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DOCTORAL SCHOOL OF MATHEMATICS AND INFORMATICS

Contributions to the study of the coupled fixed point problem

Ph.D. Thesis Summary

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Introduction

The application of fixed point theory is very important in many fields as mathematics, statistics, chemistry biology, computer science, engineering and economics in dealing with problems arising in approximation theory, potential theory, game theory, mathematical economics, theory of differential equations, theory of integral equations, theory of matrix equatins etc. (see K. C. Border [20], A. Cataldo, E. A. Lee, X. Liu, E. D. Matsikoudis, H. Zheng [24], Y. Guo [52], A. Hyvärinen [59], A. Noumsi, S. Derrien, P. Quinton [89], A. Yantir and S. Gulsan Topal [161]). Fixed point theorems are useful in order to prove the existence of various types of Nash equilibria (see K. C. Border [20]) in economics, for proving the existence of weak periodic solutions for a model describing the electrical heating of a conductor taking into account the Joule-Thomson effect (see M. Badii [12]).

The classical Banach contraction principle is remarkable in its simplicity and it is perhaps the most widely applied fixed point theorem in all of analysis. This is because the contractive condition on the operator is easy to test and it requires only the structure of a complete metric space for its setting (see S. Banach [11]). Several mathematicians have been dedicated to the improvement and generalization of this principle (see A. M. Ostrowski [96], M. A. Krasnoselskii, P. P. Zabreiko [74], J. Jachymski [61], E. Rakotch [117], D. W. Boyd and J. S. W. Wong [21], J. Matkowski [81], F. E. Browder [22], A. Meir and E. Keeler [82], M. Geraghty [47], J. Jachymski [62], A. D. Arvanitakis [8], C. Mongkolkeha, W. Sintunavarat, P. Kumam [84], W. Sintunavarat, P. Kumam [144], W. Sintunavarat, P. Kumam [143]). The classical Banach contraction principle is a very useful tool in nonlinear analysis with many applications to operatorial equations, fractal theory, optimization theory and other topics.

A. C. M. Ran and M. C. B. Reurings in [118] investigated the existence of fixed points of nonlinear contraction operators in metric spaces endowed with a partial ordering and presented some applications to matrix equations. Since then, several authors have studied the problem of the existence and uniqueness of a fixed point for contraction type operators on partially ordered sets (see R. P. Agarwal, M. A. El-Gebeily and D. O'Regan [4], L. Ćirić, M. Cakić, J. S. Rajović and J. S. Ume [34], H. K. Nashine, B. Samet, C. Vetro [86], J. J. Nieto and R. R. López [87], J. J. Nieto and R. R. López [88], Y. J. Cho, R. Saadati and S. Wang [26], E. Graily, S. M. Vaezpour, R. Saadati, Y. J. Cho [50], W. Sintunavarat, Y. J. Cho and P. Kumam [145]).

The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions, see N. L. Ćirić [32], [33], N. Mizoguchi, W. Takahashi [83], B. E. Rhoades [125]. In 1969, S. B. Nadler [85] extended the Banach contraction principle from singlevalued to multivalued mapping.

Nadler's Theorem has been modified and generalized by many authors in metric fixed point theory. These generalizations weak the contractive nature of the map, but often with some additional requirements, as for instance to take compact values. See for example the fixed point results for multivalued mappings of generalized contractive type of Reich [120], L. Ćirić [32], V. M Sehgal and R. E. Smithson [142] and N. Mizoguchi, W. Takahashi [83].

Y. Feng and S. Liu defined in Y. Feng and S. Liu [42] a different kind of contractivity for multivalued mappings, which focuses the requirements on some orbits of the mapping under consideration. The main fixed point theorem in Y. Feng and S. Liu [42] is also a proper generalization of Nadler's Theorem. They also gave fixed point theorems for multivalued Caristi type mappings.

In 2007, D. Klim and D. Wardowski [72] inspired by Mizoguchi-Takahashi and Feng-Liu's work obtain a further generalization of the previous fixed point results given in Y. Feng and S. Liu [42], N. Mizoguchi, W. Takahashi [83], S. Reich [120].

O. Kada, T. Suzuki and W. Takahashi [64] introduced in 1996 the concept of ω-distance on a metric space and by using this notion they got an improvement of the Takahashi's nonconvex optimization Theorem, as well as generalizations of J. Caristi's fixed point Theorem and Ekeland's variational principle. They also gave fixed point theorems for singlevalued mappings of (ω-)contractive type.

In 2009, J. G. Falset, L. Guran and E. Llorens-Fuster [41] obtained a generalization of the fixed point results presented by D. Klim and D. Wardowski (Theorem 2.1 of [72]), for multivalued mappings of contractive type in complete metric spaces, by using the concept of ω-distance.

One of our interests is to obtain fixed point results for multivalued operators in cone metric spaces in order to generalize some results given by J. G. Falset, L. Guran and E. Llorens-Fuster in [41]. In the next paragraphs we will point out also some essential aspects regarding cone metric spaces.

In 1905, M. Fréchet [44], [45] introduced the concept of metric spaces. In 1934, his PhD student Đ. Kurepa [76] introduced more abstract metric spaces, in which the metric takes values in an ordered vector space. In the literature, the metric spaces with vector valued metric are known under various names: pseudometric spaces Đ. Kurepa [76], L. Collatz [35], K-metric spaces J. Eisenfeld [39], P. P. Zabrejko [162], I. A. Rus, A. Petruşel, M. A. Şerban [134], generalized metric spaces B. Rzepecki [137], vector-valued metric spaces I. D. Arandelović, D. J. Kečkić [7], cone-valued metric spaces K. J. Chung [30], [31], cone metric spaces L. G. Huang, X. Zhang [58], S. Janković, Z. Kadelburg and S. Radenović [63].

It is well known that cone metric spaces and cone normed spaces have deep applications in the numerical analysis and the fixed point theory. Some applications of cone metric spaces can be seen in L. Collatz [35] and P. P. Zabrejko [162]. J. Schröder [140], [141] was the first who showed the importance of cone metric spaces in numerical analysis and L. V. Kantorovich [67] showed for the first time the importance of cone normed spaces for the numerical analysis.

Starting from 2007, many authors have investigated cone metric spaces over Banach spaces and fixed point theorems in such spaces (L. G. Huang, X. Zhang [58], S. Rezapour, R. Hamlbarani [124], D. Wardowski [160], H. K. Pathak, N. Shahzad [97], I. Şahin, M. Telsi [150], A. Amini-Harandi, M. Fakhar [6], A. Sönmez [149], A. Latif, F. Y. Shaddad [78], D. Turkoglu, M. Abuloha [154], M. A. Khamsi [69], S. Radenović, Z. Kadelburg [116], M. Khani, M. Pourmahdian [71], M. Asadi, S. M. Vaezpour, H. Soleimani [9], etc.).

Banach contraction principle was extended for singlevalued contraction on spaces endowed with vector-valued metrics by A. I. Perov [98], A. I. Perov and A.V. Kibenko [99] and J. Ortega and W. Rheinboldt [95]. For some other contributions to this topic we also refer to A. Bucur, L. Guran and A. Petrusel [23], R.P. Agarwal [3], A. D. Filip and A. Petrusel [43], D. O'Regan, N. Shahzad, R. P. Agarwal [94], R. Precup, A. Viorel [112], R. Precup, A. Viorel [113], R. Precup [111], etc. The case of multivalued contractions on spaces endowed with vector-valued metrics is treated in A. Petrusel [104], I. R. Petre, A. Petruşel [101], Sh. Rezapour, P. Amiri [123], etc.

In the study of the fixed points for an operator, it is sometimes useful to consider a more general concept, namely coupled fixed point. The concept of coupled fixed point for nonlinear operators was introduced and studied by Opoitsev (see V.I. Opoitsev [90]-[92]) and then, in 1987, by D. Guo and V. Lakshmikantham (see [53]) in connection with coupled quasisolutions of an initial value problem for ordinary differential equations.

Later, a new research direction for the theory of coupled fixed points in ordered metric space was initiated by T. Gnana Bhaskar and V. Lakshmikantham in [48] and by V. Lakshmikantham and L. Ćirić in [77]. In [48] T. Gnana Bhaskar and V. Lakshmikantham introduced the notion of the mixed monotone property of a given operator. Furthermore, they proved some coupled fixed point theorems for operators which satify the mixed monotone property and presented as an application, the existence and uniqueness of a solution for a periodic boundary value problem. Their approach is based on some contractive type conditions on the operator.

In the last few decades, a wide discussion on coupled fixed point theorems attracted the interest of many mathematicians because of their important role in the study of nonlinear differential equations, nonlinear integral equations and differential inclusions. For other results on coupled fixed point theory see T. Gnana Bhaskar and V. Lakshmikantham [48], D. Guo, Y. J. Cho and J. Zhu [54], S. Hong [57], V. Lakshmikantham and L. Ćirić [77], M. D. Rus [135], V. Berinde [14], M. Berzig [17], W. Sintunavarat, P. Kumam, and Y. J. Cho [147], M. Abbas, W. Sintunavarat, P. Kumam [2], Y. J. Cho, G. He, N. J. Huang [25], Y. J. Cho, M. H. Shah, N. Hussain [27], Y. J. Cho, B. E. Rhoades, R. Saadati, B. Samet, W. Shantawi [28], M. E. Gordji, Y. J. Cho, H. Baghani [49], B. Samet, C. Vetro [139], W. Sintunavarat, Y. J. Cho and P. Kumam [146], W. Sintunavarat, A. Petrusel, P. Kumam [148], B. Samet [138].

In the present work we develop a detailed and unitary study regarding fixed point and coupled fixed point existence, uniqueness, data dependence, stability for singlevalued and multivalued operators considering the mixed monotone and limit shadowing properties. We support this study by presenting also some applications.

This thesis is devided in four chapters, each chapter containing several sections.

Chapter 1: Preliminaries

In this chapter we present the basic notions which are further considered in the next chapters of this work, allowing us to present the results of this investigation. This chapter contains the following sections:

In the **first section** we introduce the concepts of vector-valued space and convergent to zero matrices.

In the **second section** we recall some basic fixed point theorems for singlevalued and multivalued operators.

In the **third section** we present some open problems regarding fixed points and strict fixed points.

In the **fourth section** we remind the classical Cauchy's Lemma and give a generalization of it.

Chapter 2: Fixed point theorems in generalized metric spaces

In this chapter, we give some fixed point results for singlevalued and multivalued operators in spaces endowed with vector-valued metrics and in cone metric spaces. Our approach is based on Perov fixed point theorem in spaces endowed with vector-valued metrics. We study the Ulam-Hyers stability and the limit shadowing property of the fixed point problems. This chapter has three sections.

In the **first section** we give some existence, uniqueness and stability results for fixed point problem for singlevalued operators in $\mathbb{R}^m_+$ generalized metric spaces.

The original contributions in this section are the following results: Theorem 2.1.1 is an extension of Perov's Theorem and in the same time it extends some fixed point results given by V. Berinde [13], S. Reich [119] and G. E. Hardy, T. D. Rogers [55]; Theorem 2.1.2 is a result concerning the Ulam-Hyers stability of a fixed point equation.

The results presented in this section are included in the following works: A. Petrusel, G. Petrusel and C. Urs [108], A. Petrusel, C. Urs and O. Mleşniţe [109].

In the **second section** we present existence and stability results for a fixed point problem, for multivalued operators in $\mathbb{R}^m_+$ generalized metric spaces. The limit shadowing property is also investigated.

Our main results in this section are: Theorem 2.2.1 is an extension of Nadler's fixed point Theorem in a space endowed with a vector-valued metric, it is a multivalued version of Theorem 2.1.1 presented in the first section of this chapter and it is also a generalization of some results presented by M. Berinde, V. Berinde [15]; Theorem 2.2.4 is a result which concerns the Ulam-Hyers stability of a fixed point inclusion and limit shadowing property for a multivalued contraction; Theorem 2.2.5 which is a result regarding Ulam-Hyers stability of a fixed point inclusion for a multivalued operator.

The results presented in this section are contained in the following papers: A. Petrusel, G. Petrusel and C. Urs [108], A. Petrusel, C. Urs and O. Mleşniţe [109].

In the **third section** we obtained a fixed point result for a multivalued operator using the concept of c-distance in cone metric spaces. The c-distance, which was introduced by Y. J. Cho, R. Saadati and S. Wang [26], is a generalization of ω-distance given by O. Kada, T. Suzuki and W. Takahashi [64].

The most important contribution in this section is Theorem 2.3.8, which is a fixed point result for multivalued operators on cone metric spaces endowed with a c-distance. This theorem is a generalization of Theorem 3.3 given by J. G. Falset, L. Guran and E. Llorens-Fuster in [41].

The result, which is obtained in cone metric spaces is included in the paper E. Llorens-Fuster, C. Urs [79].

Chapter3: Coupled fixed point theorems

In this chapter we present existence, uniqueness and stability results for coupled fixed point of a pair of contractive type singlevalued and multivalued operators on complete metric spaces. The approach is based on Perov type fixed point theorem for contractions in spaces endowed with vector-valued metrics. The chapter is structured in three sections.

In the **first section** we give an existence, uniqueness, data dependence and Ulam-Hyers stability result for the coupled fixed point of a pair of contractive singlevalued operators on spaces endowed with vector-valued metrics.

Our main contribution in this section is Theorem 3.1.2 which is an existence, uniqueness, data dependence and Ulam-Hyers stability result for the coupled fixed point of a pair of singlevalued operators.

The result presented in this section is included in the paper C. Urs [155].

In the **second section** we provide some results in the framework of ordered metric spaces for the coupled fixed point problem of a pair of singlevalued operators, having the mixed monotone property.

Our main result in this section is Theorem 3.2.2 which is Gnana Bhaskar-Lakshmikantham type theorem for the coupled fixed point problem associated to a pair of singlevalued o-perators satisfying a generalized mixed monotone assumption.

The result obtained in this section is included in the paper A. Petruşel, G. Petruşel and C. Urs [108].

In the **third section** we investigated the existence, uniqueness, data dependence and Ulam-Hyers stability of the coupled fixed point of a pair of multivalued operators on spaces endowed with vector-valued metrics.

Our contributions in this section are: Theorem 3.3.3, which is an Ulam-Hyers stability result of a fixed point inclusion for a multivalued operator, with proximinal values in generalized metric space; Theorem 3.3.4, which is an existence, uniqueness and Ulam-Hyers stability result of a fixed point inclusion for a multivalued operator; Theorem 3.3.6, which is an existence and stability result for a system of operatorial inclusions for multivalued operators having proximinal values; Theorem 3.3.7, which is a result for existence, uniqueness and stability of a system of operatorial inclusions for multivalued operators.

The results which are presented in this section are included in the paper C. Urs [155].

Chapter 4: Applications

In this chapter we present some applications to first-order differential equations systems with periodic boundary value conditions and to systems of functional-integral equations in order to validate our previous results. The results presented in this chapter are applications of the coupled fixed point problems for contractive type singlevalued and multivalued operators on spaces endowed with vector-valued metrics. This chapter contains two sections.

In the **first section** we present an application to integral equations and boundary value problem.

Our main contribution in this section is Theorem 4.1.3, which represents a result for the existence, uniqueness and Ulam-Hyers stability of a solution to a periodic boundary value problem and it is considered as an application of the coupled fixed point Theorem 3.1.2, presented in Chapter 3.

The result given in this section is included in the paper C. Urs [156].

In the **second section** we give some applications to a first-oder differential system with periodic boundary value conditions, considering also the mixed monotone property and we present some applications to systems of functional-integral equations.

The first contribution in this section is Theorem 4.2.2, which includes the investigation of the existence and uniqueness of a solution to a periodic boundary value problem as an application of the coupled fixed point Theorem 3.2.2 presented in Chapter 3. We obtained Theorem 4.2.2 for the case of singlevalued operators with mixed monotone property in partially ordered metric space.

The second result, Theorem 4.2.3 is an application of coupled fixed point Theorem 3.2.3 exposed in Chapter 3. This application is an existence and uniqueness result for a system of functional-integral equations, which appears in traffic flow models.

The third contribution, Theorem 4.2.4 is also an application of Theorem 3.2.3 and it is an existence and uniqueness result for a system of functional-integral equations. In this case we apply the coupled fixed point theorem to an equivalent operatorial system of equations. As a consequence, we obtained the last application, Theorem 4.2.10 which is an existence and uniqueness result for a system of first-order boundary value problem with multivalued operators.

The first result, Theorem 4.2.2 is included in the paper C. Urs [157].

The author's contributions, presented in this work are part of the following scientific papers:

[155] C. Urs, *Ulam-Hyers stability for coupled fixed points of contractive type operators*, J. Nonlinear Sci. Appl., 6 (2013), 124-136, (MR 3017896).

[156] C. Urs, *Coupled fixed point theorems and applications to periodic boundary value problems*, Miskolc Mathematical Notes, 14 (2013), no. 1, 323-333, (MR 3070711), (IF: 0,304).

[108] A. Petrusel, G. Petrusel, and C. Urs, *Vector-valued metrics, fixed points and coupled fixed points for nonlinear operators*, Fixed Point Theory and Appl. (2013), 2013:218 doi:10.1186/1687-1812-2013-218, (MR 3108266), (IF: 1,87).

[109] A. Petrusel, C. Urs and O. Mleşniţe, *Vector-valued Metrics in Fixed Point Theory*, Contemporary Math. Series, Amer. Math. Soc., 2013, to appear.

[157] C. Urs, *Coupled fixed point theorems for mixed monotone operators and applications*, 2013, Studia Univ. Babeş-Bolyai Math., 2013, to appear.

[79] E. Llorens-Fuster, C. Urs, *Fixed point results for multivalued operators with respect to a c-distance*, 2013, submitted.

A significant part of the original contributions included in this thesis were also presented at the following scientific conferences:

- International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA), July 5^{th}–8^{th}, 2011, Babeş-Bolyai University of Cluj-Napoca, Romania;
- The 5^{th} International Workshop- 2012, Constructive Methods for Non-Linear Boundary Value Problems, June 28^{th}-July 1^{st},2012, Tokaj, Hungary;
- 6^{th} European Congress of Mathematics, July 2^{nd}-7^{th}, 2012, Krakow, Poland;
- The 10^{th} International Conference on Fixed Point Theory and its Applications, July 9^{th}-15^{th}, 2012, Babeş-Bolyai University of Cluj-Napoca, Romania;
- Workshop on Metric Fixed Point Theory, November 15^{th}-17^{th}, 2012, University of Valencia, Spain.
- The Fourtheenth International Conference on Applied Mathematics and Computer Science (Theodor Angheluţă Seminar), August 29^{th}-31^{st}, 2013, Cluj-Napoca, Romania.

Keywords: fixed point; contraction; singlevalued operator; multivalued operator; vector-valued metric; convergent to zero matrix; ordered metric space; coupled fixed point; Ulam-Hyers stability; limit shadowing property; mixed monotone property; cone metric spaces; boundary value problem; differential equations; system of functional-integral equations.

Chapter 1

Preliminaries

The aim of this chapter is to present the basic concepts and results, which are further considered in the next chapters, allowing us to present the results of this work. In the first section of this chapter we recall the notions of generalized metric space and convergent to zero matrix. The purpose of the second section is to present some important fixed point theorems which will be the basis of our study. In the third section we present some open problems regarding fixed points and strict fixed points. The aim of the fourth section is to recall the classical Cauchy's Lemma and to give a generalization of it.

The main references for this chapter are the papers of W. A. J. Luxemburg, A. C. Zaanen [80]; I. A. Rus [126]; A. C. Zaanen [163]; P. P. Zabrejko [162]; R. S. Varga [158]; R. Precup [110]; A. Granas, J. Dugundji [51]; L.-G. Huang, X. Zhang [58]; G. Allaire and S. M. Kaber [5]; I. A Rus, A. Petruşel and G. Petruşel [131].

1.1 Vector-valued metrics and convergent to zero matrices

In this section we introduce the notions, which are used later in our investigations. Concepts as generalized metric space, as well as convergent to zero matrices are recalled here.

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

(a) $d(x, y) \geq O$ for all $x, y \in X$; if $d(x, y) = O$, then $x = y$; (where $O := \underbrace{(0, 0, \cdots, 0)}_{m-times})$

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

A nonempty set X endowed with a vector-valued metric d is called a $\mathbb{R}^m_+$ generalized metric space and it will be denoted by (X, d). The notions of convergent sequence, Cauchy sequence, completeness, open and closed subset, open and closed ball, etc. are similar to those for usual metric spaces.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix. If $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, then, by definition:

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for } i \in \{1, 2, ..., m\}.$$

Notice that, through this work, we will make an identification between row and column vectors in $\mathbb{R}^m$.

Definition 1.1.1 *A square matrix $A \in M_{mm}(\mathbb{R}_+)$ is said to be convergent towards zero if and only if*

$$A^n \to 0 \text{ as } n \to \infty$$

A classical result in matrix analysis is the following theorem (see G. Allaire and S. M. Kaber [5], R. S. Varga [158]).

Theorem 1.1.2 *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalents:*
(i) A is convergent towards zero;
(ii) The eigenvalues of A are in the open unit disc, i.e $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;
(iii) The matrix $(I - A)$ is nonsingular and

$$(I - A)^{-1} = I + A + ... + A^n + ...; \tag{1.1}$$

(iv) The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements;
(v) $A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$;
(vi) The matrices qA and Aq are convergent towards zero, for each $q \in (1, Q)$, where $Q := \frac{1}{\rho(A)}$.

For more results regarding matrices convergent towards zero see I. A. Rus [126], A. I. Perov [98], M. Turinici [152], R. Precup [111].

Definition 1.1.3 *An operator $T : X \to X$ is said to be A-contraction (with respect to the vector-valued metric d on X) if there exists a convergent towards zero matrix A such that*

$$d(T(u), T(v)) \leq Ad(u, v)$$

for all $u, v \in X$.

1.2 Basic fixed point theorems for singlevalued and multivalued operators

In this section we present some well-known fixed point theorems which will help us further in our investigation. We recall one of the basic principles of fixed point theory on complete metric space.

We recall now Perov's fixed point theorem (see A. I. Perov [98], A. I. Perov, A. V. Kibenko [99], J. Ortega and W. Rheinboldt [95]). Perov's fixed point theorem is an extension of Banach's contraction principle for singlevalued contraction on spaces endowed with vector-valued metrics.

Theorem 1.2.1 (Perov) *Let (X, d) be a complete generalized metric space and the operator $f : X \to X$ be an A-contraction then:*
(i) $Fix(f) = \{x^*\}$;
(ii) the sequence of successive approximations $(x_n)_{n \in \mathbb{N}}$, $x_n = f^n(x_0)$ is convergent and has the limit x^*, for all $x_0 \in X$;

(iii) one has the following estimation

$$d\left(x_n,x^*\right) \le A^n \left(I-A\right)^{-1} d\left(x_0,x_1\right); \tag{1.2}$$

(iv) if $g: X \to X$ is an operator such that there exist $y^* \in Fix(g)$ and $\eta \in \left(\mathbb{R}_+^m\right)^*$ with $d\left(f\left(x\right), g\left(x\right)\right) \le \eta$, for each $x \in X$, then

$$d\left(x^*,y^*\right) \le \left(I-A\right)^{-1} \eta;$$

(v) if $g: X \to X$ is an operator and there exists $\eta \in \left(\mathbb{R}_+^m\right)^*$ such that $d\left(f\left(x\right), g\left(x\right)\right) \le \eta$, for all $x \in X$, then for the sequence $y_n := g^n\left(x_0\right)$ we have the following estimation

$$d\left(y_n,x^*\right) \le \left(I-A\right)^{-1} \eta + A^n \left(I-A\right)^{-1} d\left(x_o,x_1\right). \tag{1.3}$$

1.3 Fixed points and strict fixed points

It is of interest in fixed point theory to study the following open problems, see A. Petruşel, I. A. Rus and M. A. Şerban [107].

Problem 1.3.1 *Which are the metric conditions on T which imply that*

$$Fix(T) = SFix(T) \neq \emptyset \ ?$$

Problem 1.3.2 *Which are the metric conditions on T which imply that*

$$SFix(T) = \{x^*\}?$$

Problem 1.3.3 *Which are the metric conditions on T which imply that*

$$Fix(T) = SFix(T) = \{x^*\} \ ?$$

Problem 1.3.4 *In which metric conditions on T, the following implication holds:*

$$SFix(T) \neq \emptyset \implies Fix(T) = SFix(T) = \{x^*\} \ ?$$

It is an interesting aspect to investigate Problem 1.3.1, Problem 1.3.2, Problem 1.3.3 and Problem 1.3.4 in generalized metric spaces.

1.4 Cauchy-type lemmas

In this section we present the classical Cauchy's lemma and a result which was given by I. A. Rus [128] and using it, we obtained stability type results regarding Ulam-Hyers stability, limit shadowing property for multivalued operators in generalized metric spaces. We recall also a generalization of Cauchy's lemma given by I. A. Rus and M. A. Şerban in [133].

Chapter 2

Fixed point theorems in generalized metric spaces

In this chapter, we present some fixed point results for singlevalued and multivalued operators in spaces endowed with $\mathbb{R}^m_+$ metrics and in cone metric spaces. The approach is based on Perov type fixed point theorem in spaces endowed with vector-valued metrics. The Ulam-Hyers stability and the limit shadowing property of the fixed point problems are also discussed.

The references which we used in order to develop this chapter are the following: A. I. Perov [98]; S. B. Nadler [85]; H. Covitz and S. B. Nadler [36]; R. P. Agarwal [3]; O. Kada, T. Suzuki and W. Takahashi [64]; I. A. Rus, A. Petruşel, A. Sîntămărian [132]; A. Petruşel [104]; Y. Feng and S. Liu [42]; D. Klim and D. Wardowski [72]; L. G. Huang, X. Zhang [58]; M. Berinde and V. Berinde [15], R. Precup and A. Viorel [112]; I. A. Rus [129], [130]; R. Precup [111], A. Bucur, L. Guran and A. Petruşel [23]; J. G. Falset, L. Guran and E. Llorens-Fuster [41]; D. Wardowski [160]; S. Radenović and B. E. Rhoades [115]; A. Petruşel and I. A. Rus [106], A. D. Filip and A. Petruşel [43], M. Bota and A. Petruşel [19], P. T. Petru, A. Petruşel and J. C. Yao [102], Y. J. Cho, R. Saadati and S. Wang [26].

2.1 Fixed point theorems for singlevalued operators in vectorial metric spaces

The aim of this section is to present some existence, uniqueness and stability results for fixed point equations in $\mathbb{R}^m_+$ generalized metric spaces. The approach is based on an abstract fixed point theorem in ordered complete metric spaces. The results which are given in this section are related with other existence and stability results for the coupled fixed point problem for singlevalued operators proved in C. Urs [155], by the support of Perov's fixed point Theorem.

Notice that in R. Precup [111], as well as in A. Bucur, L. Guran and A. Petruşel [23], A. D. Filip and A. Petruşel [43] and R. Precup, A. Viorel [112] are pointed out the advantages of working with vector-valued norm, with respect to the usual scalar norms.

There is a vast literature concerning this approach, see for example R. P. Agarwal [3], D. O'Regan, N. Shahzad, R. P. Agarwal [94], A. Petruşel, I. A. Rus [106], R. Precup, A. Viorel [113], R. Precup [111], etc.

We will focus our attention to the following system of operatorial equations:

$$\begin{cases} x = T_1(x,y) \\ y = T_2(x,y) \end{cases}$$

where $T_1, T_2 : X \times X \to X$ are two given operators.

By definition, a solution $(x,y) \in X \times X$ of the above system is called a fixed point for the operators T_1 and T_2. Notice that if $S : X \times X \to X$ is an operator and we define:

$$T_1(x,y) := S(x,y) \text{ and } T_2(x,y) := S(y,x),$$

then we get the classical concept of coupled fixed point for the operator S, introduced by V.I. Opoitsev and then intensively studied in some papers by D. Guo and V. Lakshmikantham [53], T. Gnana Bhaskar and V. Lakshmikantham [48], V. Lakshmikantham and L. Ćirić [77], etc.

The case of an operatorial inclusion is defined in a similar way, namely by using the symbol $\in$ instead of $=$. The concept of coupled fixed point for a multivalued operator S is accordingly defined.

The next result is an extension of Perov's Theorem.

Theorem 2.1.1 *Let (X,d) be a generalized complete metric space and let $f : X \to X$ be an (A,B,C,D,E)-contraction, i.e., $A,B,C,D,E \in M_{mm}(\mathbb{R}_+)$ are such that the matrices E and $C+E$ or the matrices D and $B+D$ converge to zero and the matrix $M := (I-C-E)^{-1}(A+C+D)$ or the matrix $N := (I-B-D)^{-1}(A+B+E)$ converges to zero and*

$$d(f(x), f(y)) \le Ad(x,y) + Bd(y,f(x)) + Cd(x,f(y)) + Dd(x,f(x)) + Ed(y,f(y)), \quad (2.1)$$

for all $x, y \in X$.
Then, the following conclusions hold:

1. f has at least one fixed point and, for each $x_0 \in X$, the sequence $x_n := f^n(x_0)$ of successive approximations of f starting from x_0 converges to $x^*(x_0) \in Fix(f)$ as $n \to \infty$;

2. For each $x_0 \in X$ we have

$$d(x_n, x^*(x_0)) \le M^n (I-M)^{-1} d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}$$

or

$$d(x_n, x^*(x_0)) \le N^n (I-N)^{-1} d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}.$$

3. If, additionally, the matrix $A+B+C$ converges to zero, then f has a unique fixed point in X.

We remind two important abstract concepts: weakly Picard operator and ψ-weakly Picard operator (see I. A. Rus [129], [130]).

For the proof of the next theorems we need the notion of generalized Ulam-Hyers stability of a fixed point equation, which was introduced by I. A. Rus in [130] (see also I. A. Rus [127]). The concept is adapted after the definition given by S. Reich and A. J. Zaslawski in [121] in the context of a metric space.

We give now the following abstract result (see also I. A. Rus [130]) concerning the Ulam-Hyers stability of a fixed point equation.

Theorem 2.1.2 *Let (X, d) be a generalized metric space and $f : X \to X$ be a ψ-weakly Picard operator. Then, the fixed point equation $x = f(x)$ is generalized Ulam-Hyers stable.*

2.2 Fixed point theorems for multivalued operators in vectorial metric spaces

The aim of this section is to present some existence and stability results for fixed point inclusions in $\mathbb{R}^m_+$ generalized metric spaces.

We give an extension of the Nadler fixed point theorem in a space endowed with a vector-valued metric, which is also a multivalued version of Perov's Theorem. The following result is a generalization of some results given by M. Berinde, V. Berinde [15].

Theorem 2.2.1 *Let (X, d) be a generalized complete metric space and let $S : X \to P_{cl}(X)$ be a multivalued (A, B, C)-contraction, i.e., $A, B, C \in M_{mm}(\mathbb{R}_+)$ are such that the matrix $M := A + C$ converges to zero and*

$$H(S(x), S(y)) \leq Ad(x, y) + BD(y, S(x)) + CD(x, S(x)), \text{ for all } x, y \in X. \tag{2.2}$$

Then:

(i) $Fix(S) \neq \emptyset$;

(ii) for each $(x, y) \in Graph(S)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ (with $x_0 = x$, $x_1 = y$ and $x_{n+1} \in S(x_n)$, for each $n \in \mathbb{N}^*$) such that $(x_n)_{n\in\mathbb{N}}$ is convergent to a fixed point $x^* := x^*(x, y)$ of S and the following relations hold

$$d(x_n, x^*) \leq M^n(I - M)^{-1}d(x_0, x_1), \text{ for each } n \in \mathbb{N}^*$$

and

$$d(x, x^*) \leq (I - M)^{-1}d(x, y).$$

If in the previous result the matrix $C = O_m$, then we obtain a fixed point theorem for a multivalued almost contraction in generalized complete metric spaces.

We recal the notions of multivalued weakly Picard and ψ-multivalued weakly Picard operator (see I. A. Rus, A. Petruşel, A. Sîntămărian [132] and A. Petruşel [104]).

We remind two important stability concepts.

Definition 2.2.2 *Let (X, d) be a generalized metric space and $F : X \to P(X)$ be a multivalued operator. The fixed point inclusion*

$$x \in F(x), \ x \in X \tag{2.3}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ increasing, continuous in O with $\psi(O) = O$ such that for each $\varepsilon := (\varepsilon_1, ..., \varepsilon_m)$ (with $\varepsilon_i > 0$ for $i \in \{1, ..., m\}$) and for each ε-solution $y^* \in X$ of (2.3), i.e.,

$$D(y^*, F(y^*)) \leq \varepsilon, \tag{2.4}$$

there exists a solution x^* of the fixed point inclusion (2.3) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

In particular, if $\psi(t) = C \cdot t$, for each $t \in \mathbb{R}^m_+$ (where $C \in M_{mm}(\mathbb{R}_+)$), then the fixed point inclusion (2.3) is said to be Ulam-Hyers stable.

Definition 2.2.3 *Let (X, d) be a generalized metric space and $F : X \to P(X)$ be a multivalued operator. Then, the multivalued operator F is said to have the limit shadowing property if for each sequence $(y_n)_{n\in\mathbb{N}}$ in X such that $D(y_{n+1}, F(y_n)) \to O$ as $n \to +\infty$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of F such that $d(x_n, y_n) \to O$ as $n \to +\infty$.*

Using the following auxiliary result (see I. A. Rus [128]) we can get stability type results (Ulam-Hyers stability and limit shadowing property) for multivalued A-contractions.

Cauchy-type Lemma. *Let $A \in M_{mm}(\mathbb{R}_+)$ be a matrix convergent towards zero and $(B_n)_{n\in\mathbb{N}} \in \mathbb{R}^m_+$ be a sequence, such that $\lim\limits_{n\to+\infty} B_n = O_m$. Then*

$$\lim_{n\to+\infty}\left(\sum_{k=0}^{n} A^{n-k}B_k\right) = O_m.$$

We prove the Ulam-Hyers stability of the fixed point inclusion (2.3) for the case of a multivalued A-contraction, which has at least one strict fixed point. The limit shadowing property is also established.

Theorem 2.2.4 *Let (X, d) be a generalized complete metric space and let $F : X \to P_{cl}(X)$ be a multivalued A-contraction. Suppose also that $SFix(F) \neq \emptyset$, i.e., there exists $x^* \in X$ such that $\{x^*\} = F(x^*)$. Then:*

(a) $Fix(F) = SFix(F) = \{x^*\}$;

(b) the fixed point inclusion (2.3) is Ulam-Hyers stable;

(c) the multivalued operator F has the limit shadowing property.

We also have the following abstract result, concerning the Ulam-Hyers stability of the fixed point inclusion (2.3) for multivalued operators.

Theorem 2.2.5 *Let (X, d) be a generalized metric space and $F : X \to P_{cl}(X)$ be a multivalued ψ-weakly Picard operator. Suppose also that there exists a matrix $C \in \mathbb{M}_{mm}(\mathbb{R}_+)$ such that for any $\varepsilon := (\varepsilon_1, ..., \varepsilon_m)$ (with $\varepsilon_i > 0$ for $i \in \{1, ..., m\}$) and any $z \in X$ with $D(z, F(z)) \le \varepsilon$ there exists $u \in F(z)$ such that $d(z, u) \le C\varepsilon$. Then, the fixed point inclusion (2.3) is generalized Ulam-Hyers stable.*

For other examples and results regarding the Ulam-Hyers stability and the limit shadowing property of the operatorial equations and inclusions see I. A. Rus [130], [129], [127], A. Petruşel and I. A. Rus [106], M. Bota and A. Petruşel [19], P. T. Petru, A. Petruşel and J. C. Yao [102].

2.3 Fixed point theorems in cone metric spaces endowed with c-distance

The aim of this section is to present a fixed point result for a multivalued mapping in cone metric space using the concept of c-distance.

In this section we present the basic notions in cone metric spaces and we give some examples of cone metric spaces and c-distances. For basic concepts and results see P. P. Zabrejko [162].

The fixed point theory for cone metric spaces was re-discovered by L. G. Huang, X. Zhang [58] in 2007 and became a subject of interest for many authors. Cone metric spaces are generalizations of metric spaces, where the metric is replaced by the mapping with values in a cone, from a Banach space.

Y. J. Cho, R. Saadati and S. Wang [26] introduced in 2011 a new concept of c-distance in cone metric spaces, which is a cone version of ω-distance, given by O. Kada, T. Suzuki and W. Takahashi [64]. They proved in [26] some fixed point theorems for contractive type mappings in partially ordered cone metric spaces, using c-distance.

In this investigation we will use c-distance in order to obtain a fixed point theorem for a multivalued mapping, which allows us to give a generalization of Theorem 3.3 presented by J. G. Falset, L. Guran and E. Llorens-Fuster in [41].

In the following, we introduce some notions which will be used in this section.

Let E be a real Banach space and θ denote the zero element in E. Let P be a subset of E with int $P \neq \emptyset$, where int P denotes the interior of P. Then P is called a cone if the following conditions are satisfied:

(i) P is closed and $P \neq \{\theta\}$;

(ii) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$,

(iii) $x \in P \cap (-P) = \{\theta\}$ implies $x = \theta$.

For any cone $P \subset E$, the *partial ordering* $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation $x \prec y$ stands for $x \preceq y$, but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in$int P, whenever int $P \neq \emptyset$. A cone P is called *normal* if there exists a number $K > 0$ such that

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K \|y\|$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ $(n \to \infty)$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

We assume in our approach that E is a real Banach space and P is a cone in E with int $P \neq \emptyset$.We recall the notions of cone metric and cone metric space (see for example L.-G. Huang, X. Zhang [58]).

Example 2.3.1 *Let* $E = \mathbb{R}^3$, $P = \{(z_1, z_2, z_3) \in E \mid z_i \geq 0, i = 1, 2, 3\}$, $X = \mathbb{R}^2$ *and* $d : X \times X \to E$ *such that*

$$d(x, y) = (d_\infty(x, y), d_2(x, y), d_1(x, y)).$$

Then (X, d) *is a cone metric space.*

In the above example we use the Chebyshev metric, the euclidian metric and the Minkowski metric.

We assume that (X, d) is a cone metric space with respect to E and P and for the sake of brevity we will omit hereafter to mention E and P.

For further definitions and properties of cone metric spaces see L. G. Huang, X. Zhang [58], S. Radenović and B. E. Rhoades [115].

The concept of c-distance on a cone metric space (X, d), which was introduced by Y. J. Cho, R. Saadati and S. Wang [26] is a generalization of ω-distance.

Definition 2.3.2 *(Y. J. Cho, R. Saadati and S. Wang [26]) Let (X, d) be a cone metric space. Then a function $q : X \times X \to E$ is called a c-distance on X if the following are satisfied:*

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$;

(q2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q3) for any $x \in X$, if there exists $u = u_x \in P$ such that $q(x, y_n) \preceq u$ for each $n \geq 1$, then $q(x, y) \preceq u$ whenever (y_n) is a sequence in X converging to a point $y \in X$;

(q4) for any $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Now we give some examples of c-distances, where d is the cone metric, which we considered in the Example 2.3.1.

Example 2.3.3 *Let (X, d) be a cone metric space and P be a normal cone. Let $q : X \times X \to E$, defined by $q(x, y) = d(x, y)$, for all $x, y \in X$. Then q is a c-distance. (q1) and (q2) (see Definition 2.3.2) are obvious and (q3) (see Definition 2.3.2) is satisfied. Let $c := (c_1, c_2, c_3) \in \mathbb{R}^3$ with $(0, 0, 0) \ll (c_1, c_2, c_3)$ there exists $e := (e_1, e_2, e_3) = (\frac{c_1}{2}, \frac{c_2}{2}, \frac{c_3}{2}) \in \mathbb{R}^3$ with $(0, 0, 0) \ll (e_1, e_2, e_3)$, such that $q(z, x) \ll e$ and $q(z, y) \ll e$. Then $q(x, y) \preceq q(x, z) + q(z, y) \ll (e_1, e_2, e_3) + (e_1, e_2, e_3) = c$. So (q4) (see Definition 2.3.2) is satisfied. Hence q is a c-distance.*

Example 2.3.4 *Let (X, d) be a cone metric space. and P be a normal cone. Let F be a bounded and closed subset of X. Assume that F contains at least two points and c is $c := (c_1, c_2, c_3) \succeq (\sup\{d_\infty(x, y)\}, \sup\{d_2(x, y)\}, \sup\{d_1(x, y)\}) = diamF$, where $diamF$ is the diameter of F. Then $q : X \times X \to E$, defined by*

$$q(x, y) = \begin{cases} d(x, y), \ \text{if } x, y \in F \\ c, \ \text{if } x \notin F \text{ or } y \notin F \end{cases},$$

is a c-distance.

A set $A \subset X$ is called *closed* if for any sequence $\{x_n\} \subset A$ convergent to x we have $x \in A$.

A set $A \subset X$ is said to be sequentially compact if for any sequence $\{x_n\} \subset A$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ is convergent to an element of A.

We denote $N(X)$ a collection of all nonempty subsets of X, $C(X)$ a collection of all nonempty closed subsets of X and $K(X)$ a collection of all nonempty sequentially compact subsets of X.

Y. Feng and S. Liu [42] obtained an extension of Nadler's fixed point theorem in complete metric spaces under following conditions:

Let $T : X \to N(X)$ be a multivalued mapping. Define function $f : X \to \mathbb{R}$ as $f(x) = d(x, T(x))$.

For a positive constant b ($b \in (0,1)$), define the set $I_b^x \subset X$ as

$$I_b^x = \{y \in T(x) : bd(x,y) \leq d(x, T(x))\}.$$

Theorem 2.3.5 *(Y. Feng and S. Liu [42]) Let (X,d) be a complete metric space, $T : X \to C(X)$ be a multivalued mapping. If there exists a constant $c \in (0,1)$, such that for any $x \in X$ there is $y \in I_b^x$ satisfying*

$$d(y, T(y)) \leq cd(x,y),$$

then T has a fixed point in X provided $c < b$ and f is lower semi-continuous.

D. Wardowski [160], inspired by the work of Y. Feng and S. Liu [42] introduced the concept of set-valued contractions in cone metric spaces and obtained a fixed point theorem, considering the distance between a point and a set in the following way:

Let (M,d) be a cone metric space. Let $T : M \to C(M)$. For $x \in M$, we denote

$$\begin{aligned} D(x,Tx) &= \{d(x,z) : z \in Tx\}, \\ S(x,Tx) &= \{u \in D(x,Tx) : \|u\| = \inf\{\|v\| : v \in D(x,Tx)\}\}. \end{aligned}$$

A mapping $f : X \to \mathbb{R}$ is said to be lower semi-continuous at x (lsc for short), with respect to d, if for any sequence (x_n) in X and $x \in X$ with $x_n \to x$, the inequality $f(x) \leq \lim\limits_{n\to\infty} \inf f(x_n)$ holds.

Let $T : X \to K(X)$, $b \in (0,1]$ and $x \in X$. In our investigation we will consider the following set:

$$I_b^x := \{y \in T(x) : bd(x,y) \leq S(x, T(x))\},$$

We give the following definition:

Definition 2.3.6 *Let $T : X \to K(X)$ be a multivalued mapping, and let q be a c-distance on X. Define the function $f : X \to \mathbb{R}$ by $f(x) := D_q(x, T(x))$, where $D_q(x,T(x)) = \inf\limits_{y \in T(x)} \|q(x,y)\|$. For each $b \in [0,1]$, we define the set $I_{b,q}^x := \{y \in T(x) : b\,\|q(x,y)\| \leq D_q(x, T(x))\}$.*

Remark 2.3.7 *If $T : X \to K(X)$ is a multivalued mapping and $0 < b < 1$, it is clear that, for every $x \in X$, the set $I_{b,q}^x$ is nonempty.*

We present now a fixed point theorem for multivalued operators on cone metric spaces endowed with a c-distance.

Theorem 2.3.8 *Let (X,d) be a complete cone metric space, P be a regular cone, q be a c-distance on X and let $T : X \to K(X)$ be a multivalued mapping. Assume that the $g : X \to \mathbb{R}$ defined by $g(x) = \inf\limits_{y \in T(x)} \|q(x,y)\|$, $x \in X$ is lower semicontinuous. The following conditions hold:*

1. There exist $b \in (0,1)$ and $\varphi : [0,\infty[\to [0,b[$ such that

(1i) for each $t \in [0,\infty[$,

$$\lim_{r\to t^+} \sup \varphi(r) < b;$$

(1ii) for every $x \in X$, there exists $y \in I^x_{b,q}$ such that

$$D_q(y, T(y)) \leq \varphi(\|q(x, y)\|) \, \|q(x, y)\| \, ;$$

2. for every $y \in X$ with $y \notin T(y)$

$$\inf\{\|q(x, y)\| + D_q(x, T(x)) : x \in X\} > 0$$

Then T has a fixed point.

Chapter 3

Coupled fixed point theorems

In this chapter we present some coupled fixed points results for contractive type singlevalued and multivalued operators on spaces endowed with vector-valued metrics. The approach is based on Perov-type fixed point theorem for contractions in metric spaces endowed with vector-valued metrics. For related results to Perov's fixed point theorem and for some generalizations and applications of it we refer to A. Bucur, L. Guran and A. Petruşel [23], A. D. Filip and A. Petrusel [43], R. Precup [111].

In order to develop this chapter we mention here the references which were considered: D. Guo and V. Lakshmikantham [53]; D. Guo, Y. J. Cho and J. Zhu [54]; J. J. Nieto and R. R. López [87]; T. Gnana Bhaskar and V. Lakshmikantham [48]; J. J. Nieto and R. R. López [88]; S. Hong [57]; R. P. Agarwal, M. A. El-Gebeily and D. O'Regan [4]; L. Ćirić, M. Cakić, J. S. Rajović and J. S. Ume [34]; I. A. Rus [130], R. Precup [111]; V. Lakshmikantham and L. Ćirić [77]; M. D. Rus [135]; M. Bota and A. Petruşel [19], P. T. Petru, A. Petruşel and J. C. Yao [102].

3.1 Coupled fixed point theorems for singlevalued operators

The aim of this section is to give an existence, uniqueness, data dependence and Ulam-Hyers stability result for the coupled fixed point of a pair of contractive singlevalued operators on spaces endowed with vector-valued metrics.

Let (X, d) be a metric space. We will focus our attention to the following system of operatorial equations:

$$\begin{cases} x = T_1(x, y) \\ y = T_2(x, y) \end{cases}$$

where $T_1, T_2 : X \times X \to X$ are two given operators.

By definition, a solution $(x, y) \in X \times X$ of the above system is called a fixed point for the pair (T_1, T_2). In a similar way, the case of an operatorial inclusion (using the symbol $\in$ instead of $=$) could be considered.

For the proof of the main result in this section we need the notion of Ulam-Hyers stability of a system of operatorial equations (see I. A. Rus [130]).

Definition 3.1.1 *Let (X, d) be a metric space and let $T_1, T_2 : X \times X \to X$ be two operators. Then the operatorial equations system*

$$\left\{ \begin{array}{l} x = T_1(x, y) \\ y = T_2(x, y) \end{array} \right. \tag{3.1}$$

is said to be Ulam-Hyers stable, if there exist $c_1, c_2, c_3, c_4 > 0$ such that for each $\varepsilon_1, \varepsilon_2 > 0$ and each pair $(u^*, v^*) \in X \times X$ such that

$$\begin{array}{l} d(u^*, T_1(u^*, v^*)) \leq \varepsilon_1 \\ d(v^*, T_2(u^*, v^*)) \leq \varepsilon_2, \end{array} \tag{3.2}$$

there exists a solution $(x^*, y^*) \in X \times X$ of (3.1) such that

$$\begin{array}{l} d(u^*, x^*) \leq c_1\varepsilon_1 + c_2\varepsilon_2 \\ d(v^*, y^*) \leq c_3\varepsilon_1 + c_4\varepsilon_2. \end{array} \tag{3.3}$$

For examples and other considerations regarding Ulam-Hyers stability and generalized Ulam-Hyers stability of the operatorial equations and inclusions see I. A. Rus [130], M. Bota and A. Petruşel [19], P. T. Petru, A. Petruşel and J. C. Yao [102].

The main result of this section is the following existence, uniqueness, data dependence and Ulam-Hyers stability theorem for the coupled fixed point of a pair of singlevalued operators (T_1, T_2).

Theorem 3.1.2 *Let (X, d) be a complete metric space, $T_1, T_2 : X \times X \to X$ be two operators such that*

$$\begin{array}{l} d(T_1(x, y), T_1(u, v)) \leq k_1 d(x, u) + k_2 d(y, v) \\ d(T_2(x, y), T_2(u, v)) \leq k_3 d(x, u) + k_4 d(y, v) \end{array} \tag{3.4}$$

for all $(x, y), (u, v) \in X \times X$, (where $k_i \in \mathbb{R}_+$, for $i \in \{1, 2, 3, 4\}$). We suppose that $A := \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ converges to zero. Then:
(i) there exists a unique element $(x^*, y^*) \in X \times X$ such that

$$\left\{ \begin{array}{l} x^* = T_1(x^*, y^*) \\ y^* = T_2(x^*, y^*), \end{array} \right. \tag{3.5}$$

(ii) the sequence $(T_1^n(x, y), T_2^n(x, y))_{n \in \mathbb{N}}$ converges to (x^*, y^*) as $n \to \infty$, where

$$\begin{array}{l} T_1^{n+1}(x, y) := T_1^n(T_1(x, y), T_2(x, y)) \\ T_2^{n+1}(x, y) := T_2^n(T_1(x, y), T_2(x, y)), \end{array} \tag{3.6}$$

for all $n \in \mathbb{N}$.
(iii) we have the following estimation:

$$\begin{pmatrix} d(T_1^n(x_0, y_0), x^*) \\ d(T_2^n(x_0, y_0), y^*) \end{pmatrix} \leq A^n (I - A)^{-1} \begin{pmatrix} d(x_0, T_1(x_0, y_0)) \\ d(y_0, T_2(x_0, y_0)) \end{pmatrix}, \tag{3.7}$$

(iv) let $F_1, F_2 : X \times X \to X$ be two operators such that, there exist $\eta_1, \eta_2 > 0$ with

$$d\left(T_1\left(x,y\right), F_1\left(x,y\right)\right) \leq \eta_1$$
$$d\left(T_2\left(x,y\right), F_2\left(x,y\right)\right) \leq \eta_2,$$

for all $(x,y) \in X \times X$. If $(a^*, b^*) \in X \times X$ is such that

$$\begin{cases} a^* = F_1\left(a^*, b^*\right) \\ b^* = F_2\left(a^*, b^*\right), \end{cases} \tag{3.8}$$

then

$$\begin{pmatrix} d\left(a^*, x^*\right) \\ d\left(b^*, y^*\right) \end{pmatrix} \leq \left(I - A\right)^{-1} \eta, \tag{3.9}$$

where $\eta := \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$.

(v) let $F_1, F_2 : X \times X \to X$ be two operators such that, there exist $\eta_1, \eta_2 > 0$ with

$$\begin{matrix} d\left(T_1\left(x,y\right), F_1\left(x,y\right)\right) \leq \eta_1 \\ d\left(T_2\left(x,y\right), F_2\left(x,y\right)\right) \leq \eta_2, \end{matrix} \tag{3.10}$$

for all $(x,y) \in X \times X$. If we consider the sequence $\left(F_1^n\left(x,y\right), F_2^n\left(x,y\right)\right)_{n\in\mathbb{N}}$, given by

$$\begin{matrix} F_1^{n+1}\left(x,y\right) := F_1^n\left(F_1\left(x,y\right), F_2\left(x,y\right)\right) \\ F_2^{n+1}\left(x,y\right) := F_2^n\left(F_1\left(x,y\right), F_2\left(x,y\right)\right), \end{matrix} \tag{3.11}$$

for all $n \in \mathbb{N}^*$ and $\eta := \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, then

$$\begin{pmatrix} d\left(F_1^n\left(x_0, y_0\right), x^*\right) \\ d\left(F_2^n\left(x_0, y_0\right), y^*\right) \end{pmatrix} \leq \left(I - A\right)^{-1} \eta + A^n \left(I - A\right)^{-1} \begin{pmatrix} d\left(x_0, T_1\left(x_0, y_0\right)\right) \\ d\left(y_0, T_2\left(x_0, y_0\right)\right) \end{pmatrix},$$

(vi) the operatorial equations system

$$\begin{cases} x = T_1\left(x,y\right) \\ y = T_2\left(x,y\right) \end{cases} \tag{3.12}$$

is Ulam-Hyers stable.

3.2 Coupled fixed point theorems for mixed monotone type singlevalued operators

In this section the purpose is to present, in the setting of an ordered metric space, a Gnana Bhaskar-Lakshmikantham type theorem for the coupled fixed point problem associated to a pair of singlevalued operators satisfying a generalized mixed monotone assumption.

Let X be a nonempty set endowed with a partial order relation denoted by $\leq$. Then we denote

$$X_{\leq} := \{(x_1, x_2) \in X \times X \ : \ x_1 \leq x_2 \text{ or } x_2 \leq x_1\}.$$

If $f : X \to X$ is an operator, then we denote the cartesian product of f with itself as follows:

$$f \times f : X \times X \to X \times X, \text{ given by } (f \times f)(x_1, x_2) := (f(x_1), f(x_2)).$$

The following result will be an important tool in our approach.

Theorem 3.2.1 *Let $(X, d, \leq)$ be an ordered generalized metric space and let $f : X \to X$ be an operator. We suppose that:*
(1) for each $(x, y) \notin X_{\leq}$ there exists $z(x, y) := z \in X$ such that $(x, z), (y, z) \in X_{\leq}$;
(2) $X_{\leq} \in I(f \times f)$;
(3) $f : (X, d) \to (X, d)$ is continuous;
(4) the metric d is complete;
(5) there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_{\leq}$;
(6) there exists a matrix $A \in M_{mm}(\mathbb{R}_+)$ which converges to zero, such that

$$d(f(x), f(y)) \leq Ad(x, y), \text{ for each } (x, y) \in X_{\leq}.$$

Then $f : (X, d) \to (X, d)$ is a Picard operator.

We will apply the above result for the coupled fixed point problem generated by two operators.

Let X be a nonempty set endowed with a partial order relation denoted by $\leq$. If we consider $z := (x, y), w := (u, v)$ two arbitrary elements of $Z := X \times X$, then, by definition

$$z \preceq w \text{ if and only if } (x \geq u \text{ and } y \leq v).$$

Notice that $\preceq$ is a partial order relation on Z.
We denote

$$Z_{\preceq} = \{(z, w) := ((x, y), (u, v)) \in Z \times Z \ : \ z \preceq w \text{ or } w \preceq z\}.$$

Let $T : Z \to Z$ be an operator defined by

$$T(x, y) := \begin{pmatrix} T_1(x, y) \\ T_2(x, y) \end{pmatrix} = (T_1(x, y), T_2(x, y)). \tag{3.13}$$

The cartesian product of T and T will be denoted by $T \times T$ and it is defined in the following way

$$T \times T : Z \times Z \to Z \times Z, \quad (T \times T)(z, w) := (T(z), T(w)).$$

The first main result in this section is the following theorem.

Theorem 3.2.2 *Let $(X, d, \leq)$ be an ordered and complete metric space and let $T_1, T_2 : X \times X \to X$ be two operators. We suppose:*
(i) for each $z = (x, y), w = (u, v) \in X \times X$ which are not comparable with respect to the partial ordering $\preceq$ on $X \times X$, there exists $t := (t_1, t_2) \in X \times X$ (which may depend on (x, y) and (u, v)) such that t is comparable (with respect to the partial ordering $\preceq$) with both z and w, i.e.,

$$((x \geq t_1 \text{ and } y \leq t_2) \text{ or } (x \leq t_1 \text{ and } y \geq t_2)) \text{ and } ((u \geq t_1 \text{ and } v \leq t_2) \text{ or } (u \leq t_1 \text{ and } v \geq t_2));$$

(ii) for all $(x \geq u$ *and* $y \leq v)$ *or* $(u \geq x$ *and* $v \leq y)$ *we have*

$$\begin{cases} T_1(x,y) \geq T_1(u,v) \\ T_2(x,y) \leq T_2(u,v) \end{cases} \text{or} \begin{cases} T_1(u,v) \geq T_1(x,y) \\ T_2(u,v) \leq T_2(x,y) \end{cases};$$

(iii) $T_1, T_2 : X \times X \to X$ *are continuous;*
(iv) there exists $z_0 := (z_0^1, z_0^2) \in X \times X$ *such that*

$$\begin{cases} z_0^1 \geq T_1(z_0^1, z_0^2) \\ z_0^2 \leq T_2(z_0^1, z_0^2) \end{cases} \text{or} \begin{cases} T_1(z_0^1, z_0^2) \geq z_0^1 \\ T_2(z_0^1, z_0^2) \leq z_0^2 \end{cases};$$

(v) there exists a matrix $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in M_2(\mathbb{R}_+)$ *convergent toward zero such that*

$$\begin{aligned} d(T_1(x,y), T_1(u,v)) &\leq k_1 d(x,u) + k_2 d(y,v) \\ d(T_2(x,y), T_2(u,v)) &\leq k_3 d(x,u) + k_4 d(y,v), \end{aligned}$$

for all $(x \geq u$ *and* $y \leq v)$ *or* $(u \geq x$ *and* $v \leq y)$*;*
Then there exists a unique element $(x^*, y^*) \in X \times X$ *such that*

$$x^* = T_1(x^*, y^*) \text{ and } y^* = T_2(x^*, y^*)$$

and the sequence of the succesive aproximations $(T_1^n(w_0^1, w_0^2), T_2^n(w_0^1, w_0^2))$ *converges to* (x^*, y^*) *as* $n \to \infty$, *for all* $w_0 = (w_0^1, w_0^2) \in X \times X$.

For the particular case of classical coupled fixed point problems (i.e., $T_1(x,y) := S(x,y)$ and $T_2(x,y) := S(y,x)$, where $S : X \times X \to X$ is a given operator) we get the following generalization of the Gnana Bhaskar-Lakshmikantham theorem in [48].

Theorem 3.2.3 *Let* $(X, d, \leq)$ *be an ordered and complete metric space and let* $S : X \times X \to X$ *be an operator. We suppose:*

(i) for each $z = (x,y), w = (u,v) \in X \times X$ *which are not comparable with respect to the partial ordering* $\preceq$ *on* $X \times X$, *there exists* $t := (t_1, t_2) \in X \times X$ *(which may depend on* (x,y) *and* (u,v)*) such that* t *is comparable (with respect to the partial ordering* $\preceq$*) with both* z *and* w*;*

(ii) for all $(x \geq u$ *and* $y \leq v)$ *or* $(u \geq x$ *and* $v \leq y)$ *we have*

$$\begin{cases} S(x,y) \geq S(u,v) \\ S(y,x) \leq S(v,u) \end{cases} \text{or} \begin{cases} S(u,v) \geq S(x,y) \\ S(v,u) \leq S(y,x) \end{cases};$$

(iii) $S : X \times X \to X$ *is continuous;*
(iv) there exists $z_0 := (z_0^1, z_0^2) \in X \times X$ *such that*

$$\begin{cases} z_0^1 \geq S(z_0^1, z_0^2) \\ z_0^2 \leq S(z_0^2, z_0^1) \end{cases} \text{or} \begin{cases} S(z_0^1, z_0^2) \geq z_0^1 \\ S(z_0^2, z_0^1) \leq z_0^2 \end{cases};$$

(v) there exist $k_1, k_2 \in \mathbb{R}_+$ *with* $k_1 + k_2 < 1$ *such that*

$$d(S(x,y), S(u,v)) \leq k_1 d(x,u) + k_2 d(y,v),$$

for all $(x \geq u$ *and* $y \leq v)$ *or* $(u \geq x$ *and* $v \leq y)$*;*
Then there exists a unique element $(x^*, y^*) \in X \times X$ *such that*

$$x^* = S(x^*, y^*) \text{ and } y^* = S(y^*, x^*),$$

and the sequence of the succesive aproximations $(S^n(w_0^1, w_0^2), S^n(w_0^2, w_0^1))$ *converges to* (x^*, y^*) *as* $n \to \infty$*, for all* $w_0 = (w_0^1, w_0^2) \in X \times X$*.*

For some recent fixed point results for mixed monotone operators on ordered metric spaces see R. P. Agarwal, M. A. El-Gebeily and D. O'Regan [4], L. Ćirić, M. Cakić, J. S. Rajović and J. S. Ume [34], T. Gnana Bhaskar and V. Lakshmikantham [48], V. Lakshmikantham and L. Ćirić [77], J. J. Nieto and R. R. López [87], [88].

3.3 Coupled fixed point theorems for multivalued operators

The aim of this section is to present an existence, uniqueness, data dependence and Ulam-Hyers stability result for the coupled fixed point of a pair of multivalued operators on complete metric spaces.

If (X, d) is a metric space and $S : X \times X \to P(X)$ is a multivalued operator, then, by definition, a coupled fixed point for S is a pair $(x^*, y^*) \in X \times X$ satisfying

$$\begin{cases} x^* \in S(x^*, y^*) \\ y^* \in S(y^*, x^*). \end{cases} \tag{3.14}$$

We will consider now the case of multivalued operators.
The main result in this section is an existence, uniqueness, data dependence and Ulam-Hyers stability theorem for the coupled fixed point of a pair of multivalued operators (T_1, T_2). For the proof of our main result, we give the following theorem.

Theorem 3.3.1 *Let* (X, d) *be a complete generalized metric space and let* $T : X \to P_{cl}(X)$ *be a multivalued A-contraction, i.e. there exists* $A \in M_{mm}(\mathbb{R}_+)$ *which converges towards zero as* $n \to \infty$ *and for each* $x, y \in X$ *and each* $u \in T(x)$ *there exists* $v \in T(y)$ *such that* $d(u, v) \leq A \cdot d(x, y)$*. Then* T *is a MWP-operator, i.e.* $FixT \neq \emptyset$ *and for each* $(x, y) \in Graph(T)$ *there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *of succesive approximations for* T *starting from* (x, y) *which converges to a fixed point* x^* *of* T*. Moreover* $d(x, x^*) \leq (I - A)^{-1} d(x, y)$*, for all* $(x, y) \in Graph(T)$*.*

Definition 3.3.2 *Let* (X, d) *generalized metric space and* $F : X \to P(X)$*. The fixed point inclusion*

$$x \in F(x), \ x \in X \tag{3.15}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ *increasing, continuous in o with* $\psi(0) = 0$ *such that for each* $\varepsilon := (\varepsilon_1, ..., \varepsilon_m) > 0$ *and for each* ε*-solution* y^* *of (3.15), i.e.*

$$D_d(y^*, F(y^*)) \leq \varepsilon$$

there exists a solution x^* *of the fixed point inclusion (3.15) such that*

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

In particular, if $\psi(t) = C \cdot t$, for each $t \in \mathbb{R}^m_+$ (where $C \in M_{mm}(\mathbb{R}_+)$), then (3.15) is said to be Ulam-Hyers stable.

Theorem 3.3.3 *Let (X, d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multivalued A-contraction with proximinal values. Then, the fixed point inclusion (3.15) is Ulam-Hyers stable.*

Theorem 3.3.4 *Let (X, d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multivalued A-contraction such that there exists $x^* \in X$ with $T(x^*) = \{x^*\}$. Then the fixed point inclusion (3.15) is Ulam-Hyers stable.*

Let (X, d) be a metric space. We will focus our attention to the following system of operatorial inclusions:

$$\begin{cases} x \in T_1(x, y) \\ y \in T_2(x, y) \end{cases} \tag{3.16}$$

where $T_1, T_2 : X \times X \to P(X)$ are two given multivalued operators.

By definition, a solution $(x, y) \in X \times X$ of the above system is called a coupled fixed point for the pair (T_1, T_2).

Definition 3.3.5 *Let (X, d) be a metric space and let $T_1, T_2 : X \times X \to P(X)$ are two multivalued operators. Then the operatorial inclusions system (3.16) is said to be Ulam-Hyers stable if there exist $c_1, c_2, c_3, c_4 > 0$ such that for each $\varepsilon_1, \varepsilon_2 > 0$ and each pair $(u^*, v^*) \in X \times X$ which satisfies the relations*

$$\begin{aligned} &d(u^*, w) \leq \varepsilon_1 \text{ , for all } w \in T_1(u^*, v^*) \\ &d(v^*, z) \leq \varepsilon_2 \text{ , for all } z \in T_2(u^*, v^*) \end{aligned} \tag{3.17}$$

there exists a solution $(x^*, y^*) \in X \times X$ of (3.16) such that

$$\begin{aligned} &d(u^*, x^*) \leq c_1\varepsilon_1 + c_2\varepsilon_2 \\ &d(v^*, y^*) \leq c_3\varepsilon_1 + c_4\varepsilon_2. \end{aligned} \tag{3.18}$$

Now we give our main result in this section.

Theorem 3.3.6 *Let (X, d) be a complete metric space and let $T_1, T_2 : X \times X \to P_{cl}(X)$ be two multivalued operators. Suppose that T_1 has proximinal values with respect to the first variable and T_2 with respect to the second one. For each $(x, y), (u, v) \in X \times X$ and each $z_1 \in T_1(x, y)$, $z_2 \in T_2(x, y)$ there exist $w_1 \in T_1(u, v)$, $w_2 \in T_2(u, v)$ satisfying*

$$\begin{aligned} d(z_1, w_1) &\leq k_1 d(x, u) + k_2 d(y, v) \\ d(z_2, w_2) &\leq k_3 d(x, u) + k_4 d(y, v), \end{aligned}$$

where $k_i \in \mathbb{R}_+$, for $i \in \{1, 2, 3, 4\}$. We suppose that $A := \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ converges to zero. Then:

(i) there exists $(x^*, y^*) \in X \times X$ a solution for (3.16).

(ii) the operatorial system (3.16) is Ulam-Hyers stable.

Theorem 3.3.7 *Let (X, d) be a complete metric space and let $T_1, T_2 : X \times X \to P_{cl}(X)$ be two multivalued operators. Suppose there exist $x^*, y^* \in X$ such that*

$$T_1(x^*, y^*) = \{x^*\}, \quad T_2(x^*, y^*) = \{y^*\}. \tag{3.19}$$

For each $(x, y), (u, v) \in X \times X$ and each $z_1 \in T_1(x, y)$, $z_2 \in T_2(x, y)$ there exist $w_1 \in T_1(u, v)$, $w_2 \in T_2(u, v)$ satisfying

$$\begin{aligned} d(z_1, w_1) &\leq k_1 d(x, u) + k_2 d(y, v) \\ d(z_2, w_2) &\leq k_3 d(x, u) + k_4 d(y, v), \end{aligned}$$

where $k_i \in \mathbb{R}_+$, for $i \in \{1, 2, 3, 4\}$. We suppose that $A := \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ converges to zero. Then:

(i) there exists $(x^*, y^*) \in X \times X$ a solution for (3.16),

(ii) the operatorial system (3.16) is Ulam-Hyers stable.

Chapter 4

Applications

In this chapter we present some applications to periodic boundary value systems and to systems of functional-integral equations. These applications of the coupled fixed point results for contractive type singlevalued and multivalued operators on spaces endowed with vector-valued metrics were established in order to validate our previous investigations.

In order to develop this chapter the following references were used: A. C. M. Ran, M. C. B. Reurings [118]; T. Gnana Bhaskar and V. Lakshmikantham [48], J. J. Nieto and R. R. López [88]; V. Lakshmikantham and L. Ćirić [77]; V. Berinde, M. Borcut [16]; W. Sintunavarat, P. Kumam, and Y. J. Cho [143]; M. D. Rus [136].

4.1 Application to a periodic boundary value problem

In this section we study the existence, uniqueness and Ulam-Hyers stability of a solution to a periodic boundary value problem as an application of the coupled fixed point Theorem 3.1.2 presented in Chapter 3. The approach is based on the application presented in T. Gnana Bhaskar and V. Lakshmikantham [48].

For more applications, see for example: V. Lakshmikantham and L. Ćirić [77], W. Sintunavarat, P. Kumam, and Y. J. Cho [143], J. J. Nieto and R. R. López [88], A. C. M. Ran, M. C. B. Reurings [118].

We consider now the following periodic boundary value problem

$$\begin{cases} u' = f(t,u) + g(t,v) \\ v' = f(t,v) + g(t,u) \\ u(0) = u(T) \\ v(0) = v(T) \end{cases} \tag{4.1}$$

assuming that f, g are continuous functions and satisfy certain assumptions, which we will present later.

In general, a problem of this type does not have solution. Take, for example the following problem

$$\begin{cases} x'(t) = 1 \\ x(0) = x(T). \end{cases}$$

As a consequence, we can point out that the periodic boundary value system (4.1) has in general no solutions.

Then, in order to obtain existence results we rewrite the system (4.1) in the following form and we study the existence of its solution:

$$\begin{cases} u' + \lambda_1 u - \lambda_2 v = f(t,u) + g(t,v) + \lambda_1 u - \lambda_2 v, \\ v' + \lambda_1 v - \lambda_2 u = f(t,v) + g(t,u) + \lambda_1 v - \lambda_2 u, \end{cases} \tag{4.2}$$

together with the periodicity conditions,

$$\begin{cases} u(0) = u(T), \\ v(0) = v(T). \end{cases} \tag{4.3}$$

This problem is equivalent to the following integral equations system:

$$\begin{cases} u(t) = \int_0^T G_1(t,s)[f(s,u) + g(s,v) + \lambda_1 u - \lambda_2 v] \\ \quad +G_2(t,s)[f(s,v) + g(s,u) + \lambda_1 v - \lambda_2 u]ds \\ \\ v(t) = \int_0^T G_1(t,s)[f(s,v) + g(s,u) + \lambda_1 v - \lambda_2 u] \\ \quad +G_2(t,s)[f(s,u) + g(s,v) + \lambda_1 u - \lambda_2 v]ds \end{cases}$$

where

$$G_1(t,s) = \begin{cases} \frac{1}{2}\left[\frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}} + \frac{e^{\sigma_2(t-s)}}{1-e^{\sigma_2 T}}\right] & 0 \le s < t \le T \\ \\ \frac{1}{2}\left[\frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}} + \frac{e^{\sigma_2(t+T-s)}}{1-e^{\sigma_2 T}}\right] & 0 \le t < s \le T \end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{1}{2}\left[\frac{e^{\sigma_2(t-s)}}{1-e^{\sigma_2 T}} - \frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}}\right] & 0 \le s < t \le T \\ \\ \frac{1}{2}\left[\frac{e^{\sigma_2(t+T-s)}}{1-e^{\sigma_2 T}} - \frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}}\right] & 0 \le t < s \le T \end{cases}$$

Here, $\sigma_1 = -(\lambda_1 + \lambda_2)$ and $\sigma_2 = (\lambda_2 - \lambda_1)$.
We need to guarantee that $G_1(t,s) \ge 0$, $0 \le t,\ s \le T$, and $G_2(t,s) \le 0$, $0 \le t, s \le T$, by choosing λ_1, λ_2 suitably.
We make the following appropriate assumption:

Assumption There exist $\lambda_1 > 0$, $\lambda_2 > 0$ and $\mu_1 > 0$, $\mu_2 > 0$, such that for all $u, v \in \mathbb{R}$, $v \le u$,

$$0 \le (f(t,u) + \lambda_1 u) - (f(t,v) + \lambda_1 v) \le \mu_1(u - v) \tag{4.4}$$

$$-\mu_2(u - v) \le (g(t,u) - \lambda_2 u) - (g(t,v) - \lambda_2 v) \le 0, \tag{4.5}$$

where $S := \begin{pmatrix} \frac{\mu_1}{\lambda_1+\lambda_2} & \frac{\mu_2}{\lambda_1+\lambda_2} \\ \frac{\mu_2}{\lambda_1+\lambda_2} & \frac{\mu_1}{\lambda_1+\lambda_2} \end{pmatrix}$ is a matrix convergent to zero.

The following lemma answers to the above problem, regarding guaranteeing the conditions for $G_1(t,s)$ and $G_2(t,s)$.

Lemma 4.1.1 *(T. Gnana Bhaskar and V. Lakshmikantham [48]) If*

$$\begin{aligned} \ln\left(\frac{2e-1}{e}\right) &\leq (\lambda_2-\lambda_1)T \qquad (4.6)\\ (\lambda_1+\lambda_2)T &\leq 1 \qquad (4.7)\end{aligned}$$

then $G_1(t,s)\geq 0$ *for* $0\leq t,s\leq T$, *and* $G_2(t,s)\leq 0$ *for* $0\leq t,s\leq T$.

Let $X=C(I,\mathbb{R})$ be the metric space of all continuous functions $u:I\to\mathbb{R}$, endowed with the metric $d(u,v)=\sup\limits_{t\in I}|u(t)-v(t)|$, for $u,v\in X$.

For $x,y,u,v\in X$, we also denote $\widetilde{d}((x,y),(u,v)):=\left(\begin{array}{c} d(x,u)\\ d(y,v)\end{array}\right)$.

Let us define $A:X\times X\to X$ for $t\in I$, by

$$\begin{aligned} A(u,v)(t) &= \int_0^T G_1(t,s)[f(s,u)+g(s,v)+\lambda_1u-\lambda_2v]\\ &\quad +G_2(t,s)[f(s,v)+g(s,u)+\lambda_1v-\lambda_2u]ds.\end{aligned}$$

Note that if $(u,v)\in X\times X$ is a coupled fixed point of A, then we have

$$u(t)=A(u,v)(t) \text{ and } v(t)=A(v,u)(t), \text{ for all } t\in I.$$

Thus, (u,v) is a solution of (4.2)- (4.3).

For the proof of our main result we need the following notion.

Definition 4.1.2 *The system*

$$\begin{cases} u(t)=\displaystyle\int_0^T G_1(t,s)[f(s,u)+g(s,v)+\lambda_1u-\lambda_2v]\\ \qquad +G_2(t,s)[f(s,v)+g(s,u)+\lambda_1v-\lambda_2u]ds\\ v(t)=\displaystyle\int_0^T G_1(t,s)[f(s,v)+g(s,u)+\lambda_1v-\lambda_2u]\\ \qquad +G_2(t,s)[f(s,u)+g(s,v)+\lambda_1u-\lambda_2v]ds\end{cases} \qquad (4.8)$$

is said to be Ulam-Hyers stable if there exist $c_1,c_2>0$ *such that for each* $\varepsilon_1,\varepsilon_2>0$ *and each solution* (x^*,y^*) *of the following inequations system*

$$\begin{cases} \left| x^*(t)-\displaystyle\int_0^T G_1(t,s)[f(s,x^*)+g(s,y^*)+\lambda_1x^*-\lambda_2y^*]\right.\\ \qquad \left.+G_2(t,s)[f(s,y^*)+g(s,x^*)+\lambda_1y^*-\lambda_2x^*]ds\right|\leq\varepsilon_1\\ \left| y^*(t)-\displaystyle\int_0^T G_1(t,s)[f(s,y^*)+g(s,x^*)+\lambda_1y^*-\lambda_2x^*]\right.\\ \qquad \left.+G_2(t,s)[f(s,x^*)+g(s,y^*)+\lambda_1x^*-\lambda_2y^*]ds\right|\leq\varepsilon_2\end{cases} \qquad (4.9)$$

there exists a solution (u^*,v^*) *of (4.8) such that*

$$\begin{aligned} |u^*(t)-x^*(t)| &\leq c_1\varepsilon_1+c_2\varepsilon_2\\ |v^*(t)-y^*(t)| &\leq c_3\varepsilon_1+c_4\varepsilon_2.\end{aligned}$$

Our main result is the following existence, uniqueness and Ulam-Hyers stability of a solution to a periodic boundary value problem.

Theorem 4.1.3 *Consider the problem (4.1) with $f, g \in C(I \times \mathbb{R}, \mathbb{R})$ and suppose that the Assumption is satisfied. If (4.6) and (4.7) are fulfilled, then:*

(i) there exists a unique solution (u^*, v^*) of the periodic boundary value problem (4.1).

(ii) let $f_1, g_1 \in C(I \times \mathbb{R}, \mathbb{R})$ such that, there exist $\eta_1, \eta_2 > 0$ with

$$\begin{cases} |f(t,u) - f_1(t,u)| \leq \eta_1 \\ |g(t,u) - g_1(t,u)| \leq \eta_2 \end{cases}$$

for all $(t, u) \in I \times \mathbb{R}$. Let $(a^*, b^*) \in X \times X$ be a solution of the problem (4.1) with f replaced by f_1 and g replaced by g_1. Then

$$\widetilde{d}((u^*, v^*), (a^*, b^*)) = \begin{pmatrix} d(a^*, u^*) \\ d(b^*, v^*) \end{pmatrix} \leq (I - S)^{-1}\eta,$$

where $\eta := \begin{pmatrix} (\eta_1 + \eta_2)\frac{1}{\lambda_2 - \lambda_1} \\ (\eta_1 + \eta_2)\frac{1}{\lambda_2 - \lambda_1} \end{pmatrix}$.

(iii) the system (4.8) is Ulam-Hyers stable.

4.2 Applications to systems of differential and functional-integral equations

In this section we provide some applications to first-oder differential systems with periodic boundary value conditions, considering also the mixed monotone property and we present some applications to systems of functional-integral equations. In the first part we investigate the existence and uniqueness of a solution to a periodic boundary value problem, as an application of the coupled fixed point Theorem 3.2.2 for mixed monotone singlevalued operators. In the second part of this section we present two applications of Theorem 3.2.3, which are existence and uniqueness results for systems of functional-integral equations, which appear in traffic flow models. The last application which is presented in this section is an existence and uniqueness result for a system of first-order boundary value problem with multivalued operators.

We study now the existence and uniqueness of the solution to a periodic boundary value system, as an application to coupled fixed point Theorem 3.2.2 for mixed monotone type singlevalued operators, in the framework of partially ordered metric space.

We denote the partial order relation by $\preceq$ on $C(I) \times C(I)$. If we consider $z := (x, y)$ and $w := (u, w)$ two arbitrary elements of $C(I) \times C(I)$, then by definition

$$z \preceq w \text{ if and only if } (x \geq u \text{ and } y \leq v),$$

where $x \geq u$ means that $x(t) \geq u(t)$, for all $t \in I$.

We consider the first-order periodic boundary value system:

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) \\ \quad x(0) = x(T) \\ \quad y(0) = y(T) \end{cases} \quad \text{for all } t \in I := [0, T] \tag{4.10}$$

where $T > 0$ and $f_1, f_2 : I \times \mathbb{R}^2 \to \mathbb{R}$ under the assumptions:
(a1) f_1, f_2 are continuous;
(a2) there exist $\lambda > 0$ and $\mu_1, \mu_2, \mu_3, \mu_4 > 0$ such that

$$0 \leq f_1(t, x, y) - f_1(t, u, v) + \lambda(x - u) \leq \lambda[\mu_1(x - u) + \mu_2(y - v)]$$

$$-\lambda \left[\mu_3(x - u) + \mu_4(y - v)\right] \leq f_2(t, x, y) - f_2(t, u, v) + \lambda(y - v) \leq 0,$$

for all $t \in I$ and $x, y, u, v \in \mathbb{R}$.
(a3) for each $z := (x, y), w := (u, w) \in C(I) \times C(I)$ which are not comparable with respect to the partial ordering $\preceq$ on $C(I)\times C(I)$ there exists $p := (p_1, p_2) \in C(I)\times C(I)$ such that p is comparable (with respect to the partial ordering $\preceq$) with both z and w, i.e.

$$\begin{aligned} ((x &\geq p_1 \text{ and } y \leq p_2) \text{ or } (x \leq p_1 \text{ and } y \leq p_2)) \text{ and} \\ (u &\geq p_1 \text{ and } v \leq p_2) \text{ or } (u \leq p_1 \text{ and } v \leq p_2)). \end{aligned}$$

(a4) for all ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$) we have

$$\begin{cases} f_1(t, x, y) \geq f_1(t, u, v) \\ f_2(t, x, y) \leq f_1(t, u, v) \end{cases} \text{ or } \begin{cases} f_1(t, u, v) \geq f_1(t, x, y) \\ f_2(t, u, v) \leq f_2(t, x, y) \end{cases}.$$

(a5) there exists $z_0 := (z_0^1, z_0^2) \in C(I) \times C(I)$ such that the following relations hold:

(a5')

$$\begin{cases} z_0^1(t) \geq f_1(t, z_0^1(t), z_0^2(t)) \\ z_0^2(t) \geq f_2(t, z_0^1(t), z_0^2(t)) \end{cases} \text{ or } \begin{cases} f_1(t, z_0^1(t), z_0^2(t)) \geq z_0^1(t) \\ f_2(t, z_0^1(t), z_0^2(t)) \leq z_0^2(t) \end{cases}$$

(a5")

$$\begin{aligned} (1 + \lambda) \int_0^T G_\lambda(t, s) z_0^1(s) ds &\geq z_0^1(t) \\ (1 + \lambda) \int_0^T G_\lambda(t, s) z_0^2(s) ds &\leq z_0^2(t) \end{aligned}$$

for all $t \in I$.
(a6) the matrix $S := \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ is convergent to zero.

Lemma 4.2.1 *Let $x \in C^1(I)$ be such that it satisfies the periodic boundary value problem*

$$\begin{cases} x'(t) = h(t) \\ x(0) = x(T) \end{cases} \quad t \in I,$$

with $h \in C(I)$. Then for some $\lambda \neq 0$ the above problem is equivalent to

$$x(t) = \int_0^T G_\lambda(t,s)(h(s) + \lambda x(s))ds, \text{ for all } t \in I,$$

where

$$G_\lambda(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & \text{if } 0 \leq s < t \leq T \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & \text{if } 0 \leq t < s \leq T \end{cases}.$$

The problem (4.10) is equivalent to the coupled fixed point problem $\begin{cases} x = F_1(x,y) \\ y = F_2(x,y) \end{cases}$, with $X = C(I)$ and $F_1, F_2 : X^2 \to X$ defined by

$$\begin{aligned} F_1(x,y)(t) &= \int_0^T G_\lambda(t,s)\left[f_1(s,x(s),y(s)) + \lambda x(s)\right] ds \\ F_2(x,y)(t) &= \int_0^T G_\lambda(t,s)\left[f_2(s,x(s),y(s)) + \lambda y(s)\right] ds \end{aligned}$$

We consider the complete metric d induced by the sup-norm on X,

$$d(x,y) = \sup_{t\in I} |x(t) - y(t)|, \text{ for } x,y \in C(I).$$

For $x,y,u,v \in X$, we also denote $\widetilde{d}((x,y),(u,v)) := \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix}$.
Note that if $(x,y) \in X \times X$ is a coupled fixed point of F, then we have

$$x(t) = F_1(x,y)(t) \text{ and } y(t) = F_2(x,y)(t) \text{ for all } t \in I,$$

where $F(x,y)(t) := (F_1(x,y)(t), F_2(x,y)(t))$.

Theorem 4.2.2 *Consider the problem (4.10) under the assumptions (a1)-(a6). Then there exists a unique solution (x^*, y^*) of the first-order boundary value problem (4.10).*

As an application of Theorem 3.2.3, we present now an existence and uniqueness result for a system of functional-integral equations which appears in some traffic flow models.

$$\begin{cases} x(t) = f\left(t, x(t), \int_0^T k(t,s,x(s),y(s))ds\right) \\ y(t) = f\left(t, y(t), \int_0^T k(t,s,x(s),y(s))ds\right) \end{cases} \tag{4.11}$$

By a solution of system (4.11) we understand a couple $(x,y) \in C[0,T] \times C[0,T]$, which satisfies the system for all $t \in [0,T]$.

As before, we consider on $X := C[0,T]$ the following partial ordering relation

$$x \leq_C y \text{ if and only if } x(t) \leq y(t), \text{ for all } t \in [0,T]$$

and the max-norm

$$\|x\|_C := \max_{t\in[0,T]} |x(t)|.$$

Notice that, as before, the partial ordering relation $\leq_C$ generates on $X \times X$ a partial ordering $\preceq_C$.
If we define

$$S : X \times X \to X, \ \ (x, y) \longmapsto S(x, y), \ \text{where } S(x, y)(t) := f(t, x(t), \int_0^T k(t, s, x(s), y(s))ds),$$

then, the above system can be represented as a coupled fixed point problem:

$$\begin{cases} x = S(x, y) \\ y = S(y, x) \end{cases}$$

An existence and uniqueness result for the system (4.11) is the following theorem.

Theorem 4.2.3 *Let $k : [0, T] \times [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two continuous mappings. We suppose:*

(i) there exists $z_0 := (z_0^1, z_0^2) \in C[0, T] \times C[0, T]$ such that

$$\begin{cases} z_0^1(t) \geq f(t, z_0^1(t), \int_0^T k(t, s, z_0^1(t), z_0^2(t))ds) \\ z_0^2(t) \leq f(t, z_0^2(t), \int_0^T k(t, s, z_0^2(t), z_0^1(t))ds) \end{cases} \ \ or \ \ \begin{cases} z_0^1(t) \leq f(t, z_0^1(t), \int_0^T k(t, s, z_0^1(t), z_0^2(t))ds) \\ z_0^2(t) \geq f(t, z_0^2(t), \int_0^T k(t, s, z_0^2(t), z_0^1(t))ds) \end{cases};$$

(ii) (a) $f(t, \cdot, z)$ is increasing, for all $t \in [0, T]$, $z \in \mathbb{R}$ and $k(t, s, \cdot, z)$ is increasing, $k(t, s, w, \cdot)$ is decreasing and $f(t, w, \cdot)$ is increasing, for all $t, s \in [0, T]$, $w, z \in \mathbb{R}$

or

(b) $f(t, \cdot, z)$ is decreasing, for all $t \in [0, T]$, $z \in \mathbb{R}$ and $k(t, s, \cdot, z)$ is decreasing, $k(t, s, w, \cdot)$ is increasing and $f(t, w, \cdot)$ is decreasing, for all $t, s \in [0, T]$, $w, z \in \mathbb{R}$;

(iii) there exist $k_1, k_2 \in \mathbb{R}_+$ such that

$$|f(t, w_1, z_1) - f(t, w_2, z_2)| \leq k_1|w_1 - w_2| + k_2|z_1 - z_2|,$$

for all $t \in [0, T]$ and $w_1, w_2, z_1, z_2 \in \mathbb{R}$;

(iv) there exist $\alpha, \beta \in \mathbb{R}_+$ such that, for all $t, s \in [0, T]$ and $w_1, w_2, z_1, z_2 \in \mathbb{R}$ we have

$$|k(t, s, w_1, z_1) - k(t, s, w_2, z_2)| \leq \alpha|w_1 - w_2| + \beta|z_1 - z_2|;$$

(v) $k_1 + k_2T(\alpha + \beta) < 1$.

Then, there exists a unique solution (x^*, y^*) of the system (4.11).

We present now another application of Theorem 3.2.3, an existence and uniqueness result for a system of functional-integral equations. We apply a coupled fixed point theorem to an equivalent system of operatorial equations

$$\begin{cases} x(t) = f\left(t, x(t), \int_0^t k(t, s, x(s), y(s))ds\right) \\ y(t) = f\left(t, y(t), \int_0^t k(t, s, x(s), y(s))ds\right) \end{cases} \tag{4.12}$$

By a solution of system (4.12) we understand a couple $(x, y) \in C[0, T] \times C[0, T]$, which satisfies the system for all $t \in [0, T]$.
We consider on $X := C[0, T]$ the following partial ordering relation

$$x \leq_B y \text{ if and only if } x(t) \leq y(t), \ \text{for all } t \in [0, T]$$

and the Bielecki norm

$$\|x\|_B := \max_{t\in[0,T]} \left(|x(t)|e^{-\tau t}\right),$$

for some suitable $\tau > 0$.

Notice that, the partial ordering relation $\leq_B$ generates on $X \times X$ a partial ordering $\preceq_B$.
If we define the operator

$$S : X \times X \to X, \quad (x, y) \longmapsto S(x, y), \text{ where } S(x, y)(t) := f(t, x(t), \int_0^t k(t, s, x(s), y(s))ds),$$

then, the above system can be represented as a coupled fixed point problem:

$$\begin{cases} x = S(x, y) \\ y = S(y, x) \end{cases}$$

An existence and uniqueness result for the system (4.12) is the following theorem.

Theorem 4.2.4 *Let $k : [0, T] \times [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two continuous mappings. We suppose:*

(i) there exists $z_0 := (z_0^1, z_0^2) \in C[0, T] \times C[0, T]$ such that

$$\begin{cases} z_0^1(t) \geq f(t, z_0^1(t), \int_0^t k(t, s, z_0^1(t), z_0^2(t))ds) \\ z_0^2(t) \leq f(t, z_0^2(t), \int_0^t k(t, s, z_0^2(t), z_0^1(t))ds) \end{cases} \quad or \quad \begin{cases} z_0^1(t) \leq f(t, z_0^1(t), \int_0^t k(t, s, z_0^1(t), z_0^2(t))ds) \\ z_0^2(t) \geq f(t, z_0^2(t), \int_0^t k(t, s, z_0^2(t), z_0^1(t))ds) \end{cases};$$

(ii) (a) $f(t, \cdot, z)$ is increasing, for all $t \in [0, T]$, $z \in \mathbb{R}$ and $k(t, s, \cdot, z)$ is increasing, $k(t, s, w, \cdot)$ is decreasing and $f(t, w, \cdot)$ is increasing, for all $t, s \in [0, T]$, $w, z \in \mathbb{R}$

or

(b) $f(t, \cdot, z)$ is decreasing, for all $t \in [0, T]$, $z \in \mathbb{R}$ and $k(t, s, \cdot, z)$ is decreasing, $k(t, s, w, \cdot)$ is increasing and $f(t, w, \cdot)$ is decreasing, for all $t, s \in [0, T]$, $w, z \in \mathbb{R}$;

(iii) there exist $k_1, k_2 \in \mathbb{R}_+$ such that

$$|f(t, w_1, z_1) - f(t, w_2, z_2)| \leq k_1|w_1 - w_2| + k_2|z_1 - z_2|,$$

for all $t \in [0, T]$ and $w_1, w_2, z_1, z_2 \in \mathbb{R}$;

(iv) there exist $\alpha, \beta \in \mathbb{R}_+$ such that, for all $t, s \in [0, T]$ and $w_1, w_2, z_1, z_2 \in \mathbb{R}$ we have

$$|k(t, s, w_1, z_1) - k(t, s, w_2, z_2)| \leq \alpha|w_1 - w_2| + \beta|z_1 - z_2|;$$

(v) $k_1 < 1$.

Then, there exists a unique solution (x^*, y^*) of the system (4.12).

As the consequence, of the above results we can obtain an existence and uniqueness result for a system of first-order boundary value problem with multivalued operators.

In what follows, we will consider a basic selection theorem. For other results regarding continuous selections for lower semicontinuous and upper semicontinuous multi-function with convex values see A. Petruşel [103].

Definition 4.2.5 *Let X, Y be nonempty sets and $F : X \to P(Y)$. Then the singlevalued operator $f : X \to Y$ is called a selection of F if and only if $f(x) \in F(x)$, for each $x \in X$.*

Theorem 4.2.6 (Michael's Selection Theorem) *Let (X, d) be a metric space, Y be a Ba-nach space and $F : X \to P_{cl,cv}(\mathbb{R})$ be lower semicontinuous on X. Then there exists $f : X \to Y$ a continuous selection of F.*

We consider the following first-order periodic boundary value system:

$$\begin{cases} x'(t) \in F_1(t, x(t), y(t)) \\ y'(t) \in F_2(t, x(t), y(t)) \\ \quad x(0) = x(T) \\ \quad y(0) = y(T) \end{cases} \text{for all } t \in I := [0, T], \tag{4.13}$$

where $T > 0$ and $F_1, F_2 : [0, T] \times \mathbb{R}^2 \to P_{cl,cv}(\mathbb{R})$.
For all $t \in I$ and $x, y \in C^1(I)$ we denote

$$\begin{aligned} G_1 &: [0, T] \to P_{cl,cv}(\mathbb{R}),\ G_1(t) := F_1(t, x(t), y(t)) \text{ and} \\ G_2 &: [0, T] \to P_{cl,cv}(\mathbb{R}),\ G_2(t) := F_2(t, x(t), y(t)). \end{aligned}$$

Remark 4.2.7 *If G_1 and G_2 are lower semicontinuous, then G_1, G_2 have (by Michael's Selection Theorem) continuous selections.*

Thus there exist

$$\begin{aligned} g_{xy}^{(1)} &: [0, T] \to \mathbb{R}, \\ g_{xy}^{(2)} &: [0, T] \to \mathbb{R}, \end{aligned}$$

continuous selections for G_1 and G_2 (i.e. $g_{xy}^{(1)}(t) \in G_1(t) = F_1(t, x(t), y(t))$, $g_{xy}^{(2)}(t) \in G_2(t) = F_2(t, x(t), y(t))$).
We consider now the periodic boundary value problem:

$$\begin{cases} x'(t) = g_{xy}^{(1)}(t) \\ y'(t) = g_{xy}^{(2)}(t) \\ x(0) = x(T) \\ y(0) = y(T) \end{cases} \text{for all } t \in I := [0, T]. \tag{4.14}$$

Remark 4.2.8 *Any solution for (4.14) is a solution for (4.13).*

Remark 4.2.9 *The periodic boundary value problem (4.14) is equivalent to the following system*

$$\begin{cases} x(t) = \int_0^T G_\lambda(t, s) \left[g_{xy}^{(1)}(s) + \lambda x(s) \right] ds \\ \\ y(t) = \int_0^T G_\lambda(t, s) \left[g_{xy}^{(2)}(s) + \lambda y(s) \right] ds \end{cases}$$

where

$$G_\lambda(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & \text{if } 0 \le s < t \le T \\ \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & \text{if } 0 \le t < s \le T \end{cases}$$

As before, the problem (4.14) is equivalent to the coupled fixed point problem $\begin{cases} x = G^{(1)}_{xy} \\ y = G^{(2)}_{xy} \end{cases}$, where $X = C(I)$ and $G^{(1)}_{xy}, G^{(2)}_{xy} : X^2 \to X$ are defined by

$$\begin{aligned} G^{(1)}_{xy}(t) &= \int_0^T G_\lambda(t,s)\left[g^{(1)}_{xy}(s) + \lambda x(s)\right] ds \\ G^{(2)}_{xy}(t) &= \int_0^T G_\lambda(t,s)\left[g^{(2)}_{xy}(s) + \lambda y(s)\right] ds. \end{aligned}$$

We consider the complete metric d induced by the sup-norm on X,

$$d(x,y) = \sup_{t\in I} |x(t) - y(t)|\,, \text{ for } x, y \in C(I).$$

For $x, y, u, v \in X$, we also denote $\widetilde{d}((x,y),(u,v)) := \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix}$.

Theorem 4.2.10 *We consider the problem (4.13) and we suppose that:*
(i) $G_1 : [0,T] \to P_{cl,cv}(\mathbb{R})$, $G_1(t) := F_1(t, x(t), y(t))$ is lower semicontinuous, $G_2 : [0,T] \to P_{cl,cv}(\mathbb{R})$, $G_2(t) := F_2(t, x(t), y(t))$ is lower semicontinuous, for each $x, y \in C[0,T]$,
(ii) for any $g^{(1)}_{xy}$ and $g^{(2)}_{xy}$ continuous selection for G_1, respectively G_2, there exist $\lambda > 0$ and $\mu_1, \mu_2, \mu_3, \mu_4 > 0$ such that

$$0 \;\le\; g^{(1)}_{xy}(t) - g^{(1)}_{uv}(s) + \lambda(x-u) \le \lambda[\mu_1(x-u) + \mu_2(y-v)]$$

$$0 \;\le\; g^{(2)}_{xy}(t) - g^{(2)}_{uv}(t) + \lambda(y-v) \le \lambda\left[\mu_3(x-u) + \mu_4(y-v)\right],$$

(iii) the matrix $S := \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ is convergent to zero,
Then there exists a unique solution of the first-order boundary value problem (4.13).

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