a_2,n,\alpha})$, $z \in \Omega_{a_1,n,\alpha}$, $w \in \Omega_{a_2,n,\alpha}$, $\lambda \in [0,1]$, where $a_1, a_2 > 0$, $a_1 + a_2 \leq 1$.*

Taking $\alpha = \frac{1}{n+1}$ in Theorem 3.5.7, Chirilă [14] obtained the following result regarding ε-starlikeness for the Pfaltzgraff-Suffridge extension operator Ψ_n (cf. [48] for $\varepsilon = 1$; compare with [35]).

Corollary 3.5.9 ([14]) *Let $\varepsilon \in [0,1]$ and let $f : B^n \to \mathbb{C}^n$ be a normalized ε-starlike mapping. Also let $F = \Psi_n(f)$ and let $a_1, a_2 > 0$ be such that $a_1 + a_2 \leq 1$. Then*

$$(1-\lambda)F(z) + \lambda\varepsilon F(w) \in F(\Omega_{a_1+a_2,n,1/(n+1)}),$$

for all $z \in \Omega_{a_1,n,1/(n+1)}$, $w \in \Omega_{a_2,n,1/(n+1)}$ and $\lambda \in [0,1]$.

Taking $a_1 = a_2 = \frac{1}{2}$ in Corollary 3.5.9 and using the fact that $\Omega_{1,n,1/(n+1)} = B^{n+1}$, we obtain the following result due to Chirilă [14]. In the case $\varepsilon = 1$, see [48].

Corollary 3.5.10 ([14]) *If f is a normalized ε-starlike mapping on B^n, $\varepsilon \in [0,1]$, and $F = \Psi_n(f)$, then*

$$(1-\lambda)F(z) + \lambda\varepsilon F(w) \in F(B^{n+1}),\ z, w \in \Omega_{1/2,n,1/(n+1)},\ \lambda \in [0,1].$$

Remark 3.5.11 The operator $\Psi_{n,\alpha}$ was generalized on some Reinhardt domains in $\mathbb{C}^n$ by Chirilă [15], and certain properties similar with those above were also obtained.

3.5.3 Subordination and the operator $\Psi_{n,\alpha}$

We next obtain a subordination preserving result under the operator $\Psi_{n,\alpha}$, and we give some particular cases of this result. Theorem 3.5.12 is due to Chirilă [17] (see [56] for $\alpha = 1/(n+1)$).

Theorem 3.5.12 ([17]) *Let $f, g : B^n \to \mathbb{C}^n$ be two locally biholomorphic mappings such that $f(0) = g(0) = 0$, $Df(0) = aI_n$, $Dg(0) = bI_n$, where $0 < a \leq b$. If $\alpha \in [0, 1/(n+1)]$ and $f \prec g$, then $\Psi_{n,\alpha}(f) \prec \Psi_{n,\alpha}(g)$. We choose the branches of the power functions such that*

$$[J_f(z')]^\alpha|_{z'=0} = a^{n\alpha} \text{ and } [J_g(z')]^\alpha|_{z'=0} = b^{n\alpha}.$$

The following consequences of Theorem 3.5.12 are due to Chirilă [17]. These results were obtained in [56] for $\alpha = 1/(n+1)$.

Corollary 3.5.13 ([17]) *Let $f \in \mathcal{L}S_n(B^n)$ and $M \geq 1$ be such that $\|f(z')\| \leq M$, $z' \in B^n$. Then $\|\Psi_{n,\alpha}(f)(z)\| \leq M$, $z \in B^{n+1}$, whenever $\alpha \in [0, 1/(n+1)]$. Moreover, $\Psi_{n,\alpha}(f)$ is biholomorphic on the ball B_ρ^{n+1} where $\rho = 1/(mM^n)$ and $m = \min\limits_{r \in [0,1]} (2 - r^2)/(r(1 - r^2))$.*

Corollary 3.5.14 ([17]) *Let $f : B^n \to \mathbb{C}^n$ be a locally biholomorphic mapping on B^n such that $f(0) = 0$ and $Df(0) = aI_n$, $a \in (0, 1]$. Let $g \in S^0(B^n)$ and assume that $f \prec g$. Then $\|\Psi_{n,\alpha}(f)(z)\| \leq \|z\|/(1 - \|z\|)^2$, $z \in B^{n+1}$, whenever $\alpha \in [0, 1/(n+1)]$.*

Chapter 4

Subclasses of biholomorphic mappings associated with g-Loewner chains

In this chapter we use the method of Loewner chains to generate certain subclasses of normalized biholomorphic mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$, which have interesting geometric characterizations. We present the classes of g-starlike mappings, g-spirallike mappings of type $\alpha \in (-\pi/2, \pi/2)$, and g-almost starlike mappings of order $\alpha \in [0, 1)$ on B^n. We provide examples of this type of mappings and we obtain their characterization by using g-Loewner chains. We will use these results to prove that, under certain assumptions, the mapping $F : B^n \to \mathbb{C}^n$ given by $F(z) = P(z)z$ is g-starlike, g-spirallike of type $\alpha \in (-\pi/2, \pi/2)$ and g-almost starlike of order $\alpha \in [0, 1)$ on B^n, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$. More generally, we consider conditions under which F has g-parametric representation on B^n. Various applications of these results are also provided. In this way we obtain concrete examples of mappings which have g-parametric representation on the Euclidean unit ball B^n in $\mathbb{C}^n$.

We remark that this chapter contains original results obtained in [13].

4.1 g-starlikeness, g-spirallikeness and g-almost starlikeness on B^n

In this section we present the definition of g-starlike mappings, g-spirallike mappings of type α and g-almost starlike mappings of order α and we give their characterization by using g-Loewner chains. We also provide examples for these subclasses of biholomorphic mappings. The results presented in this section are due to Chirilă [13].

4.1.1 Definitions and examples

We first present the set of g-starlike mappings on B^n, where g satisfies the requirements of Definition 2.4.13. This notion was introduced by Graham, Hamada and Kohr [38] and by Hamada and Honda [52]. This notion was also studied by Xu and Liu in the case of complex Banach spaces [115].

Definition 4.1.1 A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *g-starlike* on B^n if

$$\frac{1}{\|z\|^2}\langle [Df(z)]^{-1}f(z), z\rangle \in g(U),\ z \in B^n \setminus \{0\}. \tag{4.1.1}$$

We denote the class of g-starlike mappings on B^n by $S_g^*(B^n)$. When $n = 1$, we denote this class by S_g^*. If $g(\zeta) = (1-\zeta)/(1+\zeta)$, then this class reduces to the class of starlike mappings on B^n.

We now give particular examples of $S_g^*(B^n)$ (cf. [52]).

Remark 4.1.2 If $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$ and $\gamma \in (0, 1)$, then g maps the unit disc U onto the open disc of center $1/(2\gamma)$ and radius $1/(2\gamma)$. Then the class $S_g^*(B^n)$ reduces to the class of starlike mappings of order γ on B^n.

Remark 4.1.3 If $g(\zeta) = (1-\alpha)\frac{1-\zeta}{1+\zeta} + \alpha$, $|\zeta| < 1$ and $0 \leq \alpha < 1$, relation (4.1.1) can be rewritten as

$$\text{Re}\ \langle [Df(z)]^{-1}f(z), z\rangle > \alpha\|z\|^2,\ z \in B^n \setminus \{0\}.$$

Therefore the class $S_g^*(B^n)$ reduces to the class of almost starlike mappings of order α on B^n.

Let

$$q_\rho(\zeta) = 1 + \frac{4(1-\rho)}{\pi^2}\left(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2,\ 0 \leq \rho < 1.$$

We choose the branch of the square root such that $\sqrt{\zeta}|_{\zeta=1} = 1$ and the branch of the logarithm function such that $\log 1 = 0$.

Definition 4.1.4 ([53]) A normalized locally biholomorphic mapping $f \in H(B^n)$ is said to be *parabolic starlike of order ρ* if

$$\frac{1}{\|z\|^2}\langle [Df(z)]^{-1}f(z), z\rangle \in g(U),\ z \in B^n \setminus \{0\},$$

where $g = \dfrac{1}{q_\rho}$.

Hamada, Honda and Kohr [53] proved that $g(U)$ is starlike with respect to 1.

For various results regarding g-starlike mappings, such as growth and covering theorems and coefficient estimates, see [38], [52], [115].

Next we define the set of g-spirallike mappings of type $\alpha \in (-\pi/2, \pi/2)$ on B^n, where g satisfies the requirements of Definition 2.4.13. This notion was introduced by Chirilă [13].

Definition 4.1.5 ([13]) A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *g-spirallike of type $\alpha \in (-\pi/2, \pi/2)$* if

$$i\frac{\sin\alpha}{\cos\alpha} + \frac{e^{-i\alpha}}{\cos\alpha}\left\langle [Df(z)]^{-1}f(z), \frac{z}{\|z\|^2}\right\rangle \in g(U),\ z \in B^n \setminus \{0\}. \tag{4.1.2}$$

If $g(\zeta) = (1-\zeta)/(1+\zeta)$, this class becomes the class of spirallike mappings of type α on B^n and when $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0,1)$, we obtain the class of spirallike mappings of type α and order γ on B^n. When $\alpha = 0$, the class of g-spirallike mappings of type 0 on B^n reduces to the set of g-starlike mappings on B^n.

Obviously, if f is g-spirallike of type α, then f is also spirallike of type α, and hence f is biholomorphic on B^n. On the other hand, the motivation for introducing the subclass of g-spirallike mappings of type α is provided by Corollary 4.2.13.

We next present the set of g-almost starlike mappings of order $\alpha \in [0,1)$ on B^n, where g satisfies the requirements of Definition 2.4.13. This class was introduced by Chirilă [13].

Definition 4.1.6 (**[13]**) A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is said to be *g-almost starlike of order* $\alpha \in [0,1)$ if

$$\frac{1}{1-\alpha}\left\langle [Df(z)]^{-1}f(z), \frac{z}{\|z\|^2}\right\rangle - \frac{\alpha}{1-\alpha} \in g(U),\ z \in B^n \setminus \{0\}. \tag{4.1.3}$$

If $g(\zeta) = (1-\zeta)/(1+\zeta)$, this class reduces to the class of almost starlike mappings of order α on B^n and when $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in (0,1)$, we obtain the class of almost starlike mappings of order α and type γ on B^n. When $\alpha = 0$, the class of g-almost starlike mappings of order 0 on B^n reduces to the set of g-starlike mappings on B^n.

If f is g-almost starlike of order α, then f is also almost starlike of order α, and hence biholomorphic on B^n. The motivation for introducing the subclass of g-almost starlike mappings of order α on B^n is provided by Corollary 4.2.15.

We next provide examples of g-starlike mappings, g-spirallike mappings of type α and g-almost starlike mappings of order α on the unit disc U and on the Euclidean unit ball B^n in $\mathbb{C}^n$ (cf. [52]).

We first give an example of a g-starlike function on the unit disc U (see [52]).

Example 4.1.7 Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition 2.4.13. Also, let $b \in S_g^*$ be defined by $b(0) = 0$, $b'(0) = 1$ and

$$\frac{\zeta b'(\zeta)}{b(\zeta)} = \frac{1}{g(\zeta)},\ |\zeta| < 1.$$

Then

$$b(\zeta) = \zeta \exp \int_0^\zeta \left[\frac{1}{g(x)} - 1\right] \frac{dx}{x},\ |\zeta| < 1. \tag{4.1.4}$$

The function b given by relation (4.1.4) is g-starlike on U, therefore b has g-parametric representation on U (see Theorem 4.1.11).

We next give examples of g-starlike mappings on the Euclidean unit ball B^n in $\mathbb{C}^n$ (cf. [52]).

Example 4.1.8 Assume that g satisfies the requirements of Definition 2.4.13.

(i) Assume that g is convex. If $f_1, \dots, f_n \in S_g^*$, then $f \in S_g^*(B^n)$, where $f(z) = (f_1(z_1), \dots, f_n(z_n))$, $z = (z_1, \dots, z_n) \in B^n$. Moreover, $f \in S_g^0(B^n)$.

(ii) If $f \in S_g^*$, then the mapping $F(z) = \dfrac{f(z_1)}{z_1} z$, $z = (z_1, \dots, z_n) \in B^n$, is g-starlike on B^n. Hence $F \in S_g^0(B^n)$.

We now give examples of g-spirallike mappings of type α on the Euclidean unit ball B^n in $\mathbb{C}^n$, $\alpha \in (-\pi/2, \pi/2)$. These examples were provided by Chirilă [13] (cf. [52]).

Example 4.1.9 Assume that g satisfies the requirements of Definition 2.4.13.

(i) Assume that g is convex. If $f_1, \dots, f_n$ are g-spirallike of type α on U, $\alpha \in (-\pi/2, \pi/2)$, then $f(z) = (f_1(z_1), \dots, f_n(z_n))$ is g-spirallike of type α on B^n. Moreover, $f \in S_g^0(B^n)$.

(ii) If f is g-spirallike of type $\alpha \in (-\pi/2, \pi/2)$ on U , then the mapping $F(z) = \dfrac{f(z_1)}{z_1} z$, $z = (z_1, \dots, z_n) \in B^n$, is g-spirallike of type α on B^n. Hence $F \in S_g^0(B^n)$.

Finally we give examples of g-almost starlike mappings of order $\alpha \in [0, 1)$ on the Euclidean unit ball B^n in $\mathbb{C}^n$. These examples are due to Chirilă [13] (cf. [52]).

Example 4.1.10 Assume that g satisfies the requirements of Definition 2.4.13.

(i) Assume that $g(U)$ is starlike with respect to 1. If f_1 is g-almost starlike of order α on U, $\alpha \in [0, 1)$, then $f(z) = (f_1(z_1), z_2, \dots, z_n)$ is g-almost starlike of order α on B^n. Moreover, $f \in S_g^0(B^n)$.

(ii) Assume that g is convex. If $f_1, \dots, f_n$ are g-almost starlike of order α on U, $\alpha \in [0, 1)$, then $f(z) = (f_1(z_1), \dots, f_n(z_n))$ is g-almost starlike of order α on B^n. Moreover, $f \in S_g^0(B^n)$.

(iii) If f is g-almost starlike of order $\alpha \in [0, 1)$ on U , then the mapping $F(z) = \dfrac{f(z_1)}{z_1} z$, $z = (z_1, \dots, z_n) \in B^n$, is g-almost starlike of order α on B^n. Hence $F \in S_g^0(B^n)$.

4.1.2 Characterizations by using g-Loewner chains

In this section we obtain the characterizations of g-starlikeness, g-spirallikeness of type α, and g-almost starlikeness of order α, in terms of g-Loewner chains. We first present the characterization of g-starlike mappings by using g-Loewner chains, where g satisfies the requirements of Definition 2.4.13. This result is due to Chirilă [13]. In the case $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$, we obtain the usual characterization of starlikeness in terms of Loewner chains (see [90]; see also Theorem 2.4.6).

Theorem 4.1.11 ([13]) *A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is g-starlike if and only if $f(z, t) = e^t f(z)$ is a g-Loewner chain, where g satisfies the requirements of Definition* 2.4.13.

The characterization of g-spirallike mappings of type $\alpha \in (-\pi/2, \pi/2)$ on B^n by using g-Loewner chains was obtained by Chirilă [13], where g satisfies the requirements of Definition 2.4.13 (compare with [54]). In the case $g(\zeta) = \frac{1-\zeta}{1+\zeta}$ in Theorem 4.1.12, we obtain the usual characterization of spirallikeness of type α in terms of Loewner chains (see Theorem 2.4.7).

Theorem 4.1.12 ([13]) *A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is g-spirallike of type $\alpha \in (-\pi/2, \pi/2)$ if and only if $f(z,t) = e^{(1-ia)t} f(e^{iat} z)$ is a g-Loewner chain, where $a = \tan\alpha$ and g satisfies the requirements of Definition* 2.4.13.

Chirilă [13] also obtained the characterization of g-almost starlike mappings of order $\alpha \in [0,1)$ by using g-Loewner chains, where g satisfies the requirements of Definition 2.4.13. If we take $g(\zeta) = \frac{1-\zeta}{1+\zeta}$ in Theorem 4.1.13, we obtain the characterization of almost starlike mappings of order α using Loewner chains (see [114]; see also Theorem 2.4.8).

Theorem 4.1.13 ([13]) *A normalized locally biholomorphic mapping $f : B^n \to \mathbb{C}^n$ is g-almost starlike of order $\alpha \in [0,1)$ if and only if $f(z,t) = e^{\frac{1}{1-\alpha}t} f(e^{\frac{\alpha}{\alpha-1}t} z)$ is a g-Loewner chain, where g satisfies the requirements of Definition* 2.4.13.

4.2 A subclass of biholomorphic mappings on B^n generated by g-Loewner chains

Next, using the characterizations given in the previous section, we prove that, under certain assumptions, the mapping $F(z) = P(z)z$, $z \in B^n$, is g-starlike, g-spirallike of type $\alpha \in (-\pi/2, \pi/2)$ and g-almost starlike of order $\alpha \in [0,1)$ on B^n, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$. More generally, we consider conditions under which F has g-parametric representation on B^n, where g satisfies the requirements of Definition 2.4.13. Several applications of these results are provided.

We remark that this section is based on original results obtained in [13].

Theorem 4.2.1 due to Chirilă [13] represents the main result of this section. In this theorem we consider conditions such that a mapping $F : B^n \to \mathbb{C}^n$ given by $F(z) = P(z)z$ belongs to $S_g^0(B^n)$, where $P \in H(B^n, \mathbb{C})$ with $P(0) = 1$. Various particular cases and applications of Theorem 4.2.1 will be also obtained.

Theorem 4.2.1 ([13]) *Let $P : B^n \to \mathbb{C}$ be a holomorphic function on B^n such that $P(0) = 1$ and let $F(z) = P(z)z$, $z \in B^n$. Let $F(z,t) = P(z,t)z$, $z \in B^n$, $t \geq 0$, where $P(z,t) : B^n \times [0,\infty) \to \mathbb{C}$ satisfies the following conditions:*

(i) *$P(\cdot,t) \in H(B^n, \mathbb{C})$, $P(0,t) = e^t$, $t \geq 0$, $P(\cdot,0) = P$, $P(z,t) \neq 0$, $z \in B^n$, $t \geq 0$, and*
$$1 + \frac{DP(z,t)(z)}{P(z,t)} \neq 0, \text{ for } z \in B^n \text{ and } t \geq 0;$$

(ii) *$P(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in B^n$.*

(iii) $\{e^{-t}P(\cdot,t)\}_{t\geq 0}$ *is a normal family on* B^n.

If g *satisfies the requirements of Definition* 2.4.13 *and*

$$\frac{\dfrac{\partial P}{\partial t}(z,t)}{P(z,t)\left(1+\dfrac{DP(z,t)(z)}{P(z,t)}\right)} \in g(U), \quad \textit{a.e. } t\geq 0,\ \forall z\in B^n, \tag{4.2.1}$$

then $F(z,t)$ *is a* g*-Loewner chain. Moreover,* $F\in S_g^0(B^n)$.

It is easy to see that the converse of Theorem 4.2.1 also holds (see [13]).

Proposition 4.2.2 *Let* $P: B^n\to\mathbb{C}$ *be a holomorphic function on* B^n *such that* $P(0)=1$ *and let* $F(z)=P(z)z$, $z\in B^n$. *Let* $F(z,t)=P(z,t)z$, $z\in B^n$, $t\geq 0$, *where* $P(z,t): B^n\times[0,\infty)\to\mathbb{C}$ *satisfies the conditions* $(i)-(iii)$ *from Theorem* 4.2.1. *If* g *satisfies the requirements of Definition* 2.4.13 *and* $F(z,t)$ *is a* g*-Loewner chain, then the relation* (4.2.1) *holds.*

In view of Theorem 4.2.1, Chirilă [13] obtained the following particular cases.

Corollary 4.2.3 was obtained by Pfaltzgraff and Suffridge [91] in the case $g(\zeta)=\frac{1-\zeta}{1+\zeta}$ and by Graham, Hamada and Kohr [38] in the case of functions $g: U\to\mathbb{C}$ satisfying the requirements of Definition 2.4.13. Corollary 4.2.3 was also obtained by Chirilă [13].

Corollary 4.2.3 *Let* $g: U\to\mathbb{C}$ *satisfy the requirements of Definition* 2.4.13. *Also, let* $F(z)=P(z)z$, $z\in B^n$, *where* $P: B^n\to\mathbb{C}$ *is a holomorphic function such that* $P(0)=1$. *If* $1+\frac{DP(z)(z)}{P(z)}\in\frac{1}{g}(U)$, $z\in B^n$, *then* $F(z,t)=e^tP(z)z$, $z\in B^n$, $t\geq 0$, *is a* g*-Loewner chain. Moreover,* $F\in S_g^*(B^n)$.

If we take $g(\zeta)=\frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta|<1$, $\gamma\in[0,1)$, in Corollary 4.2.3, we obtain the following consequence, in view of Remark 4.1.2 (see [13]).

Corollary 4.2.4 *Let* $F(z)=P(z)z$, $z\in B^n$, *where* $P: B^n\to\mathbb{C}$ *is a holomorphic function such that* $P(0)=1$. *If*

$$\operatorname{Re}\left[1+\frac{DP(z)(z)}{P(z)}\right]>\gamma, \quad z\in B^n,$$

then F *is starlike of order* γ *on* B^n, *where* $\gamma\in[0,1)$.

Remark 4.2.5 Under the same assumptions as in Corollary 4.2.3, by taking the value of g as in Remark 4.1.3 and Definition 4.1.4, we obtain that F is almost starlike of order $\alpha\in[0,1)$ and parabolic starlike of order $\rho\in[0,1)$ on B^n, respectively (see [13]).

We shall give some applications of Corollary 4.2.3. To this end, we need the following result. In the case $g(\zeta)\equiv(1-\zeta)/(1+\zeta)$, Proposition 4.2.6 was obtained by Pfaltzgraff and Suffridge [91] and when g satisfies the requirements of Definition 2.4.13, this result was obtained by Graham, Hamada and Kohr [38]. Xu and Liu [115] obtained a generalization of Proposition 4.2.6 in the case of complex Banach spaces.

Proposition 4.2.6 *Let $g : U \to \mathbb{C}$ satisfy the requirements from Definition* 2.4.13*, such that $1/g$ is a convex function on U. Let $f_j \in S_g^*$, $j = 1, 2, \ldots, n$. If $\lambda_j \geq 0$ and $\sum_{j=1}^n \lambda_j = 1$, then*

$$F(z) = z \prod_{j=1}^{n} \left(\frac{f_j(z_j)}{z_j} \right)^{\lambda_j}, \ z \in B^n \tag{4.2.2}$$

belongs to $S_g^*(B^n)$.

If we take $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in [0, 1)$ in Proposition 4.2.6, we obtain the following consequence. This result was obtained by Xu and Liu [115] in the case of complex Banach spaces (see also [13]).

Corollary 4.2.7 *If $f_j \in S_\gamma^*$, $j \in \{1, \ldots, n\}$, $\gamma \in [0, 1)$ and $\lambda_j \geq 0$, $\sum_{j=1}^n \lambda_j = 1$, then*

$$F(z) = z \prod_{j=1}^{n} \left(\frac{f_j(z_j)}{z_j} \right)^{\lambda_j}$$

is also starlike of order γ on B^n. Moreover,

$$\frac{1 - (1 - 2\gamma)\|z\|}{(1 + \|z\|)^{2n(1-\gamma)+1}} \leq |J_F(z)| \leq \frac{1 + (1 - 2\gamma)\|z\|}{(1 - \|z\|)^{2n(1-\gamma)+1}}, \ z \in B^n. \tag{4.2.3}$$

The estimate is sharp.

We formulate the following conjecture (see [13]).

Conjecture. If $f \in S_\gamma^*(B^n)$, where $\gamma \in [0, 1)$, then the relation $(4.2.3)$ holds.

From Corollary 4.2.7, we obtain the following consequence (cf. [91], [115, Theorem 5]; see also [13]).

Corollary 4.2.8 *If $f_j \in K$, $j \in \{1, \ldots, n\}$ and F is given by* $(4.2.2)$*, then $F \in S_{1/2}^*(B^n)$ and*

$$\frac{1}{(1 + \|z\|)^{n+1}} \leq |J_F(z)| \leq \frac{1}{(1 - \|z\|)^{n+1}}, \ z \in B^n. \tag{4.2.4}$$

The result is sharp.

Remark 4.2.9 Corollary 4.2.8 does not hold for the full family $K(B^n)$ (see e.g. [44]). In fact, sharp bounds for $|J_F(z)|$, $F \in K(B^n)$, are not known. General discussions about J_F, where $F \in K(B^n)$, can be found in [32] and [44].

Remark 4.2.10 Using Proposition 4.2.6, Remark 4.1.3 and Definition 4.1.4, we obtain that the mapping F given by (4.2.2) preserves the notions of almost starlikeness of order $\alpha \in [0, 1)$ and parabolic starlikeness of order $\rho \in [0, 1)$ on B^n, respectively (see [13]).

We will now give another application of Corollary 4.2.3. The following result was considered by Chirilă [13].

Corollary 4.2.11 *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition 2.4.13 and assume that $1/g$ is a convex function on U. Let $p_j(\zeta)$ be a normalized holomorphic function on U such that $1 + \frac{\zeta p_j''(\zeta)}{p_j'(\zeta)} \prec \frac{1}{g}(\zeta)$, for $\zeta \in U$, $j = 1, 2, \ldots, n$. If $\lambda_j \geq 0$ and $\sum_{j=1}^{n} \lambda_j = 1$, then*

$$F(z) = z \prod_{j=1}^{n} (p_j'(z_j))^{\lambda_j}, \; z \in B^n \tag{4.2.5}$$

is a mapping in $S_g^*(B^n)$.

If we take $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in [0, 1)$ in Corollary 4.2.11, we obtain the following consequence due to Chirilă [13].

Corollary 4.2.12 *If $f_j \in K(\gamma)$, $j \in \{1, \ldots, n\}$, $\gamma \in [0, 1)$ and $\lambda_j \geq 0$, $\sum_{j=1}^{n} \lambda_j = 1$, then*

$$F(z) = z \prod_{j=1}^{n} (f_j'(z_j))^{\lambda_j}$$

is starlike of order γ on B^n.

In view of Theorem 4.2.1, we obtain the following consequence regarding g-spirallike mappings of type α on B^n. The following corollary is due to Chirilă [13].

Corollary 4.2.13 ([13]) *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition 2.4.13. Also, let $F(z) = P(z)z$, $z \in B^n$, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$ and $1 + \frac{DP(z)(z)}{P(z)} \neq 0$, $z \in B^n$. If $\frac{1+ia\frac{DP(z)(z)}{P(z)}}{1+\frac{DP(z)(z)}{P(z)}} \in g(U)$, $z \in B^n$, then $F(z, t) = e^t P(e^{iat}z)z$, $z \in B^n$, $t \geq 0$, is a g-Loewner chain, where $a = \tan\alpha$ and $\alpha \in (-\pi/2, \pi/2)$. Moreover, F is g-spirallike of type α on B^n.*

By taking $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in [0, 1)$ in Corollary 4.2.13, Chirilă [13] obtained the following consequence.

Corollary 4.2.14 *Let $F(z) = P(z)z$, $z \in B^n$, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$ and $1 + \frac{DP(z)(z)}{P(z)} \neq 0$, $z \in B^n$. If*

$$\mathrm{Re}\left[\frac{1 + \frac{DP(z)(z)}{P(z)}}{1 + ia\frac{DP(z)(z)}{P(z)}}\right] > \gamma, \quad z \in B^n,$$

then F is spirallike of type α and order γ on B^n, where $a = \tan\alpha$, $\alpha \in (-\pi/2, \pi/2)$ and $\gamma \in [0, 1)$.

In view of Theorem 4.2.1, Chirilă [13] obtained the following consequence regarding g-almost starlike mappings of order α on B^n.

Corollary 4.2.15 ([13]) *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition 2.4.13. Also, let $F(z) = P(z)z$, $z \in B^n$, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$ and $1 + \frac{DP(z)(z)}{P(z)} \neq 0$, $z \in B^n$. If $\frac{1+\frac{\alpha}{\alpha-1}\frac{DP(z)(z)}{P(z)}}{1+\frac{DP(z)(z)}{P(z)}} \in g(U)$, $z \in B^n$, $\alpha \in [0,1)$, then $F(z,t) = e^t P(e^{\frac{\alpha}{\alpha-1}t}z)z$, $z \in B^n$, $t \geq 0$, is a g-Loewner chain. Moreover, F is g-almost starlike of order α on B^n.*

If we take $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $|\zeta| < 1$, $\gamma \in [0,1)$ in Corollary 4.2.15, we obtain the following consequence due to Chirilă [13].

Corollary 4.2.16 *Let $F(z) = P(z)z$, $z \in B^n$, where $P : B^n \to \mathbb{C}$ is a holomorphic function such that $P(0) = 1$ and $1 + \frac{DP(z)(z)}{P(z)} \neq 0$, $z \in B^n$. If*

$$\operatorname{Re}\left[\frac{1+\frac{DP(z)(z)}{P(z)}}{1+\frac{\alpha}{\alpha-1}\frac{DP(z)(z)}{P(z)}}\right] > \gamma, \quad z \in B^n,$$

then F is almost starlike of order α and type γ on B^n, where $\gamma \in [0,1)$ and $\alpha \in [0,1)$.

Chapter 5

Extreme points and support points for the family $\overline{S_g^0(B^n)}$

In this chapter we are concerned with extreme points and support points associated with the compact family $\overline{S_g^0(B^n)}$, where $g : U \to \mathbb{C}$ is a univalent function which satisfies certain natural assumptions. Various consequences and applications will be also presented. Finally, we discuss the case of extreme points and support points associated with extension operators which preserve Loewner chains. Recent contributions in this direction have been obtained in [18], [41], [83], [84], [106].

In the case of one complex variable, Pell [87] and Kirwan [61] proved that if f is an extreme point (respectively, f is a support point) for the family S of normalized univalent functions on the unit disc U, and if $f(z,t)$ is a Loewner chain such that $f = f(\cdot,0)$, then $e^{-t}f(\cdot,t)$ is an extreme point of S (respectively, $e^{-t}f(\cdot,t)$ is a support point of S), for all $t \geq 0$.

A very good treatment concerning extremal problems related to various compact subsets of univalent functions on the unit disc U may be found in [50], [97], [104].

A generalization of Pell's and Kirwan's results to several complex variables was obtained by Graham, Kohr and Pfaltzgraff [48], in the case of the compact family $\Phi_n(S)$, where Φ_n is the Roper-Suffridge extension operator. Graham, Hamada, Kohr and Kohr [41] and Schleissinger [106] obtained generalizations of the above results to the case of mappings which have parametric representation on B^n. Muir and Suffridge [84] gave various characterizations of extreme points for convex mappings on B^n. On the other hand, Muir [83] considered extreme points and support points for compact subsets associated with a large family of extension operators.

In this chapter we are concerned with the n-dimensional versions of Pell's [87] and Kirwan's [61] results in the case of the family $\overline{S_g^0(B^n)}$.

This chapter contains original results obtained in [18] and [17].

5.1 Preliminary results

We start this section by recalling the notions of extreme points and support points for compact subsets of $H(B^n)$, where B^n is the Euclidean unit ball in $\mathbb{C}^n$.

Definition 5.1.1 (e.g. **[50]**) Let $\mathcal{F}$ be a subset of $H(B^n)$.

(i) A point $f \in \mathcal{F}$ is called an *extreme point* of $\mathcal{F}$ provided $f = tg + (1-t)h$, where $t \in (0,1)$, $g, h \in \mathcal{F}$, implies $f = g = h$. In other words, $f \in \mathcal{F}$ is an extreme point of $\mathcal{F}$ if f is not a proper convex combination of two points in $\mathcal{F}$.

(ii) A point $g \in \mathcal{F}$ is called a *support point* of $\mathcal{F}$ if there exists a continuous linear functional $L : H(B^n) \to \mathbb{C}$ such that $\operatorname{Re} L|_{\mathcal{F}}$ is not constant and $\operatorname{Re} L(g) = \max_{h \in \mathcal{F}} \operatorname{Re} L(h)$.

We denote by $\operatorname{ex} \mathcal{F}$ and $\operatorname{supp} \mathcal{F}$ the subsets of $\mathcal{F}$ consisting of extreme points of $\mathcal{F}$ and support points of $\mathcal{F}$, respectively.

Remark 5.1.2 Pell [87] and Kirwan [61] proved that if f is an extreme point of S (respectively, f is a support point of S) and if $f(z,t)$ is a Loewner chain such that $f = f(\cdot, 0)$, then $e^{-t} f(\cdot, t)$ is an extreme point of S (respectively, $e^{-t} f(\cdot, t)$ is a support point of S), for all $t \geq 0$.

Graham, Kohr and Pfaltzgraff [48] studied extreme points and support points for families of univalent mappings on B^n constructed using the Roper-Suffridge extension operator.

Graham, Hamada, Kohr and Kohr [41] proved the following result concerning extreme points for the compact family $S^0(B^n)$. In the case of one complex variable, the following result is due to Pell [87] and Kirwan [61], since $S^0(B^1) = S$.

Theorem 5.1.3 *Let $f \in \operatorname{ex} S^0(B^n)$ and $f(z,t)$ be a Loewner chain such that $f = f(\cdot, 0)$ and $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is a normal family on B^n. Then $e^{-t} f(\cdot, t) \in \operatorname{ex} S^0(B^n)$ for $t \geq 0$.*

Graham, Hamada, Kohr and Kohr [41] obtained the following result related to support points for the family $S^0(B^n)$. In the case of one complex variable, see [61] and [87]. Schleissinger [106] proved that Theorem 5.1.4 holds for all $t \geq 0$.

Theorem 5.1.4 *Let $f \in \operatorname{supp} S^0(B^n)$ and let $f(z,t)$ be a Loewner chain such that $f = f(\cdot, 0)$ and $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is a normal family on B^n. Then there exists $t_0 > 0$ such that $e^{-t} f(\cdot, t) \in \operatorname{supp} S^0(B^n)$ for $0 \leq t < t_0$.*

Graham, Hamada, Kohr and Kohr [41] proved analogous results for mappings which belong to $S^0(B^n)$ and which are bounded in the norm by a fixed constant. They also considered extreme and support points for biholomorphic mappings of B^n generated by using extension operators that preserve Loewner chains. On the other hand, Muir [83] considered extreme points and support points for compact families generated by a large class of extension operators.

In this chapter we obtain various results related to extreme points and support points for the family $\overline{S_g^0(B^n)}$, where g satisfies the requirements of Definition 2.4.13. We note that $\overline{S_g^0(B^n)}$ is a compact subset of $H(B^n)$, since $S_g^0(B^n)$ is a locally uniformly bounded family (see [38, Corollary 2.3]). Also, $\overline{S_g^0(B^n)} \subseteq S^0(B^n)$, since $S_g^0(B^n) \subseteq S^0(B^n)$ and $S^0(B^n)$ is a compact, and thus a closed subset of $S(B^n)$ (see [47]).

Certain applications and consequences will be obtained. We also consider extreme points and support points associated with extension operators which preserve Loewner chains. In particular, we consider extreme points and support points for the compact family $\Psi_n(\overline{S_g^0(B^n)})$, where Ψ_n is the Pfaltzgraff-Suffridge extension operator given by (3.1.4).

The results presented in this chapter are original results due to Chirilă, Hamada and Kohr [18] and Chirilă [17].

5.2 Extreme points for the family $\overline{S_g^0(B^n)}$

We next consider extreme points associated with the compact family $\overline{S_g^0(B^n)}$. First, we obtain the following result (cf. [41], for $g(\zeta) = (1-\zeta)/(1+\zeta)$, $|\zeta| < 1$). This result is due to Chirilă, Hamada and Kohr [18].

Lemma 5.2.1 ([18]) *Let $g : U \to \mathbb{C}$ be a univalent function, which satisfies the requirements of Definition* 2.4.13. *Let $f \in S_g^0(B^n)$ and let $f(z,t)$ be a g-Loewner chain such that $f = f(\cdot, 0)$. Also, let $v_{s,t}(z) = v(z,s,t)$ be the transition mapping associated with $f(z,t)$ and let $v_t(z) = v(z,t) = v_{0,t}(z)$ for $z \in B^n$ and $t \geq 0$. If $r \in S_g^0(B^n)$, then $e^t r(v(\cdot,t)) \in S_g^0(B^n)$ for $t \geq 0$.*

We next present the following result concerning extreme points for the compact family $\overline{S_g^0(B^n)}$. When $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$, this result was obtained by Graham, Hamada, Kohr and Kohr [41, Theorem 2.1] (cf. [61] and [87], in the case $n = 1$). Note that Theorem 5.2.2 was recently obtained by Chirilă, Hamada and Kohr [18].

Theorem 5.2.2 ([18]) *Let $g : U \to \mathbb{C}$ be a univalent function, which satisfies the requirements of Definition* 2.4.13. *Let $f \in \operatorname{ex} \overline{S_g^0(B^n)}$. Then there exists a Loewner chain $f(z,t)$ such that $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family, $f = f(\cdot,0)$, and $e^{-t} f(\cdot,t) \in \operatorname{ex} \overline{S_g^0(B^n)}$ for $t \geq 0$.*

We close this section with the following result, which is a generalization of [41, Proposition 2.2] to the case of mappings with g-parametric representation on B^n.

Proposition 5.2.3 ([18]) *Let $g : U \to \mathbb{C}$ be a univalent function, which satisfies the requirements of Definition* 2.4.13. *Also, let $f \in \overline{S_g^0(B^n)}$. Then there exists a Loewner chain $f(z,t)$ such that $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family, $f = f(\cdot,0)$, $e^{-t} f(\cdot,t) \in \overline{S_g^0(B^n)}$ and $e^t v(\cdot,t) \in \overline{S_g^0(B^n)} \setminus \operatorname{ex} \overline{S_g^0(B^n)}$ for $t \geq 0$, where $v_{s,t}(z) = v(z,s,t)$ is the transition mapping associated with $f(z,t)$ and $v_t(z) = v(z,t) = v(z,0,t)$ for $z \in B^n$ and $t \geq 0$. In particular, the identity mapping id_{B^n} is not an extreme point of $\overline{S_g^0(B^n)}$.*

5.3 Support points for the family $\overline{S_g^0(B^n)}$

In this section we consider support points associated with the compact family $\overline{S_g^0(B^n)}$, where $g: U \to \mathbb{C}$ is a univalent function on the unit disc U that satisfies the requirements of Definition 2.4.13. First, we present the following result related to support points for the family $\overline{S_g^0(B^n)}$. This result was obtained by Graham, Hamada, Kohr and Kohr [41, Theorem 2.5], when $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$ (cf. [48, Theorem 3.3]). We remark that Schleissinger [106] proved that the following result holds for all $t \in [0,\infty)$, when $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$. Note that Theorem 5.3.1 is due to Chirilă, Hamada and Kohr [18], and is a generalization to higher dimensions of a result due to Pell [87] and Kirwan [61].

Theorem 5.3.1 ([18]) *Let $g: U \to \mathbb{C}$ be a univalent function which satisfies the requirements of Definition 2.4.13. Also, let $f \in \operatorname{supp} \overline{S_g^0(B^n)}$. Then there exist a Loewner chain $f(z,t)$ and $t_0 > 0$ such that $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family, $f = f(\cdot,0)$, and $e^{-t} f(\cdot,t) \in \operatorname{supp} \overline{S_g^0(B^n)}$ for $0 \leq t < t_0$.*

Remark 5.3.2 In a forthcoming paper [18], we shall prove that the mapping $e^t v(\cdot,t)$ is not a support point of $\overline{S_g^0(B^n)}$ for $t \geq 0$. Hence the conclusion of Theorem 5.3.1 will be that $e^{-t} f(\cdot,t) \in \operatorname{supp} \overline{S_g^0(B^n)}$ for $t \geq 0$.

We also obtain a generalization to the n-dimensional case of an extremal principle due to Kirwan and Schober [62] (see also [104]). In the case $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$, this result was obtained by Graham, Hamada, Kohr and Kohr [41].

Theorem 5.3.3 ([18]) *Let $g: U \to \mathbb{C}$ be a univalent function, which satisfies the requirements of Definition 2.4.13. Let $\lambda : \overline{S_g^0(B^n)} \to \mathbb{R}$ be a continuous real-valued functional. Assume that $f \in \overline{S_g^0(B^n)}$ provides the maximum for λ over the set $\overline{S_g^0(B^n)}$. Then there is a Loewner chain $f(z,t)$ such that $\{e^{-t} f(\cdot,t)\}_{t\geq 0}$ is a locally uniformly bounded family, $f = f(\cdot,0)$, and $e^{-t} f(\cdot,t) \in \overline{S_g^0(B^n)}$ provides the maximum for the associated functional $\lambda_t : \overline{S_g^0(B^n)} \to \mathbb{R}$ given by*

$$\lambda_t(r) = \lambda(e^t r \circ v_t), \quad r \in \overline{S_g^0(B^n)}, \quad t \geq 0. \tag{5.3.1}$$

Here $v_t = v_{0,t}$ and $v_{s,t} = v(\cdot,s,t)$ is the transition mapping associated with $f(z,t)$.

The following compactness result of independent interest is useful in proving Corollary 5.3.5 (see [18]).

Lemma 5.3.4 ([18]) *Let $g: U \to \mathbb{C}$ be a univalent function, which satisfies the requirements of Definition 2.4.13. Then the family $S_g^*(B^n)$ is compact.*

If $f \in S_g^*(B^n)$ provides the maximum for a continuous real-valued functional λ over the set $\overline{S_g^0(B^n)}$, then we obtain the following consequence of Theorem 5.3.3.

Corollary 5.3.5 ([18]) *Let $g : U \to \mathbb{C}$ be a univalent function, which satisfies the requirements of Definition* 2.4.13. *Let $\lambda : \overline{S_g^0(B^n)} \to \mathbb{R}$ be a continuous real-valued functional. If $f \in S_g^*(B^n)$ provides the maximum for λ over the set $\overline{S_g^0(B^n)}$, then f also provides the maximum for the associated functional λ_t given by* (5.3.1), *and $\lambda(f) = \lambda_t(f)$ for $t \geq 0$.*

5.4 Extreme and support points associated with extension operators

In this section we continue the work in [41] and [48], and we prove that if $g : U \to \mathbb{C}$ is a univalent function, which satisfies the requirements of Definition 2.4.13, and if $\Phi : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(B^{n+1})$ is an extension operator which preserves Loewner chains (see Definition 3.1.17), then we may consider the compact family $\Phi(\overline{S_g^0(B^n)})$. We shall consider extreme points and support points for this family. In particular, we consider extreme points and support points for the compact family $\Psi_n(\overline{S_g^0(B^n)})$, where Ψ_n is the Pfaltzgraff-Suffridge extension operator given by (3.1.4).

Graham, Hamada, Kohr and Kohr [41] (cf. Muir [83]) proved the following result, which provides examples of extreme points and support points associated with the compact family $\Phi(\overline{S_g^0(B^n)})$.

Lemma 5.4.1 *If $\Phi : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(B^{n+1})$ is an extension operator and $\mathcal{F} \subseteq \mathcal{L}S_n(B^n)$ is a nonempty compact set, then $\Phi(\operatorname{ex}\mathcal{F}) \subseteq \operatorname{ex}\Phi(\mathcal{F})$ and $\Phi(\operatorname{supp}\mathcal{F}) \subseteq \operatorname{supp}\Phi(\mathcal{F})$.*

In view of Lemma 5.4.1, we deduce that (see [18]; cf. [83] and [41])

$$\Phi(\operatorname{ex}\overline{S_g^0(B^n)}) \subseteq \operatorname{ex}\Phi(\overline{S_g^0(B^n)}) \text{ and } \Phi(\operatorname{supp}\overline{S_g^0(B^n)}) \subseteq \operatorname{supp}\Phi(\overline{S_g^0(B^n)}). \tag{5.4.1}$$

The following result due to Chirilă, Hamada and Kohr [18] is a direct consequence of the relation (5.4.1) and Theorem 5.2.2. In the case $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$, this result was obtained by Graham, Kohr and Pfaltzgraff [48].

Lemma 5.4.2 ([18]) *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition* 2.4.13. *Also, let $\Phi : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(B^{n+1})$ be an extension operator which preserves Loewner chains. Let $f \in \operatorname{ex}\overline{S_g^0(B^n)}$. Also, let $f(z', t)$ be a Loewner chain which satisfies the statements of Theorem* 5.2.2, *and let $F(z, t)$ be the Loewner chain given by* (3.1.5). *Then $e^{-t}F(\cdot, t) \in \operatorname{ex}\Phi(\overline{S_g^0(B^n)})$ for $t \geq 0$.*

Next, we present the following generalization of [48, Theorem 3.1] to the case of extreme points associated with the compact family $\Psi_n(\overline{S_g^0(B^n)})$, where Ψ_n is the Pfaltzgraff-Suffridge extension operator given by (3.1.4). The following theorem is due to Chirilă, Hamada and Kohr [18]. If $n = 1$ and $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$, this result was obtained by Graham, Kohr and Pfaltzgraff [48].

Theorem 5.4.3 ([18]) *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition* 2.4.13. *Let $f \in \overline{S_g^0(B^n)}$ and let $F = \Psi_n(f)$. Assume that $F \in \operatorname{ex}\Psi_n(\overline{S_g^0(B^n)})$. Then there exists a Loewner chain $F(z, t) : B^{n+1} \times [0, \infty) \to \mathbb{C}^{n+1}$ such that $F = F(\cdot, 0)$, $\{e^{-t}F(\cdot, t)\}_{t \geq 0}$ is a normal family on B^{n+1}, and $e^{-t}F(\cdot, t) \in \operatorname{ex}\Psi_n(\overline{S_g^0(B^n)})$ for $t \geq 0$.*

Note that Theorem 5.4.3 can be generalized to the case of the extension operator $\Psi_{n,\alpha}$ given by (3.5.1). Moreover, a similar result also holds for the extension operator $\Phi_{n,\alpha,\beta}$ given by (3.1.3) (see [17]).

In the case of support points associated with $\overline{S_g^0(B^n)}$, we have the following analogous result to Lemma 5.4.2 (see [18]).

Lemma 5.4.4 ([18]) *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition 2.4.13. Also, let $\Phi : \mathcal{L}S_n(B^n) \to \mathcal{L}S_{n+1}(B^{n+1})$ be an extension operator which preserves Loewner chains. Let $f \in \operatorname{supp} \overline{S_g^0(B^n)}$. Also, let $f(z',t)$ be a Loewner chain which satisfies the statements of Theorem 5.3.1, and let $F(z,t)$ be the Loewner chain given by (3.1.5). Then there exists $t_0 > 0$ such that $e^{-t}F(\cdot,t) \in \operatorname{supp} \Phi(\overline{S_g^0(B^n)})$ for $0 \le t < t_0$.*

We next discuss the case of support points associated with the compact family $\Psi_n(\overline{S_g^0(B^n)})$ (see [18]). If $n = 1$ and $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$, this result was obtained by Graham, Kohr and Pfaltzgraff [48]. Note that if $g(\zeta) = \frac{1-\zeta}{1+\zeta}$ and the operator Ψ_n is replaced by the Roper-Suffridge extension operator Φ_n, then Theorem 5.4.5 holds for all $t \in [0,\infty)$, in view of a result due to Schleissinger [106].

Theorem 5.4.5 ([18]) *Let $g : U \to \mathbb{C}$ satisfy the requirements of Definition 2.4.13. Let $f \in \overline{S_g^0(B^n)}$ and let $F = \Psi_n(f)$. Assume that $F \in \operatorname{supp} \Psi_n(\overline{S_g^0(B^n)})$. Then there exist a Loewner chain $F(z,t) : B^{n+1} \times [0,\infty) \to \mathbb{C}^{n+1}$ and $t_0 > 0$ such that $F = F(\cdot,0)$, $\{e^{-t}F(\cdot,t)\}_{t\ge 0}$ is a normal family on B^{n+1}, and $e^{-t}F(\cdot,t) \in \operatorname{supp} \Psi_n(\overline{S_g^0(B^n)})$ for $0 \le t < t_0$.*

Remark 5.4.6 It would be interesting to see if the result contained in Theorem 5.4.5 remains true for all $t \ge 0$. This may represent a starting point for further investigations.

Remark 5.4.7 It would be interesting to study extreme points and support points for bounded mappings in the family $\overline{S_g^0(B^n)}$, where $g : U \to \mathbb{C}$ is a univalent function, which satisfies the requirements of Definition 2.4.13. This may represent another starting point for further investigations. Recent results in this direction were obtained in [41] for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < 1$ (see [104] and [97] in the case $n = 1$).

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