UNIVERSITY BABES-BOLYAI OF CLUJ-NAPOCA, ROMANIA
UNIVERSITY OF PERPIGNAN VIA DOMITIA, FRANCE

THESIS

To obtain the degree of
DOCTOR OF THE UNIVERSITY BABES-BOLYAI
&
DOCTOR OF THE UNIVERSITY OF PERPIGNAN VIA DOMITIA

Discipline: Applied Mathematics

defended
by

Anca FARCAS

15th November 2013

Positive Linear Operators and History-Dependent Operators in Contact Mechanics

Scientific Advisors: Professor Octavian AGRATINI
Professor Mircea SOFONEA

The public defence will take place on the 15th of November 2013 in Tiberiu Popoviciu room.

President of the Committee: Professor Radu Precup

Referees:

Associate Professor Mikaël Barboteu (University of Perpignan)

Professor Mircea Ivan (Technical University Cluj-Napoca)

Professor Sanda Ţigoiu (University of Bucharest)

Scientific Advisors:

Professor Octavian Agratini

Professor Mircea Sofonea

Acknowledgements

This thesis was realised jointly at the Babeş-Bolyai University, Cluj-Napoca (Romania) and the University of Perpignan Via Domitia (France), under the supervision of Professors O. Agratini (Cluj-Napoca) and M. Sofonea (Perpignan). The author addresses her depth gratitude for their support in writing this manuscript. Special thanks are also adressed to all members of the *Laboratory of Mathematics and Physics* of the University of Perpignan and in the same time to all members of the *Department of Mathematics* of Babeş-Bolyai University. The author is also grateful to the members of the *Department of Mathematics and Computer Science* of the University of Medicine and Pharmacy in Cluj- Napoca and to the colleagues she worked with, both in Perpignan and Cluj-Napoca. Special thanks are extended to Joëlle Sulian who prepared all the necessary documents throughout my stay in France and to all the loved ones, for their great moral support during the preparation of this thesis.

The author acknowledges that the financial support of this thesis was provided by **The Sectoral Operational Programme Human Resources Development, Contract POSDRU 105/1.5/S/76841- "Innovative Doctoral Studies in a Knowledge Based Society" (Babeş-Bolyai University)**.

Cluj-Napoca November 2013

Contents

Introduction

The classical approximation theory represents an older topic of mathematical analysis which still remains an area with active implications in the present. One of the most modern chapters of the approximation theory refers to the function approximation by positive linear operators, which represents our first interest in this work. Besides the aspects concerning approximation theory, the second interest in this thesis is the modelling and analysis of different problems which arise in Contact Mechanics. Contact phenomena involving deformable bodies are often encountered in industry and everyday life and, for this reason, the literature dedicated to this field is extensive, since considerable efforts have been made in their modelling and analysis. For example the interaction between road and tyres, breaking pads with wheels, hip implants, artificial knee joints or the impact analysis of cars represent just few real life problems covered by the theory of Contact Mechanics.

The aim of this thesis is to present some results concerning both positive linear operators and history-dependent operators, respectively. The first part of the thesis is dedicated to the study of the function approximation by positive linear operators while the second part is centered on history-dependent operators and their applications in Contact Mechanics.

The present thesis is structured into two parts and seven chapters which are listed bellow.

Part I contains Chapters 1–2 and represents a brief introduction in the study of function approximation by positive linear operators. More precisely, we present here a study of the various convergence properties for such kind of operators.

Part II contains Chapters 3–6 and it presents background results on modelling of contact problems, as well as new results obtained in the analysis of frictionless and frictional contact problems. More precisely, we study three contact problems for which we obtain existence, uniqueness and convergence results. The common feature of these problems arise in the fact that all of them are governed by history-dependent operators which appear either in the constitutive law or in contact conditions.

A detailed description of the chapters is as follows.

Chapter 1 presents the specific framework for the problems we studied in Chapter 2. In other words, we give representative notions concerning positive linear operators, modules of continuity and different type of convergence. Finally, we present some Voronovskaja-type theorems.

Chapter 2 hosts the main results obtained in the first part of the thesis. In this chapter we emphasize the conditions required for a certain type of positive linear operators in order to fulfil a Korovkin-type theorem. The results in this chapter were published in the papers [46], [47] and [48]

Chapter 3 is devoted to preliminary material used throughout the second part of the manuscript. More specific, in the first part of this chapter we start with a survey of the basic properties of Banach and Hilbert space. Then, we introduce the notion of history-dependent operator, provide some examples and state an existence and uniqueness result for variational inequalities with history-dependent operators. In addition, we emphasize some function spaces we need in the study of the contact problems we approach in the rest of the manuscript. The second part of this chapter represents an introduction to the modelling of the contact problems which are presented in the following chapters. In order to deal with this kind of problems, we present the constitutive laws we use, we describe the contact conditions and, finally, we point out the frictionless and frictional conditions, including the well known Coulomb's law of dry friction.

Chapter 4 is devoted to the study of a quasistatic frictional contact problem in which the material's behavior is modelled with a viscoelastic constitutive law with long memory. The contact is modelled with normal compliance and memory term and the friction is modelled with Coulomb's law of dry friction. For this problem we present both the classical formulation and the variational formulation, respectively. The main result of this chapter is given by Theorem II.4.1 which states the unique weak solvability of the problem. The proof is based on arguments of variational inequalities with history-dependent operators. The second main result of the chapter is Theorem II.4.4. It states the continuous dependence of the solution with respect to the data. The material presented in this chapter made the object of the paper [117].

Chapter 5 deals with a mathematical model which describes the contact between a viscoplastic body and a foundation. In this case the contact is frictionless and is modelled with normal compliance, unilateral constraint and memory term. The novelty in this chapter lies in the contact condition. As in the previous chapter, we obtain a variational formulation of the problem and we state and prove the unique weak solvability of the problem (Theorem II.5.1). The proof of Theorem II.5.1 is based on a Banach fixed point argument combined with arguments on variational inequalities with history-dependent operators. We also give a result which states the existence and uniqueness of the solution of a penalized problem (Theorem II.5.4). This theorem also guarantees that the solution of the penalized problem converges to the solution of the variational problem obtained earlier in this chapter, as the penalization parameter converges to zero. The content of this chapter was written following [49] and [100].

Chapter 6 presents a frictionless contact problem in which, as in Chapter 5, the contact is modelled with normal compliance, unilateral constraint and memory term. The novelty of this chapter arises in the fact that here the material's behavior is modelled with a rate-type constitutive law with internal state variable. As usual, we present the classical and the variational formulation of the problem together with two main results concerning the existence and uniqueness of the solution of the problem and its convergence, respectively. Theorem II.6.1 represents the first main result in this chapter. Its proof is based on arguments on variational inequalities with history-dependent operators. Theorem II.6.4 guarantees the continuous dependence of the solution with respect to the data. This chapter follows our paper [122].

The manuscript ends with a list of references in which various details, comments and complements on the topics related to the material in this thesis can be found.

The original results presented in this thesis have been distributed in Chapter 2, Chapter 4, Chapter 5 and Chapter 6 as follows.

In Section 2.1: Lemma I.2.2 and Theorem I.2.3 that we find in paper [47].

Section 2.2: Lemma I.2.6, Lemma I.2.7, Lemma I.2.8, Lemma I.2.9 and Theorem I.2.10 published in [46].

Section 2.3: Lemma I.2.11, Theorem I.2.12 and Theorem I.2.14 that follows our paper [48].

Section 4.2: Theorem II.4.1, Lemma II.4.2, Lemma II.4.3. Section 4.3: Theorem II.4.4. All these results were published in the paper [117].

Section 5.2: Theorem II.5.1, Lemma II.5.2, Lemma II.5.3 published in [49].

Section 5.3: Theorem II.5.4, Lemma II.5.5, Lemma II.5.6, Lemma II.5.7, Lemma II.5.8, Lemma II.5.9, all published in [100].

Section 6.2 and 6.3: Theorems II.6.1, Lemma II.6.2, Lemma II.6.3 and II.6.4, respectively, which follow our paper [122].

The author's contribution to this thesis is also part of the following scientific papers:

A. Farcaş, An asymptotic formula for Jain's operators, *Studia Univ. Babes - Bolyai, Mathematica*, **57**(4)(2012), 511-517 (MR3034099).

A. Farcaş, On some fuzzy positive and linear operators, Proceedings of the Second International Conference "*Modelling and Development of Intelligent Systems*", September 29-October 02, 2011, Sibiu, Romania, ed. Dana Simian, Lucian Blaga Univ. Press, 45-52.

A. Farcaş, An approximation property of the generalized Jain's operators of two variables (accepted in *Mathematical Sciences & Applications E-Notes (MSAEN)*.

A. Farcaş, F. Pătrulescu and M. Sofonea, A history-dependent contact problem with unilateral constraint, Ann. Acad. Rom. Sci. Ser. Math. Appl. 4, no. **1** (2012), 90-96 (MR2959899).

F. Pătrulescu, **A. Farcaş**, A. Ramadan, On a penalized viscoplastic contact problem with unilateral constraint, *Ann. Univ. Buchar. Math. Ser*, 4 (**LXII**) (2013), 213-227.

M. Sofonea and **A. Farcaş**, Analysis of a History-dependent Frictional Contact Problem, *Applicable Analysis*, DOI: 10.1080/00036811.2013.778981 (IF. 0.849).

M. Sofonea, F. Pătrulescu, **A. Farcaş**, A viscoplastic contact problem with normal compliance, unilateral constraint and memory term, *Applied Mathematics and Optimization*, DOI: 10.1007/s00245- 013-9216-2 (IF 0.859).

The author participated at the following international conferences:

Second International Conference "Modelling and Development of Intelligent Systems", September 29–October 02, 2011, Sibiu, Romania with the paper *On some fuzzy positive and linear operators.*

The Second Conference of PhD Students in Mathematics, 28-30 June, Szeged, Hungary, with the paper *On some modified Szàsz-Mirakjan operators.*

XIme Colloque Franco Roumain de Mathmatiques Appliques, Bucarest, 24-30 Aot 2012, with the paper *Analyse dun problme de contact viscoplastique sans frottement.*

21eme Séminaire Franco-Polonais de Méchaniqueé, 13-15 Juin 2013, Perpignan, France with the paper *A frictional contact problem involving history-dependent operators.*

Keywords and phrases: positive linear operators, Voronovskaja-type theorems, Korovkin-type theorems, variational inequalities, history-

dependent operators, constitutive law, viscoplastic materials, viscoelastic materials, Coulomb's law of dry friction.

Part I

Positive linear operators

1
On positive linear operators

In this chapter we present some important notion concerning positive linear operators, modules of continuity and aspects regarding different types of convergence as well as Voronovskaja-type theorems for positive linear operators. All these notion will be used later in Chapter 2.

1.1 Basic notion

This section is devoted to both, properties of positive linear operators and the description of the summability methods. First of all we introduce the concept of positive linear operator.

1.2 Rate of convergence

In this section we present two types of convergence and we refer to the general framework for the rate of convergence of positive linear operators regarding an asymptotic method of Voronovskaja type.

1.3 Fuzzy aspects on positive linear operators

In this section we collect some basic elements regarding the fuzzy positive linear operators.

2

New results on some classes of positive linear operators

In this chapter we present our results that we obtained for the first part of this thesis. First section aims to present a result concerning the fuzzy version of a class of positive linear operators while in the second section we give an asymptotic formula for a class of positive linear operators defined on an unbounded interval. Finally the last part of this chapter refers to an approximation property for a generalization of the operators presented in the second section. The results presented in this chapter are based on the papers [46], [47] and [48].

2.1 On some fuzzy positive and linear operators

In this section we prove that the fuzzy Bernstein-Stancu operators satisfy the A-statistical version of fuzzy Korovkin theorem and we provide an example that fits this case.

Definition I.2.1 *Let $f \in C([0,1], \mathbb{R}_{\mathcal{F}}),\ m \in \mathbb{N},\ 0 \leq \alpha \leq \beta$. We define*

$$({}^{\mathcal{F}}L_m^{\alpha,\beta} f)(x) = \sum_{k=0}^{m}{}^{*}\, p_{m,k}(x) \odot f\left(\frac{k+\alpha}{m+\beta}\right), \quad x \in [0,1], \qquad \text{(I.2.1)}$$

where $p_{m,k}(x) = \begin{pmatrix} m \\ k \end{pmatrix} x^k (1-x)^{m-k}$.

Here $\mathbb{R}_{\mathcal{F}}$ represents the set of real fuzzy numbers and $\sum^*$ stands for fuzzy summation.

Lemma I.2.2 *The fuzzy Bernstein-Stancu operators defined by* (I.2.1) *are positive and linear operators.*

Theorem I.2.3 *If the sequence* $({}^{\mathcal{F}}L_m^{(\alpha,\beta)}f)_{m\in\mathbb{N}}$ *of operators defined by* (I.2.1) *satisfies the conditions*

$$st_A - \lim_m \|\widetilde{L_m}^{(\alpha,\beta)}(e_i) - e_i\| = 0, \ i = 0, 1, 2, \tag{I.2.2}$$

then

$$st_A - \lim_m D^*({}^{\mathcal{F}}L_m^{(\alpha,\beta)}, f) = 0. \tag{I.2.3}$$

2.2 An asymptotic formula for Jain's operators

In this section we prove a Voronovskaja type result for a class of linear positive operators of discrete type depending on a real parameter.

Lemma I.2.4 ([70]) *For* $0 < \alpha < \infty, \ \ |\beta| < 1,$ *let*

$$\omega_\beta(k,\alpha) = \alpha(\alpha + k\beta)^{k-1}e^{-(\alpha+k\beta)}/k! \ ; \ k \in \mathbb{N}_0. \tag{I.2.4}$$

then

$$\sum_{k=0}^{\infty} \omega_\beta(k,\alpha) = 1. \tag{I.2.5}$$

Lemma I.2.5 ([70]) *Let*

$$S(r,\alpha,\beta) = \sum_{k=0}^{\infty} (\alpha + \beta k)^{k+r-1}e^{-(\alpha+\beta k)}/k!, \ r = 0, 1, 2, ... \tag{I.2.6}$$

and

$$\alpha S(0,\alpha,\beta) = 1. \tag{I.2.7}$$

Then

$$S(r,\alpha,\beta) = \sum_{k=0}^{\infty} \beta^k(\alpha + k\beta)S(r-1, \alpha + k\beta, \beta). \tag{I.2.8}$$

Lemma I.2.6 *Let* S *be the function defined in Lemma I.*2.5*. Then, one has*

(i) $$S(3,\alpha,\beta) = \frac{\alpha^3}{(1-\beta)^3} + \frac{3\alpha\beta^2}{(1-\beta)^4} + \frac{\beta^3 + 2\beta^4}{(1-\beta)^5},$$

(ii) $$S(4,\alpha,\beta) = \frac{\alpha^3}{(1-\beta)^4} + \frac{6\alpha^2\beta^2}{(1-\beta)^5} + \frac{\alpha\beta^3(11\beta+4)}{(1-\beta)^6} + \frac{6\beta^6 + 8\beta^5 + \beta^4}{(1-\beta)^7}.$$

The operator defined by Jain is given by

$$(P_n^{[\beta]} f)(x) = \sum_{k=0}^{\infty} \omega_\beta(k, nx) \cdot f\left(\frac{k}{n}\right), \quad f \in C[0,\infty), \tag{I.2.9}$$

Lemma I.2.7 *The operators defined by* $(I.2.9)$ *verify the following identities.*

(i) $(P_n^{[\beta]} e_3)(x) = \dfrac{x^3}{(1-\beta)^3} + \dfrac{3x^2}{n(1-\beta)^4} - \dfrac{x(6\beta^4 - 6\beta^3 - 2\beta - 1)}{n^2(1-\beta)^5}$.

(ii) $(P_n^{[\beta]} e_4)(x) = \dfrac{x^4}{(1-\beta)^4} + \dfrac{6x^3}{n(1-\beta)^5} - \dfrac{x^2(36\beta^4 - 72\beta^3 + 36\beta^2 - 8\beta - 7)}{n^2(1-\beta)^6} + \dfrac{x(105\beta^5 - 14\beta^4 - 2\beta^3 + 12\beta^2 + 8\beta + 1)}{n^3(1-\beta)^7}$.

Lemma I.2.8 *Let the operator* $P_n^{[\beta_n]}$ *be defined by relation* $(I.2.9)$ *and let* φ_x *be given by*

$$\varphi_x \in C_2[0,\infty),\ \varphi_x(t) = t - x. \tag{I.2.10}$$

Then

(i)
$$(P_n^{[\beta_n]} \varphi_x^3)(x) = \frac{x^3}{(1-\beta_n)^3} - \frac{3x^3}{(1-\beta_n)^2} + \frac{3x^3}{1-\beta_n} - x^3 + \frac{3x^2}{n(1-\beta_n)^4} - \frac{3x^2}{n(1-\beta_n)^3} - \frac{x(6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1)}{n^2(1-\beta_n)^5}.$$

(ii)
$$\begin{aligned}(P_n^{[\beta_n]} \varphi_x^4)(x) = {} & \frac{x^4}{(1-\beta_n)^4} - \frac{4x^4}{(1-\beta_n)^3} + \frac{6x^4}{(1-\beta_n)^2} - \frac{4x^4}{1-\beta_n} + x^4 \\ & + \frac{6x^3}{n(1-\beta_n)^5} - \frac{12x^3}{n(1-\beta_n)^4} + \frac{6x^3}{n(1-\beta_n)^3} \\ & - \frac{x^2(36\beta_n^4 - 72\beta_n^3 + 36\beta_n^2 - 8\beta_n - 7)}{n^2(1-\beta_n)^6} \\ & + \frac{4x^2(6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1)}{n^2(1-\beta_n)^5} \\ & + \frac{x(105\beta_n^5 - 14\beta_n^4 - 2\beta_n^3 + 12\beta_n^2 + 8\beta_n + 1)}{n^3(1-\beta_n)^7}.\end{aligned}$$

Lemma I.2.9 *Let* $P_n^{[\beta_n]}$ *be the Jain operator and let* φ_x *be defined in* $(I.2.8)$. *In addition, if*

$$\lim_{n\to\infty} \beta_n = 0 \tag{I.2.11}$$

holds, then

$$P_n^{[\beta_n]}\varphi_x^4 \leq \frac{12x^3}{n(1-\beta_n)^5} + \frac{24x^2}{n^2(1-\beta_n)^5} + \frac{106x}{n^3(1-\beta_n)^7}.$$

Theorem I.2.10 *Let $f \in C_2([0,\infty))$ and let the operator $P_n^{[\beta_n]}$ be defined as in $(I.2.9)$. If $(I.2.9)$ holds, then*

$$\lim_{n\to\infty} n\left(P_n^{[\beta_n]}(f;x) - f(x)\right) = \frac{x}{2}f''(x), \quad \forall x > 0.$$

2.3 An approximation property of the generalized Jain's operators of two variables

The purpose of this section is to introduce a new class of double positive linear operators which depend on a parameter β. For these operators we prove a Korovkin type theorem and we present some associated convergence properties.

We present a new class of operators namely a generalization of Jain's operators on the nodes $\left(\frac{k_1+\alpha_1}{m+\gamma_1}, \frac{k_2+\alpha_2}{m+\gamma_2}\right)$.

$$^{[\beta]}\mathcal{J}_{m,n}^{\alpha,\gamma} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \omega_\beta^1(k_1, mx)\omega_\beta^2(k_2, ny) f\left(\frac{k_1+\alpha_1}{m+\gamma_1}, \frac{k_2+\alpha_2}{n+\gamma_2}\right) \quad \text{(I.2.12)}$$

with $(x,y) \in D, f \in C(D)$ and $\alpha = (\alpha_1, \alpha_2),\ \gamma = (\gamma_1, \gamma_2)$.

Lemma I.2.11 *Let $(x,y) \in C(D)$ and let $f_{00}(x,y) = 1, f_{10}(x,y) = x, f_{01}(x,y) = y, f_{20}(x,y) = x^2, f_{02}(x,y) = y^2$. Then for the operators described in relation* (I.2.12) *we have:*

$$^{[\beta]}\mathcal{J}_{m,n}^{\alpha,\gamma}(f_{0,0};x,y)=1 \tag{I.2.13}$$

$$^{[\beta]}\mathcal{J}_{m,n}^{\alpha,\gamma}(f_{1,0};x,y)=\frac{mx}{(m+\gamma_1)(1-\beta)}+\frac{\alpha_1}{m+\gamma_1} \tag{I.2.14}$$

$$^{[\beta]}\mathcal{J}_{m,n}^{\alpha,\gamma}(f_{0,1};x,y)=\frac{ny}{(n+\gamma_2)(1-\beta)}+\frac{\alpha_2}{n+\gamma_2} \tag{I.2.15}$$

$$\begin{aligned}^{[\beta]}\mathcal{J}_{m,n}^{\alpha,\gamma}(f_{2,0}+f_{0,2};x,y)&=\frac{mx^2}{(m+\gamma_1)^2(1-\beta)^2}\\ &+\frac{mx}{(m+\gamma_1)^2}\left[\frac{1}{(1-\beta^3)}+\frac{2\alpha_1}{1-\beta}\right]+\frac{\alpha_1^2}{(m+\gamma_1)^2}\\ &+\frac{ny^2}{(n+\gamma_2)^2(1-\beta)^2}+\frac{ny}{(n+\gamma_2)^2}\left[\frac{1}{(1-\beta^3)}+\frac{2\alpha_2}{1-\beta}\right]\\ &+\frac{\alpha_2^2}{(n+\gamma_2)^2}.\end{aligned} \tag{I.2.16}$$

Theorem I.2.12 *Let $f\in C(D)$ and $\beta_n\to 0$ when $n\to\infty$. Then the sequence $^{[\beta_n]}\{\mathcal{J}_{m,n}^{\alpha,\gamma}(f;x,y)\}$ converges uniformly to $f(x,y)$ on $K\subset D$, where $K=[0,A]\times[0,A]$, $0<A<\infty$, that is*

$$\lim_{m,n\to\infty}\left\|{}^{[\beta_n]}\mathcal{J}_{m,n}^{\alpha,\gamma}(f;x,y)-f(x,y)\right\|_K=0.$$

Theorem I.2.13 [34] *Let $A=(a_{j,k,m,n})$ be a nonnegative RH-regular summability matrix method. Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C(D)$ into itself. Then, for all $f\in C(D)$,*

$$st_A^2-\lim_{m,n}\|L_{m,n}f-f\|_{C(D)}=0$$

if and only if

$$st_A^2-\lim_{m,n}\|L_{m,n}f_i-f_i\|_{C(D)}=0,(i=0,1,2,3), \tag{I.2.17}$$

where $f_0(x,y)=1, f_1(x,y)=x, f_2(x,y)=y$ and $f_3(x,y)=x^2+y^2$.

Theorem I.2.14 *Let $^{[\beta]}\mathcal{J}_{m,n}^{\alpha,\gamma}$ be the operators defined in* (I.2.12). *In additon we take $\beta=\beta_n$ and $\alpha=\alpha_n,\gamma=\gamma_n$ with the properties*

$$\beta_n\to 0, n\to\infty \tag{I.2.18}$$

and,

$$\alpha_n\to 0, n\to\infty, \gamma_n\to 0, n\to\infty, \tag{I.2.19}$$

respectively. Then we have

$$\left|{}^{[\beta_n]}\mathcal{J}_{m,n}^{\alpha_n,\gamma_n}(f;x,y)-f(x,y)\right| \le \omega(f;\delta)\left\{1+\frac{1}{\delta^2}(2x^2+2y^2)\right\}. \quad \text{(I.2.20)}$$

Part II

History-Dependent Operators in Contact Mechanics

3
Preliminaries

In this chapter we gather both preliminary material which will be used later in the study of boundary value problems presented in Chapters 4–6 and preliminary material which is necessary in the modelling of the boundary value problems we present in Chapters 4–6 of the thesis. This chapter is divided into two main sections. First section is devoted to background results from functional analysis and the second section aims to present some preliminary results concerning the modelling of contact problems.

3.1 Backgrounds on functional analysis

In this section we first recall important notion concerning the convergence on normed spaces and some details regarding the Hilbert spaces. Then we introduce the concept of history-dependent operators. We provide some examples, describe their properties, and present an existence and uniqueness result for variational inequalities with history-dependent operators. We continue with a short description of the function spaces we need in the study of contact problems. Most of the results are stated without proof since they are standard and they can be found in many other references while for the results which are frequently used throughout the manuscript we give the proofs, too.

3.1.1 Basic notion

In this subsection one can find information regarding the Hilbert space and some theorems which we use later in the manuscript.

3.1.2 History-dependent operators

In this subsection we present the definition and some examples of history-dependent operators as well as fixed point results used later in the manuscript. Finally, we present an abstract existence and uniqueness result concerning a class of history-dependent variational inequalities.

3.1.3 Function spaces in Contact Mechanics

In order to introduce a mathematical model that describes a contact process, we need to describe first the spaces to which the data and the unknowns belong.

3.2 Modelling of contact problems

We start this section by presenting the physical setting of contact processes. Then we make a survey of the constitutive laws used in the literature, including the viscoelastic and the viscoplastic constitutive laws we use in this manuscript. We proceed with a description of the contact conditions. Finally, we discuss the frictionless and the frictional conditions, including the Coulomb law of dry friction. More details on topics related to the material presented in this chapter can be found in [8, 35, 39, 42, 54, 59, 61, 73, 86, 101, 114, 118, 128].

3.2.1 Physical setting

We consider a general physical setting presented in Figure 3.1 and we describe it in what follows. A deformable body occupies, in the reference configuration, an open bounded connected set $\Omega \subset \mathbb{R}^d$ with boundary Γ, composed of three sets $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$, with the mutually disjoint relatively open sets Γ_1, Γ_2 and Γ_3. The body is clamped on Γ_1. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and volume forces of density $\boldsymbol{f}_0$ act in Ω. In the reference configuration the body is in contact on Γ_3 with an obstacle, the so-called *foundation*.

We are interested in mathematical models which describe the equilibrium of the mechanical state of the body, in the physical setting above, in the framework of small strain theory. To this end, we denote by $\boldsymbol{u}$,

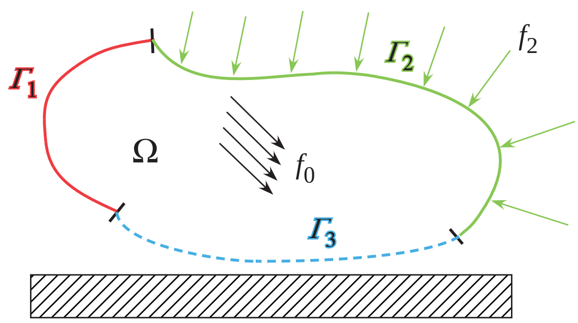

FIGURE 3.1. The physical setting; Γ_3 is the contact surface

$\boldsymbol{\sigma}$, and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ the displacement vector, the stress tensor, and the linearized strain tensor, respectively. These are functions which depend on the spatial variable $\boldsymbol{x}$ and on the time variable t. Nevertheless, in what follows we do not indicate explicitly the dependence of these quantities on $\boldsymbol{x}$ and t i.e., for instance, we write $\boldsymbol{\sigma}$ instead of $\boldsymbol{\sigma}(\boldsymbol{x}, t)$. Also, recall that the components of the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are given by

$$\varepsilon_{ij}(\boldsymbol{u}) = (\boldsymbol{\varepsilon}(\boldsymbol{u}))_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{(II.3.1)}$$

where $u_{i,j} = \partial u_i/\partial x_j$. Finally, note that, here and below, all variables are assumed to have sufficient degree of smoothness consistent with developments they are involved in. To present a mathematical model for a specific contact process, we need to precise the constitutive law, the balance equation, the boundary conditions, the contact conditions and, eventually, the initial conditions.

3.2.2 *Constitutive laws*

A constitutive law represents a relationship between the stress $\boldsymbol{\sigma}$ and the strains $\boldsymbol{\varepsilon}$ and their derivatives, eventually, which characterizes a specific material. It describes the deformations of the body resulting from the action of forces and tractions. Though the constitutive laws must satisfy some basic axioms and invariance principles, they originate mostly from experiments.

3.2.3 *Contact conditions*

In order to present a mathematical model for a specific contact process, beside the constitutive law we need to describe the balance equation, the boundary conditions and the contact conditions.

We describe here the basic equations and boundary conditions which are used in the study of contact problems with viscoelastic and viscoplastic materials. We assume in what follows that the data and the unknowns depend on time and we study the contact process in unbounded time interval $\mathbb{R}_+$.

3.2.4 Friction laws

We present now some conditions in the tangential directions, called also *frictional conditions* or *friction laws*. The simplest one is the so-called *frictionless* condition in which the tangential part of the stress, (also named *the friction force*) vanishes, i.e.

$$\boldsymbol{\sigma}_\tau = \mathbf{0}. \qquad \text{(II.3.2)}$$

4

A history-dependent frictional contact problem

In this chapter we consider a mathematical model which describes the contact between a viscoelastic body and a foundation. The contact is frictional and is modelled with normal compliance and memory term, associated to the Coulomb's law of dry friction. We derive a variational formulation of the problem which is in a form of a variational inequality for the displacement field or, equivalently, in a form of a history-dependent variational inequality for the velocity field. Our main results in this chapter are Theorem II.4.1 and II.4.4. Theorem II.4.1 states the unique solvability of the problem and is obtained in several steps, based on the arguments presented in Section 3.1.2. Theorem II.4.4 states the continuous dependence of the solution with respect to the data. It is obtained using various estimates and monotonicity arguments. The material presented in this chapter made the object of the article [117].

4.1 Problem statement

For the problem studied in this chapter the contact is frictional. It is modelled with normal compliance and memory term, associated to the Coulomb's law of dry friction. The material's behavior is described with a viscoelastic constitutive law with long memory. The classical formulation of the contact problem is the following.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{K}(t-s)\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\,ds, \quad \text{in} \quad \Omega, \quad \text{(II.4.1)}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \quad \text{(II.4.2)}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \quad \text{(II.4.3)}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \quad \text{(II.4.4)}$$

$$-\sigma_\nu(t) = p(u_\nu(t)) + \int_0^t b(t-s)u_\nu^+(s)\,ds \quad \text{on} \quad \Gamma_3, \quad \text{(II.4.5)}$$

$$\left.\begin{array}{l} \|\boldsymbol{\sigma}_\tau(t)\| \le \mu\,|\sigma_\nu(t)|, \\ -\boldsymbol{\sigma}_\tau(t) = \mu\,|\sigma_\nu(t)|\frac{\dot{\boldsymbol{u}}_\tau(t)}{\|\dot{\boldsymbol{u}}_\tau(t)\|} \ \text{ if } \ \dot{\boldsymbol{u}}_\tau(t) \neq \boldsymbol{0} \end{array}\right\} \quad \text{on} \quad \Gamma_3 \quad \text{(II.4.6)}$$

for all $t \in \mathbb{R}_+$ and, moreover,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \quad \text{(II.4.7)}$$

4.2 Existence and Uniqueness

First we assume that the viscosity operator satisfies the condition

$$\left.\begin{array}{l}
\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\quad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\quad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}}\,\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\qquad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\quad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\boldsymbol{0}_{\mathbb{S}_d}) \text{ belongs to } Q.
\end{array}\right\} \quad \text{(II.4.8)}$$

Also, the elasticity operator and the relaxation tensor satisfy the following conditions.

$$\left.\begin{array}{l}
\text{(a) } \mathcal{B}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\
\text{(b) There exists } L_{\mathcal{B}}>0 \text{ such that}\\
\qquad \|\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{B}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\
\qquad\quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(c) The mapping } \boldsymbol{x}\mapsto\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\
\qquad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\
\text{(e) The mapping } \boldsymbol{x}\mapsto\mathcal{B}(\boldsymbol{x},\mathbf{0}_{\mathbb{S}_d}) \text{ belongs to } Q.
\end{array}\right\}\qquad \text{(II.4.9)}$$

$$\mathcal{K}\in C(\mathbb{R}_+;\mathbf{Q}_\infty). \qquad \text{(II.4.10)}$$

The densities of body forces and surface tractions are such that

$$\boldsymbol{f}_0\in C(\mathbb{R}_+;L^2(\Omega)^d),\quad \boldsymbol{f}_2\in C(\mathbb{R}_+;L^2(\Gamma_2)^d) \qquad \text{(II.4.11)}$$

and the normal compliance function p satisfies

$$\left.\begin{array}{l}
\text{(a) } p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+.\\
\text{(b) There exists } L_p>0 \text{ such that}\\
\qquad |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\le L_p\,|r_1-r_2|\\
\qquad\quad \forall\,r_1,\ r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\
\text{(c) The mapping } \boldsymbol{x}\mapsto p(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\\
\qquad \text{for any } r\in\mathbb{R}.\\
\text{(d) } p(\boldsymbol{x},r)=0 \text{ for all } r\le 0,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.
\end{array}\right\}\qquad \text{(II.4.12)}$$

Finally, the surface memory function, the coefficient of friction and the initial data verify

$$b\in C(\mathbb{R}_+;L^\infty(\Gamma_3)),\quad b(t,\boldsymbol{x})\ge 0 \qquad \text{(II.4.13)}$$
$$\text{for all } t\in\mathbb{R}_+,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3,$$

$$\mu\in L^\infty(\Gamma_3),\quad \mu(\boldsymbol{x})\ge 0\quad \text{a.e. } \boldsymbol{x}\in\Gamma_3, \qquad \text{(II.4.14)}$$

$$\boldsymbol{u}_0\in V. \qquad \text{(II.4.15)}$$

Problem $\mathcal{P}^V$. *Find a displacement field* $\boldsymbol{u} : \mathbb{R}_+ \to V$ *such that* $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *and the following inequality holds, for all* $t \in \mathbb{R}_+$:

$$\begin{aligned}
&\Big(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\Big)_Q \qquad \text{(II.4.16)}\\
&\quad + \Big(\int_0^t \mathcal{K}(t-s)\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\Big)_Q\\
&\quad + \Big(p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\,ds, v_\nu - \dot{u}_\nu(t)\Big)_{L^2(\Gamma_3)}\\
&\quad + \Big(\mu\big(p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\,ds\big), \|\boldsymbol{v}_\tau\| - \|\dot{\boldsymbol{u}}_\tau(t)\|\Big)_{L^2(\Gamma_3)}\\
&\quad \geq (\boldsymbol{f}_0(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{L^2(\Gamma_2)^d} \quad \forall\, \boldsymbol{v} \in V.
\end{aligned}$$

A couple of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (II.4.1) and (II.4.16) is called a *weak solution* for the frictional contact problem, Problem $\mathcal{P}$.

We have the following existence and uniqueness result.

Theorem II.4.1 *Assume that* (II.4.8)–(II.4.15) *hold. Then, Problem* $\mathcal{P}^V$ *has a unique solution which satisfies*

$$\boldsymbol{u} \in C^1(\mathbb{R}_+; V). \qquad \text{(II.4.17)}$$

Problem $\mathcal{Q}^V$. *Find a velocity field* $\boldsymbol{w} : \mathbb{R}_+ \to V$ *such that the following inequality holds, for all* $t \in \mathbb{R}_+$:

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{w}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{w}(t)))_Q \qquad \text{(II.4.18)}\\
&\qquad + \varphi(\mathcal{R}\boldsymbol{w}(t), \boldsymbol{v}) - \varphi(\mathcal{R}\boldsymbol{w}(t), \boldsymbol{w}(t)))\\
&\qquad\qquad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t))_Q \quad \forall\, \boldsymbol{v} \in V.
\end{aligned}$$

Lemma II.4.2 *Let* $\boldsymbol{u} \in C^1(\mathbb{R}_+; V)$ *and* $\boldsymbol{w} \in C(\mathbb{R}_+; V)$ *be given functions such that* $\boldsymbol{u} = \mathcal{S}\boldsymbol{w}$. *Then* $\boldsymbol{u}$ *is a solution to the Problem* $\mathcal{P}^V$ *if and only if* $\boldsymbol{w}$ *is a solution to the Problem* $\mathcal{Q}^V$.

Lemma II.4.3 *There exists a unique solution* $\boldsymbol{w}$ *to Problem* $\mathcal{Q}^V$ *and, moreover, it satisfies* $\boldsymbol{w} \in C(\mathbb{R}_+; V)$.

4.3 A convergence result

Consider the following assumptions:

$$\mathcal{K}_\rho \to \mathcal{K} \qquad \text{in} \;\; C(\mathbb{R}_+; \mathbf{Q}_\infty) \quad \text{as} \quad \rho \to 0. \tag{II.4.19}$$

$$\left.\begin{array}{l} \text{There exists } F : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } L_0 \geq 0 \text{ such that} \\ \text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \leq F(\rho)(|r| + 1) \\ \qquad \forall\, r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for each } \rho > 0. \\ \text{(b) } F(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \\ \text{(c) } L_\rho \leq L_0 \quad \text{as} \quad \rho \to 0. \end{array}\right\} \tag{II.4.20}$$

$$b_\rho \to b \qquad \text{in} \;\; C(\mathbb{R}_+; L^\infty(\Gamma_3)) \quad \text{as} \quad \rho \to 0. \tag{II.4.21}$$

$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in} \;\; C(\mathbb{R}_+; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0. \tag{II.4.22}$$

$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in} \;\; C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0. \tag{II.4.23}$$

Theorem II.4.4 *Assume that* (II.4.19)–(II.4.23) *hold. Then the solution* $\boldsymbol{u}_\rho$ *of Problem* $\mathcal{P}_\rho^V$ *converges to the solution* $\boldsymbol{u}$ *of Problem* $\mathcal{P}^V$*, i.e.*

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad \text{in} \quad C^1(\mathbb{R}_+; V) \tag{II.4.24}$$

as $\rho \to 0$.

5

A history-dependent frictionless contact problem

In this chapter we consider a first mathematical model which describes the quasistatic contact between a viscoplastic body and a foundation. The contact is frictionless and is modelled with normal compliance, unilateral constraint and memory effects. We derive a variational formulation of the problem, then we prove its unique weak solvability. The two main theorems which state the unique solvability and the convergence result in this chapter are Theorem II.5.1 and Theorem II.5.4, respectively. The proof of Theorem II.5.1 is based on arguments presented in Section 3.1.2. Theorem II.5.4 states that the solution of the penalized problem converges to the solution of the variational problem, obtained previously. The results obtained in this chapter were published in [49] and [100].

5.1 Problem statement

For the problem analyzed in this chapter the contact is modelled with normal compliance, unilateral constraint and memory term. The material's behavior is described with a viscoplastic constitutive law. The classical formulation of the problem is the following.

Problem $\mathcal{M}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in} \quad \Omega, \tag{II.5.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{II.5.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{II.5.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{II.5.4}$$

for all $t \in \mathbb{R}_+$, *there exists* $\xi : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\left.\begin{array}{l}
u_\nu(t) \le g, \ \ \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\
(u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t) + \xi(t)\big) = 0, \\
0 \le \xi(t) \le \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \\
\xi(t) = 0 \ \text{ if } \ u_\nu(t) < 0, \\
\xi(t) = \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds \ \text{ if } \ u_\nu(t) > 0
\end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{II.5.5}$$

for all $t \in \mathbb{R}_+$ *and, moreover,*

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{II.5.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{II.5.7}$$

5.2 Existence and uniqueness

In the study of problem $\mathcal{M}$ we assume that the elasticity tensor $\mathcal{E}$ and the nonlinear constitutive function $\mathcal{G}$ satisfy the following conditions.

$$\left.\begin{array}{l}\text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d.\\ \text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that}\\ \qquad \mathcal{E}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge m_{\mathcal{E}}\|\boldsymbol{\tau}\|^2\ \ \forall\boldsymbol{\tau}\in\mathbb{S}^d,\ \text{a.e. in } \Omega.\end{array}\right\} \qquad \text{(II.5.8)}$$

$$\left.\begin{array}{l}\text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that}\\ \quad \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\|\\ \qquad \le L_{\mathcal{G}}\left(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\right)\\ \quad \forall\,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) The mapping } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\ \quad \text{for any } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{0}_{\mathbb{S}^d},\boldsymbol{0}_{\mathbb{S}^d}) \text{ belongs to } Q.\end{array}\right\} \qquad \text{(II.5.9)}$$

Also, the normal compliance function p satisfy

$$\left.\begin{array}{l}\text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+.\\ \text{(b) There exists } L_p > 0 \text{ such that}\\ \quad |p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2)| \le L_p\,|r_1-r_2|\\ \qquad\qquad \forall\, r_1,\ r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(c) } (p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2))(r_1-r_2)\ge 0\\ \qquad\qquad \forall\, r_1,\ r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto p(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\\ \quad \text{for any } r\in\mathbb{R}.\\ \text{(e) } p(\boldsymbol{x},r) = 0 \text{ for all } r\le 0,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right\} \qquad \text{(II.5.10)}$$

Finally, the densities of body forces and surface traction, the memory function and the initial data are such that

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+;L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in C(\mathbb{R}_+;L^2(\Gamma_2)^d), \qquad \text{(II.5.11)}$$

$$b \in C(\mathbb{R}_+;L^\infty(\Gamma_3)), \qquad b(t,\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x}\in\Gamma_3, \qquad \text{(II.5.12)}$$

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q. \qquad \text{(II.5.13)}$$

Consider now the subset $U \subset V$, the operators $P : V \to V$, $\mathcal{B} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; L^2(\Gamma_3))$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \le g \text{ on } \Gamma_3 \,\}, \tag{II.5.14}$$

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{II.5.15}$$

$$(\mathcal{B}\boldsymbol{u}(t), \xi)_{L^2(\Gamma_3)} = \Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \xi \Big)_{L^2(\Gamma_3)} \tag{II.5.16}$$

$$\forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \ \xi \in L^2(\Gamma_3),\ \ t \in \mathbb{R}_+,$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx \tag{II.5.17}$$

$$+ \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+.$$

Problem $\mathcal{M}^V$. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$ and a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ such that, for all $t \in \mathbb{R}_+$,*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{II.5.18}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \tag{II.5.19}$$

$$+ (\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.$$

A couple $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (II.5.1) and (II.5.19) is called a *weak solution* for the frictionless contact problem, Problem $\mathcal{M}$.

The unique solvability of Problem $\mathcal{M}^V$ is given by the following result.

Theorem II.5.1 *Assume that* (II.5.8)–(II.5.13) *hold. Then Problem $\mathcal{M}^V$ has a unique solution, which satisfies $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ and $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$.*

Lemma II.5.2 *For each function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ there exists a unique function $\mathcal{S}\boldsymbol{u} \in C(\mathbb{R}_+; Q)$ such that*

$$\mathcal{S}\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \tag{II.5.20}$$

$$+ \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall\, t \in \mathbb{R}_+.$$

Moreover, the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ is a history-dependent operator, i.e. it satisfies the following property: for every $n \in \mathbb{N}$ there *exists* $s_n > 0$ *such that*

$$\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_Q \le s_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds \quad \text{(II.5.21)}$$
$$\forall\, \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V),\ \forall\, t \in [0, n].$$

Lemma II.5.3 *Let* $(\boldsymbol{u}, \boldsymbol{\sigma})$ *be a couple of functions such that* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. *Then,* $(\boldsymbol{u}, \boldsymbol{\sigma})$ *is a solution of Problem* $\mathcal{M}^V$ *if and only if*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}\boldsymbol{u}(t) \quad \forall\, t \in \mathbb{R}_+, \quad \text{(II.5.22)}$$

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \quad \text{(II.5.23)}$$
$$+(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V$$
$$\ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U,\ \forall\, t \in \mathbb{R}_+$$

hold.

5.3 A convergence result

In this section we prove a convergence result in the study of Problem $\mathcal{M}^V$. To this end, everywhere in this section we restrict to the homogenous case i.e. assume that the function p does not depend on $\boldsymbol{x} \in \Gamma_3$. Moreover, we assume that p satisfies

$$\left.\begin{array}{l}
\text{(a) } p : \mathbb{R} \to \mathbb{R}_+. \\
\text{(b) There exists } L_p > 0 \text{ such that} \\
\qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\
\text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \ge 0 \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\
\text{(d) } p(r) = 0 \ \text{ for all } r < 0.
\end{array}\right\} \quad \text{(II.5.24)}$$

Let q be a function which satisfies

$$\left.\begin{array}{l}
\text{(a) } q :\ [g, +\infty[\to \mathbb{R}_+. \\
\text{(b) There exists } L_q > 0 \text{ such that} \\
\qquad |q(r_1) - q(r_2)| \le L_q |r_1 - r_2| \quad \forall\, r_1,\, r_2 \ge g. \\
\text{(c) } (q(r_1) - q(r_2))(r_1 - r_2) > 0 \quad \forall\, r_1,\, r_2 \ge g,\ r_1 \ne r_2. \\
\text{(d) } q(g) = 0.
\end{array}\right\} \quad \text{(II.5.25)}$$

Let $\mu > 0$ and consider the function p_μ defined by

$$p_\mu(r) = \begin{cases} p(r) & \text{if} \quad r \le g, \\ \frac{1}{\mu}\, q(r) + p(g) & \text{if} \quad r > g. \end{cases} \tag{II.5.26}$$

Using assumption (II.5.25) it follows that the function p_μ satisfies condition (II.5.24), i.e.

$$\left.\begin{array}{l} \text{(a) } p_\mu : \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_{p_\mu} > 0 \text{ such that} \\ \qquad |p_\mu(r_1) - p_\mu(r_2)| \le L_{p_\mu} |r_1 - r_2| \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) } (p_\mu(r_1) - p_\mu(r_2))(r_1 - r_2) \ge 0 \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(d) } p_\mu(r) = 0 \ \text{ for all } r < 0. \end{array}\right\} \tag{II.5.27}$$

This allows us to consider the operator $P_\mu : V \to V$ defined by

$$(P_\mu \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\mu(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V \tag{II.5.28}$$

and, moreover, we note that P_μ is a monotone, Lipschitz continuous operator.

With these notation, we consider the following contact problem.

Problem $\mathcal{M}_\mu$. *Find a displacement field* $\boldsymbol{u}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}}_\mu(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\mu(t)) + \mathcal{G}(\boldsymbol{\sigma}_\mu(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t))) \quad \text{in} \quad \Omega, \tag{II.5.29}$$

$$\operatorname{Div} \boldsymbol{\sigma}_\mu(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{II.5.30}$$

$$\boldsymbol{u}_\mu(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{II.5.31}$$

$$\boldsymbol{\sigma}_\mu(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{II.5.32}$$

for all $t \in \mathbb{R}_+$, *there exists* $\xi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\left.\begin{array}{l} \sigma_{\mu\nu}(t) + p_\mu(u_{\mu\nu}(t)) + \xi(t) = 0, \\ 0 \le \xi(t) \le \displaystyle\int_0^t b(t-s)\, u_{\mu\nu}^+(s)\, ds, \\ \xi(t) = 0 \ \text{ if } \ u_{\mu\nu}(t) < 0, \\ \xi(t) = \displaystyle\int_0^t b(t-s)\, u_{\mu\nu}^+(s)\, ds \ \text{ if } \ u_{\mu\nu}(t) > 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{II.5.33}$$

for all $t \in \mathbb{R}_+$ *and, moreover,*

$$\boldsymbol{\sigma}_{\mu\tau}(t) = \mathbf{0} \qquad \text{on} \quad \Gamma_3, \tag{II.5.34}$$

$$\boldsymbol{u}_\mu(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}_\mu(0) = \boldsymbol{\sigma}_0 \qquad \text{in} \quad \Omega. \tag{II.5.35}$$

Problem $\mathcal{M}^V_\mu$. *Find a displacement field* $\boldsymbol{u}_\mu : \mathbb{R}_+ \to U$ *and a stress field* $\boldsymbol{\sigma}_\mu : \mathbb{R}_+ \to Q$ *such that, for all* $t \in \mathbb{R}_+$,

$$\begin{aligned}\boldsymbol{\sigma}_\mu(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\mu(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s)))\, ds \qquad &(\text{II.5.36})\\ +\boldsymbol{\sigma}_{0\mu} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\mu}),&\end{aligned}$$

$$\begin{aligned}(\boldsymbol{\sigma}_\mu(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)))_Q + (P\boldsymbol{u}_\mu(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \qquad &(\text{II.5.37})\\ +(\mathcal{B}\boldsymbol{u}_\mu(t), v_\nu^+ - u_{\mu\nu}^+(t))_{L^2(\Gamma_3)} \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \quad \forall\, \boldsymbol{v} \in U.&\end{aligned}$$

We have the following existence, uniqueness and convergence result.

Theorem II.5.4 *Assume that* (II.5.24), (II.5.11), (II.5.13) *and* (II.5.25) *hold. Then:*

a) For each $\mu > 0$ *there exists a unique solution* $\boldsymbol{u}_\mu \in C(\mathbb{R}_+; V)$ *to Problem* $\mathcal{M}^V_\mu$.

b) The solution u_μ *of Problem* $\mathcal{M}^V_\mu$ *converges strongly to the solution* u *of Problem* $\mathcal{M}^V$, *that is*

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V + \|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \tag{II.5.38}$$

as $\mu \to 0$, *for all* $t \in \mathbb{R}_+$.

Lemma II.5.5 *There exists a unique solution* $\boldsymbol{u}_\mu \in C(\mathbb{R}_+; V)$ *to problem* $\mathcal{M}^V_\mu$.

In order to complete the proof of the main theorem we consider the auxiliary problem of finding a displacement field $\widetilde{\boldsymbol{u}}_\mu : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \ &(\text{II.5.39})\\ +(P_\mu\widetilde{\boldsymbol{u}}_\mu(t), \boldsymbol{v} - \widetilde{\boldsymbol{u}}_\mu(t)))_V + (\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - \widetilde{u}_{\mu\nu}^+(t))_{L^2(\Gamma_3)}&\\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \widetilde{\boldsymbol{u}}_\mu(t)))_V \ \ \forall\, \boldsymbol{v} \in V.&\end{aligned}$$

Lemma II.5.6 *There exists a unique function* $\widetilde{\boldsymbol{u}}_\mu \in C(\mathbb{R}_+;V)$ *which satisfies* (II.5.39), *for all* $t \in \mathbb{R}_+$.

Lemma II.5.7 *As* $\mu \to 0$,

$$\widetilde{\boldsymbol{u}}_\mu(t) \rightharpoonup \boldsymbol{u}(t) \quad \textit{in} \;\; V,$$

for all $t \in \mathbb{R}_+$.

Lemma II.5.8 *As* $\mu \to 0$,

$$\|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V \to 0,$$

for all $t \in \mathbb{R}_+$.

Lemma II.5.9 *The following convergence holds.*

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V + \|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \;\; \textit{as} \; \mu \to 0 \; \textit{for all} \; t \; \in \mathbb{R}_+.$$

6

A history-dependent frictionless contact problem with internal state variable

In this chapter we consider a second mathematical model which describes the quasistatic contact between a viscoplastic body and a foundation. Unlike the problem studied in the previous chapter here we model the material's behavior with a rate-type constitutive law with internal state variable. The contact is frictionless and is modelled with normal compliance, unilateral constraint and memory term. We present the classical formulation of the problem, list the assumptions on the data and derive a variational formulation of the model. Then, in Theorem II.6.1 we prove its unique weak solvability. The proof is based on arguments of history-dependent quasivariational inequalities. We also study the dependence of the solution with respect to the data and prove a convergence result, Theorem II.6.4. The content of this chapter is based on the paper [122].

6.1 Problem statement

For the problem studied in this chapter the contact is modelled with normal compliance, unilateral constraint and memory term. Moreover, the material's behavior is described with a rate-type constitutive law with internal state variable. The classical formulation of the problem is the following.

Problem $\mathcal{N}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *and an internal state variable* $\boldsymbol{\kappa} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ *such* *that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\kappa}(t)) \quad \text{in} \quad \Omega, \qquad \text{(II.6.1)}$$

$$\dot{\boldsymbol{\kappa}}(t) = \boldsymbol{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\kappa}(t)) \quad \text{in} \quad \Omega, \qquad \text{(II.6.2)}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \qquad \text{(II.6.3)}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \qquad \text{(II.6.4)}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \qquad \text{(II.6.5)}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \qquad \text{(II.6.6)}$$

for all $t \in \mathbb{R}_+$, *there exists* $\xi : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\left.\begin{array}{l} u_\nu(t) \le g, \ \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\ (u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)\big) = 0, \\ 0 \le \xi(t) \le \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \\ \xi(t) = 0 \ \text{ if } \ u_\nu(t) < 0, \\ \xi(t) = \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds \ \text{ if } \ u_\nu(t) > 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3, \qquad \text{(II.6.7)}$$

for all $t \in \mathbb{R}_+$ *and, moreover,*

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \boldsymbol{\kappa}(0) = \boldsymbol{\kappa}_0 \qquad \text{in} \quad \Omega. \qquad \text{(II.6.8)}$$

6.2 Existence and uniqueness

In this section we list the assumptions on the data, derive the variational formulation of the problem $\mathcal{N}$ and then we state and prove its unique weak solvability. To this end we assume that the elasticity tensor $\mathcal{E}$ satisfies

$$\left.\begin{array}{l} \text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega), \ 1 \le i,j,k,l \le d. \\ \text{(c) There exists } m_{\mathcal{E}} > 0 \ \text{ such that} \\ \qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2 \ \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \ \text{a.e. in } \Omega. \end{array}\right\} \qquad \text{(II.6.9)}$$

and the constitutive functions $\mathcal{G}$ and $\boldsymbol{G}$ satisfy the following conditions.

$$\left.\begin{array}{l}
\text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\
\qquad \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \boldsymbol{\kappa}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \boldsymbol{\kappa}_2)\| \\
\qquad\quad \leq L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2\|) \\
\qquad\qquad \forall\, \boldsymbol{\sigma}_1,\, \boldsymbol{\sigma}_2,\, \boldsymbol{\varepsilon}_1,\, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \boldsymbol{\kappa}_1,\, \boldsymbol{\kappa}_2 \in \mathbb{R}^m, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\sigma},\, \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \boldsymbol{\kappa} \in \mathbb{R}^m. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}_{\mathbb{S}^d}, \boldsymbol{0}_{\mathbb{S}^d}, \boldsymbol{0}_{\mathbb{R}^m}) \text{ belongs to } Q.
\end{array}\right\} \quad \text{(II.6.10)}$$

$$\left.\begin{array}{l}
\text{(a) } \boldsymbol{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \to \mathbb{R}^m. \\
\text{(b) There exists } L_G > 0 \text{ such that} \\
\qquad \|\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \boldsymbol{\kappa}_1) - \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \boldsymbol{\kappa}_2)\| \\
\qquad\quad \leq L_G(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2\|) \\
\qquad\qquad \forall\, \boldsymbol{\sigma}_1,\, \boldsymbol{\sigma}_2,\, \boldsymbol{\varepsilon}_1,\, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\, \boldsymbol{\kappa}_1,\, \boldsymbol{\kappa}_2 \in \mathbb{R}^m, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \boldsymbol{\kappa} \in \mathbb{R}^m. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{0}_{\mathbb{S}^d}, \boldsymbol{0}_{\mathbb{S}^d}, \boldsymbol{0}_{\mathbb{R}^m}) \\
\qquad \text{belongs to } L^2(\Omega)^m.
\end{array}\right\} \quad \text{(II.6.11)}$$

The densities of body forces and surface tractions are such that

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d) \qquad \text{(II.6.12)}$$

and the normal compliance function p satisfies

$$\left.\begin{array}{l}
\text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\
\text{(b) There exists } L_p > 0 \text{ such that} \\
\qquad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \leq L_p\, |r_1 - r_2| \\
\qquad\; \forall\, r_1,\, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\
\text{(c) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))(r_1 - r_2) \geq 0 \\
\qquad \forall\, r_1,\, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \\
\qquad \text{for any } r \in \mathbb{R}. \\
\text{(e) } p(\boldsymbol{x}, r) = 0 \text{ for all } r \leq 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}\right\} \quad \text{(II.6.13)}$$

Also, the surface memory function and the initial data verify

$$b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)), \quad b(t, \boldsymbol{x}) \geq 0 \quad \text{for all } t \in \mathbb{R}_+, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \quad \text{(II.6.14)}$$

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q, \quad \boldsymbol{\kappa}_0 \in L^2(\Omega)^m. \tag{II.6.15}$$

Problem $\mathcal{N}^V$. *Find a displacement field* $\boldsymbol{u} : \mathbb{R}_+ \to U$, *a stress field* $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ *and an internal state variable* $\boldsymbol{\kappa} : \mathbb{R}_+ \to L^2(\Omega)^m$ *such that, for all* $t \in \mathbb{R}_+$, we have

$$\boldsymbol{\sigma}(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\kappa}(s))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),$$

$$\boldsymbol{\kappa}(t) = \int_0^t \boldsymbol{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\kappa}(s))\, ds + \boldsymbol{\kappa}_0,$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V$$
$$+\Big(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t)\Big)_{L^2(\Gamma_3)} \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.$$

In the study of the problem $\mathcal{N}^V$ we have the following existence and uniqueness result.

Theorem II.6.1 *Assume that* (II.6.9)–(II.6.15) *hold. Then, Problem* $\mathcal{N}^V$ *has a unique solution which satisfies*

$$\boldsymbol{u} \in C(\mathbb{R}_+; U), \quad \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q) \quad \text{and} \quad \boldsymbol{\kappa} \in C(\mathbb{R}_+; L^2(\Omega)^m). \tag{II.6.16}$$

Lemma II.6.2 *For each* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ *there exists a unique function* $\mathcal{S}\boldsymbol{u} = (\mathcal{S}_1\boldsymbol{u}, \mathcal{S}_2\boldsymbol{u}) \in C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ *such that*

$$\mathcal{S}_1\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}_1\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \mathcal{S}_2\boldsymbol{u}(s))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{II.6.17}$$

$$\mathcal{S}_2\boldsymbol{u}(t) = \int_0^t \boldsymbol{G}(\mathcal{S}_1\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \mathcal{S}_2\boldsymbol{u}(s))\, ds + \boldsymbol{\kappa}_0 \tag{II.6.18}$$

for all $t \in \mathbb{R}_+$. *Moreover, the operator* $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ *is a history-dependent operator, i.e. it satisfies the following property: for every* $n \in \mathbb{N}^*$ *there exists* $s_n > 0$ *which depends only on* n, d, $\mathcal{G}$, $\boldsymbol{G}$ *and* $\mathcal{E}$, *such that*

$$\|\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{v}(t)\|_{Q \times L^2(\Omega)^m} \leq s_n \int_0^t \|\boldsymbol{u}(s) - \boldsymbol{v}(s)\|_V\, ds \tag{II.6.19}$$
$$\forall\, \boldsymbol{u}, \boldsymbol{v} \in C(\mathbb{R}_+; V) \quad \forall\, t \in [0, n].$$

Lemma II.6.3 *Let* $(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\kappa})$ *be a triple of functions which satisfy* (II.6.16). *Then* $(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\kappa})$ *is a solution of* $\mathcal{N}^V$ *if and only if*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}_1(\boldsymbol{u}(t)), \tag{II.6.20}$$

$$\boldsymbol{\kappa}(t) = \mathcal{S}_2\boldsymbol{u}(t), \tag{II.6.21}$$

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{S}_1\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q\\ +(P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + (\mathcal{B}\boldsymbol{u}(t)\,, v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)}\\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V, \quad \forall\, \boldsymbol{v} \in U,\end{aligned} \tag{II.6.22}$$

for all $t \in \mathbb{R}_+$.

6.3 A convergence result

We now study the dependence of the solution of Problem $\mathcal{N}^V$ with respect to perturbations of the data. To this end, we assume in what follows that (II.6.9)–(II.6.15) hold and we denote by $(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\kappa})$ the solution of Problem $\mathcal{N}^V$ obtained in Theorem II.6.1. For each $\rho > 0$ let p_ρ, b_ρ, $\boldsymbol{f}_{0\rho}$, $\boldsymbol{f}_{2\rho}$, $\boldsymbol{u}_{0\rho}$, $\boldsymbol{\sigma}_{0\rho}$ and $\boldsymbol{\kappa}_{0\rho}$ represent perturbations of p, b, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$ and $\boldsymbol{\kappa}_0$, respectively, which satisfy conditions (II.6.12)–(II.6.15). In addition, for every $\rho > 0$ we define the operators $P_\rho : V \to V$, $\mathcal{B}_\rho : C(\mathbb{R}_+;V) \to C(\mathbb{R}_+;L^2(\Gamma_3))$ and the function $\boldsymbol{f}_\rho : \mathbb{R}_+ \to V$ by equalities

$$(P_\rho\boldsymbol{u},\boldsymbol{v})_V = \int_{\Gamma_3} p_\rho(u_\nu)v_\nu\, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{II.6.23}$$

$$\begin{aligned}(\mathcal{B}_\rho\boldsymbol{u}(t),\xi)_{L^2(\Gamma_3)} = \Big(\int_0^t b_\rho(t-s)\, u_\nu^+(s)\, ds, \xi\Big)_{L^2(\Gamma_3)}\\ \forall\, \boldsymbol{u} \in C(\mathbb{R}_+;V),\ \xi \in L^2(\Gamma_3),\ t \in \mathbb{R}_+,\end{aligned} \tag{II.6.24}$$

$$\begin{aligned}(\boldsymbol{f}_\rho(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_{0\rho}(t)\cdot\boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_{2\rho}(t)\cdot\boldsymbol{v}\, da\\ \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+.\end{aligned} \tag{II.6.25}$$

With these data, we consider the following perturbation of Problem $\mathcal{N}^V$.

Problem $\mathcal{N}_\rho^V$. *Find a displacement field* $\boldsymbol{u}_\rho : \mathbb{R}_+ \to U$, *a stress field* $\boldsymbol{\sigma}_\rho : \mathbb{R}_+ \to Q$ *and an internal state variable* $\boldsymbol{\kappa}_\rho : \mathbb{R}_+ \to L^2(\Omega)^m$ *such that*

$$\boldsymbol{\sigma}_\rho(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s)), \boldsymbol{\kappa}_\rho(s))\,ds + \boldsymbol{\sigma}_{0\rho} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \quad \text{(II.6.26)}$$

$$\boldsymbol{\kappa}_\rho(t) = \int_0^t \boldsymbol{G}(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s)), \boldsymbol{\kappa}_\rho(s))\,ds + \boldsymbol{\kappa}_{0\rho}, \quad \text{(II.6.27)}$$

$$(\boldsymbol{\sigma}_\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (P_\rho \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V + \Big(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_{\rho\nu}^+(t)\Big)_{L^2(\Gamma_3)} \geq (\boldsymbol{f}_{,}\boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \quad \forall\, \boldsymbol{v} \in V, \quad \text{(II.6.28)}$$

for all $t \in \mathbb{R}_+$.

It follows from Theorem II.6.1 that, for each $\rho > 0$, Problem $\mathcal{N}_\rho^V$ has a unique solution $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho, \boldsymbol{\kappa}_\rho)$ with the regularity $\boldsymbol{u}_\rho \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma}_\rho \in C(\mathbb{R}_+; Q)$ and $\boldsymbol{\kappa}_\rho \in C(\mathbb{R}_+; L^2(\Omega)^m)$. Consider now the following assumptions:

$$\left.\begin{array}{l} \text{There exists } F : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } \alpha \in \mathbb{R}_+ \text{ such that} \\ \text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \leq F(\rho)(|r| + \alpha) \\ \qquad \forall\, r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for each } \rho > 0. \\ \text{(b) } F(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \end{array}\right\} \quad \text{(II.6.29)}$$

$$b_\rho \to b \quad \text{in } C(\mathbb{R}_+; L^\infty(\Gamma_3)) \quad \text{as} \quad \rho \to 0. \quad \text{(II.6.30)}$$

$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in } C(\mathbb{R}_+; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0. \quad \text{(II.6.31)}$$

$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in } C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0. \quad \text{(II.6.32)}$$

$$\boldsymbol{u}_{0\rho} \to \boldsymbol{u}_0 \quad \text{in } V \quad \text{as} \quad \rho \to 0. \quad \text{(II.6.33)}$$

$$\boldsymbol{\sigma}_{0\rho} \to \boldsymbol{\sigma}_0 \quad \text{in } Q \quad \text{as} \quad \rho \to 0. \quad \text{(II.6.34)}$$

$$\boldsymbol{\kappa}_{0\rho} \to \boldsymbol{\kappa}_0 \quad \text{in } L^2(\Omega)^m \quad \text{as} \quad \rho \to 0. \quad \text{(II.6.35)}$$

We have the following convergence result.

Theorem II.6.4 *Assume that* (II.6.29)–(II.6.35) *hold. Then the solution* $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho, \boldsymbol{\kappa}_\rho)$ *of Problem* $\mathcal{N}_\rho^V$ *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\kappa})$ *of*

Problem $\mathcal{N}^V$, *i.e.*

$$\left.\begin{aligned} \boldsymbol{u}_\rho \to \boldsymbol{u} \quad & \text{in} \quad C(\mathbb{R}_+; V), \\ \boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad & \text{in} \quad C(\mathbb{R}_+; Q), \\ \boldsymbol{\kappa}_\rho \to \boldsymbol{\kappa} \quad & \text{in} \quad C(\mathbb{R}_+; L^2(\Omega)^m) \end{aligned}\right\} \qquad \text{(II.6.36)}$$

as $\rho \to 0$.

References

[1] U. Abel, M. Ivan, X. M. Zeng, *Asymptotic Expansion for Szàsz-Mirakjan Operators*, Proceedings of Numerical Analysis and Applied Mathematics: International Conference of Numerical Analysis and Applied Mathematics, 2007.

[2] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] O. Agratini, On a sequence of linear and positive operators, *Facta Universitatis (Niš), Ser. Mat. Inform.* **14** (1999), 41-48.

[4] G. A. Anastassiou, On basic fuzzy Korovkin theory, *Studia Univ. "Babeş - Bolyai" Math.*, **4** 2005, 3–10.

[5] G. A. Anastassiou, $\mathcal{A}$ *- summability and fuzzy Korovkin approximation, in Fuzzy Mathematics:Approximation Theory*, Springer, 2010.

[6] G. A. Anastassiou, *Transfers of real approximations to vectorial and fuzzy setting, in Intelligent Mathematics: Computational Analysis*, Springer, 2011.

[7] G. A. Anastassiou, O. Duman, Statistical fuzzy approximation by fuzzy positive linear operators, *Computers and Mathematics with Applications*, **55** 2008, 573–580.

[8] S.S. Antman, *Nonlinear Problems of Elasticity*, Springer-Verlag, New York, 1995.

[9] K. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, Texts in Applied Mathematics **39**, Springer, New York, 2001.

[10] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems*, John Wiley, Chichester, 1984.

[11] H.T. Banks, G.A. Pinter, L.K. Potter, B.C. Munoz and L.C. Yanyo, Estimation and control related issues in smart material structure and fluids, *Optimization Techniques and Applications*, L. Caccetta et al., Eds., Curtain University Press 1998, 19–34.

[12] H.T. Banks, G.A. Pinter, L.K. Potter, J.M. Gaitens and L.C. Yanyo, Modeling of quasistatic and dynamic load responses of filled viesoelastic materials, Chapter 11 in *Mathematical Modeling : Case Studies from Industry*, E. Cumberbatch and A. Fitt, Eds., Cambridge University Press 2011, 229–252.

[13] H.T. Banks, S. Hu and Z.R. Kenz, A brief review of elasticity and viscoelasticity for solids, *Adv. Appl. Math. Mech.* **3** (2011), 1–51.

[14] M. Barboteu, F. Pătrulescu, A. Ramadan and M. Sofonea, History-dependent contact models for viscoplastic materials, *IMA Journal of Applied Mathematics* (submitted).

[15] V. Barbu, *Optimal Control of Variational Inequalities*, Pitman, Boston, 1984.

[16] V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Spaces*, D. Reidel Publishing Company, Dordrecht, 1986.

[17] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books in Mathematics, Springer, New York, 2011.

[18] S. N. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités, Comm. Kharkov math. Soc. **13**(1912), 1–2.

[19] H. Brézis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier* **18** (1968), 115–175.

[20] H. Brézis, Problèmes unilatéraux, *J. Math. Pures et Appl.* **51** (1972), 1–168.

[21] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, Mathematics Studies, North Holland, Amsterdam, 1973.

[22] H. Brézis, *Analyse fonctionnelle–Théorie et applications*, Masson, Paris, 1987.

[23] P.G. Ciarlet, *Mathematical Elasticity, Volume I: Three Dimensional Elasticity*, Studies in Mathematics and its Applications **20**, North-Holland, Amsterdam, 1988.

[24] C. Ciulcu, D. Motreanu and M. Sofonea, Analysis of an elastic contact problem with slip dependent coefficient of friction, *Mathematical Inequalities & Applications* **4** (2001), 465–479.

[25] M. Cocu, Existence of solutions of Signorini problems with friction, *Int. J. Engng. Sci.* **22** (1984), 567–581.

[26] M. Cocu, E. Pratt and M. Raous, Existence d'une solution du problème quasistatique de contact unilatéral avec frottement non local, *C. R. Acad. Sci. Paris,* **320**, Série I (1995), 1413–1417.

[27] M. Cocu, E. Pratt and M. Raous, Formulation and approximation of quasistatic frictional contact, *Int. J. Engng. Sci.* **34** (1996), 783–798.

[28] M. Cocu and J.M. Ricaud, Analysis of a class of implicit evolution inequalities associated to dynamic contact problems with friction, *Int. J. Engng. Sci.* **328** (2000), 1534–1549.

[29] Wu Congxin, Z. Gong, On Henstock integral of fuzzy number valued functions, *Fuzzy Sets and Systems*, **3** 2001, 523–532.

[30] Wu Congxin, Ma Ming, On embedding problem of fuzzy number space: Part 1, *Fuzzy Sets and Systems*, **44** 1991, 33–38.

[31] C. Corduneanu, Problèmes globaux dans la théorie des équations intégrales de Volterra, *Ann. Math. Pure Appl.* **67** (1965), 349–363.

[32] N. Cristescu and I. Suliciu, *Viscoplasticity*, Martinus Nijhoff Publishers, Editura Tehnică Bucharest, 1982.

[33] Z. Denkowski, S. Migórski and A. Ochal, Existence and uniqueness to a dynamic bilateral frictional contact problem in viscoelasticity, *Acta Appl. Math.* **94** (2006), 251–276.

[34] F. Dirik, K. Demirici, Korovkin type approximation theorem for functions of two variables in statistical sense, *Turk J. Math.* **34** (2010), 73–83.

[35] I. Doghri, *Mechanics of Deformable Solids*, Springer, Berlin, 2000.

[36] O. Doğru, Weighted approximation of continuous functions on the all positive axis by modified linear positive operators, *Int. J. Comput. Numerical Anal. Appl.* **1**(2) (2002) 135-147.

[37] O. Doğru, M. A. Özarslan and F. Taşdelen, On positive operators involving a certain class of generating functions, *Studia Scientiarum Mathematicarum Hungarica* **41**(4) (2004) 415-429.

[38] S. Drabla and M. Sofonea, Analysis of a Signorini problem with friction, *IMA Journal of Applied Mathematics* **62** (1999), 1–18.

[39] A.D. Drozdov, *Finite Elasticity and Viscoelasticity—A Course in the Nonlinear Mechanics of Solids*, World Scientific, Singapore, 1996.

[40] O. Duman, C. Orhan, Statistical approximation by positive linear operators, *Studia Math.*, **161**(2) (2006), 187–197.

[41] G. Duvaut, Loi de frottement non locale, *J. Méc. Thé. Appl.* Special issue (1982), 73–78.

[42] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[43] C. Eck, J. Jarušek and M. Krbeč, *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics **270**, Chapman/CRC Press, New York, 2005.

[44] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.

[45] E. Erkuş, O. Duman, A Korovkin type approximation theorem in statistical sense, *Studia Sci. Math. Hungar.*, **43** (2006), 285–294.

[46] **A. Farcaş**, An asymptotic formula for Jain's operators, Studia Univ. Babes - Bolyai, Mathematica, **57**(4)(2012), 511–517.

[47] **A. Farcaş**, On some fuzzy positive and linear operators, *Proceedings of the Second International Conference "Modelling and Development of Intelligent Systems", September 29–October 02, 2011, Sibiu, Romania*, ed. Dana Simian, Lucian Blaga Univ. Press, 45–52.

[48] **A. Farcaş**, An approximation property of the generalized Jain's operators of two variables (accepted in Applied Mathematics, Mathematical Sciences & Applications E-Notes).

[49] **A. Farcaş**, F. Pătrulescu and M. Sofonea, A history-dependent contact problem with unilateral constraint, *Ann. Acad. Rom. Sci. Ser. Math. Appl.* **4**, no. 1 (2012), 90–96.

[50] J. Favard, Sur les multiplicateurs d'interpolation, Journal de Mathematique Pures et Appliques, **23**(9) (1944), 219–247.

[51] G. Fichera, Problemi elastostatici con vincoli unilaterali. II. Problema di Signorini con ambique condizioni al contorno, *Mem. Accad. Naz. Lincei, S. VIII, Vol. VII, Sez. I*, **5** (1964) 91–140.

[52] J. A. Fridy, On statistical convergence, *Analysis* **5** (1985), 301–313.

[53] S. G. Gal, *Approximation theory in fuzzy setting, in: Handbook of Analytic-Computational Methods in Applied Mathematics*, Chapman and Hall/CRC, 2000.

[54] P. Germain and P. Muller, *Introduction à la mécanique des milieux continus*, Masson, Paris, 1980.

[55] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[56] R. Glowinski, J.-L. Lions and R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.

[57] R. Goetschel Jr., W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets and Systems*, **18** 1986, 31–43.

[58] A. Guran, F. Pfeiffer and K. Popp, eds., *Dynamics with Friction: Modeling, Analysis and Experiment, Part I*, World Scientific, Singapore, 1996.

[59] M.E. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, New York, 1981.

[60] H. J. Hamilton, Transformations of multiple sequences, *Duke Math. J.* **2** (1936), 29–60.

[61] W. Han and B.D. Reddy, *Plasticity: Mathematical Theory and Numerical Analysis*, Springer-Verlag, New York, 1999.

[62] W. Han and M. Sofonea, Evolutionary Variational inequalities arising in viscoelastic contact problems, *SIAM Journal of Numerical Analysis* **38** (2000), 556–579.

[63] W. Han and M. Sofonea, Time-dependent variational inequalities for viscoelastic contact problems, *Journal of Computational and Applied Mathematics* **136** (2001), 369–387.

[64] J. Haslinger, I. Hlaváček and J. Nečas, Numerical methods for unilateral problems in solid mechanics, *Handbook of Numerical Analysis, Vol. IV*, P.G. Ciarlet and J.-L. Lions, eds., North-Holland, Amsterdam, 1996, 313–485.

[65] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics **30**, American Mathematical Society–International Press, 2002.

[66] J. Haslinger, I. Hlaváček and J. Nečas, Numerical methods for unilateral problems in solid mechanics, *Handbook of Numerical Analysis, Vol. IV*, P.G. Ciarlet and J.-L. Lions, eds., North-Holland, Amsterdam, 1996, 313–485.

[67] I. Hlaváček, J. Haslinger, J. Nečas and J. Lovíšek, *Solution of Variational Inequalities in Mechanics*, Springer-Verlag, New York, 1988.

[68] I.R. Ionescu and M. Sofonea, *Functional and Numerical Methods in Viscoplasticity*, Oxford University Press, Oxford, 1993.

[69] N. Ispir, Ç. Atakut , Approximation by modified Szàsz-Mirakjan operators on weighted spaces, *Proc. Indian Acad. Sci. (Math. Sci.)* **112**(4) (2002) 571-578.

[70] G. C. Jain, Approximation of functions by a new class of linear operators, *Journal of Australian Math. Society*, **13**(1972), no.3, 271–276.

[71] J. Jarušek and M. Sofonea, On the solvability of dynamic elastic-visco-plastic contact problems, *Zeitschrift für Angewandte Mathematik und Mechanik* (*ZAMM*) **88** (2008), 3–22.

[72] K.L. Johnson, *Contact Mechanics*, Cambridge University Press, Cambridge, 1987.

[73] A.M. Khludnev and J. Sokolowski, *Modelling and Control in Solid Mechanics*, Birkhäuser-Verlag, Basel, 1997.

[74] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, Siam, Philadelphia, 1988.

[75] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Classics in Applied Mathematics **31**, SIAM, Philadelphia, 2000.

[76] A. Klarbring, A. Mikelič and M. Shillor, Frictional contact problems with normal compliance, *Int. J. Engng. Sci.* **26** (1988), 811–832.

[77] A. Klarbring, A. Mikelič and M. Shillor, On friction problems with normal compliance, *Nonlinear Analysis* **13** (1989), 935–955.

[78] E. Kolk, Matrix summability of statistically convergent sequences, *Analysis*, **13** (1993), 77–83.

[79] P.P. Korovkin, *On convergence of linear positive operators in the space of continuous functions* Dokl. Akad. Nauk. SSSR (N.S.) , 90 (1953) 961-964 (In Russian)

[80] P.P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Corp., Delhi, India, 1960.

[81] A.J. Kurdila, M. Zabarankin, *Convex Functional Analysis*, Birkhäuser, Basel, 2005.

[82] T.A. Laursen, *Computational Contact and Impact Mechanics*, Springer, Berlin, 2002.

[83] J. Lemaître and J.-L. Chaboche, *Mechanics of Solids Materials*, Cambridge University Press, Cambridge, 1990.

[84] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Gauthiers-Villars, Paris, 1969.

[85] J.-L. Lions and E. Magenes, *Problèmes aux limites non-homogènes I*, Dunod, Paris, 1968.

[86] L.E. Malvern, *Introduction to the Mechanics of a Continuum Medium*, Princeton-Hall, Inc., New Jersey, 1969.

[87] J.A.C. Martins and M.D.P. Monteiro Marques, eds., *Contact Mechanics*, Kluwer, Dordrecht, 2002.

[88] J.A.C. Martins and J.T. Oden, Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws, *Nonlinear Analysis TMA* **11** (1987), 407–428.

[89] J. J. Massera and J. J. Schäffer, *Linear Differential Equations and Function Spaces*, Academic Press, New York-London, 1966.

[90] S. Migórski, A. Ochal and M. Sofonea, Analysis of lumped models with contact and friction, *Journal of Applied Mathematics and Physics* (*ZAMP*) **62** (2011), 99–113.

[91] S. Migórski, A. Ochal and M. Sofonea, Analysis of a frictional contact problem for viscoelastic materials with long memory, *Discrete and Continuous Dynamical Systems, Series B* **15** (2011), 687–705.

[92] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics, **26**, Springer, New York, 2012.

[93] G. M. Mirakjan, Approximation of continuous functions with the aid of polynomials, Dokl. Akad. Nauk SSSR, **31** (1941), 201–205.

[94] D. Motreanu and M. Sofonea, Evolutionary variational inequalities arising in quasistatic frictional contact problems for elastic materials, *Abstract and Applied Analysis* **4** (1999), 255–279.

[95] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Praha, 1967.

[96] J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction*, Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York, 1981.

[97] J.T. Oden and J.A.C. Martins, Models and computational methods for dynamic friction phenomena, *Computer Methods in Applied Mechanics and Engineering* **52** (1985), 527–634.

[98] P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.

[99] P. D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.

[100] F. Pătrulescu, **A. Farcaş**, A. Ramadan, On a penalized viscoplastic contact problem with unilateral constraint,*Ann. Univ. Buchar. Math. Ser*, 4 (LXII) (2013), 213-227.

[101] A. C. Pipkin, *Lectures in Viscoelasticity Theory*, Applied Mathematical Sciences **7**, George Allen & Unwin Ltd., London, Springer-Verlag, New York, 1972.

[102] T. Popoviciu, *On the proof of the Weierstrass theorem using interpolation polynomials* Lucr. Ses. Gen. Stiintific. , 2 : 12 (1950) 1664-1667.

[103] A. Pringsheim, Zur theorie der zweifach unendlichen zahlenfolgen, *Math. Ann.* **53** (1900), 289–321.

[104] E. Rabinowicz, *Friction and Wear of Materials*, 2^{nd} edition, Wiley, New York, 1995.

[105] M. Raous, M. Jean and J.J. Moreau, eds., *Contact Mechanics*, Plenum Press, New York, 1995.

[106] B.D. Reddy, *Introductory Functional Analysis with Applications to Boundary Value Problems and Finite Elements*, Springer, New York, 1998.

[107] L. Rempulska, M. Skorupka, The Voronovskaja theorem for some operators of the Szàsz-Mirakjan type, *Le Matematiche*, L(II)(1995), 251–261.

[108] G. M. Robinson, Divergent double sequences and series, *Amer. Math. Soc. Transl.* **28** (1926), 50–73.

[109] A. D. Rodríguez–Aros, M. Sofonea and J. M. Viaño, A class of evolutionary variational inequalities with Volterra-type integral term, *Math. Models Methods Appl. Sci.* **14** (2004), 555–577.

[110] A. D. Rodríguez–Aros, M. Sofonea and J. M. Viaño, Numerical approximation of a viscoelastic frictional contact problem, *C. R. Acad. Sci. Paris, Sér. II Méc.* **334** (2006), 279–284.

[111] A. D. Rodríguez–Aros, M. Sofonea and J. M. Viaño, Numerical analysis of a frictional contact problem for viscoelastic materials with long-term memory, *Numerische Mathematik* **198** (2007), 327–358.

[112] M. Shillor, ed., Recent advances in contact mechanics, Special issue of *Math. Comput. Modelling* **28** (4–8) (1998).

[113] M. Shillor and M. Sofonea, A quasistatic viscoelastic contact problem with friction, *Int. J. Engng. Sci.* **38** (2000), 1517–1533.

[114] M. Shillor, M. Sofonea and J.J. Telega, *Models and Analysis of Quasistatic Contact*, Lecture Notes in Physics **655**, Springer, Berlin, 2004.

[115] A. Signorini, Sopra alcune questioni di elastostatica, *Atti della Società Italiana per il Progresso delle Scienze*, 1933.

[116] M. Sofonea, C. Avramescu and A. Matei, A fixed point result with applications in the study of viscoplastic frictionless contact problems, *Communications on Pure and Applied Analysis* **7** (2008), 645–658.

[117] M. Sofonea and **A. Farcaş**, Analysis of a History-dependent Frictional Contact Problem, *Applicable Analysis*, DOI: 10.1080/00036811.2013.778981.

[118] M. Sofonea and A. Matei, *Variational Inequalities with Applications. A Study of Antiplane Frictional Contact Problems*, Advances in Mechanics and Mathematics **18**, Springer, New York, 2009.

[119] M. Sofonea and A. Matei, History-dependent quasivariational inequalities arising in Contact Mechanics, *European Journal of Applied Mathematics* **22** (2011), 471–491.

[120] M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Notes, Cambridge University Press **398**, Cambridge, 2012.

[121] M. Sofonea and F. Pătrulescu, Analysis of a history-dependent frictionless contact problem, *Mathematics and Mechanics of Solids*, **18** (2013), 409-430.

[122] M. Sofonea, F. Pătrulescu, **A. Farcaş**, A viscoplastic contact problem with normal compliance, unilateral constraint and memory term, *Applied Mathematics and Optimization*, DOI: 10.1007/s00245-013-9216-2.

[123] M. Sofonea, A. D. Rodríguez–Aros and J. M. Viaño, A class of integro-differential variational inequalities with applications to viscoelastic contact, *Mathematical and Computer Modelling* **41** (2005), 1355–1369.

[124] D. D. Stancu, *A new class of uniform approximating polynomial operators in two and several variables*, Proceedings of the Conference on the Constructive Theory of Functions (Approximation Theory) (Budapest, 1969) pp. (Budapest: Akadémiai Kiadó) (1972) 443-455.

[125] H. Steinhaus, Sur al convergence ordinaire et la convergence asymptotique, *Colloq. Math.* **2** (1951), 73–74.

[126] O. Szász, Generalization of S. Bernsteins polynomials to the infinite interval, J. of Research of the Nat. Bur. of Standards, 45 (1950), 239-245.

[127] J.J. Telega, Topics on unilateral contact problems of elasticity and inelasticity, in J.J. Moreau and P.D. Panagiotopoulos, eds., *Nonsmooth Mechanics and Applications*, Springer-Verlag, Wien, 1988, 340–461.

[128] R. Temam and A. Miranville, *Mathematical Modeling in Continuum Mechanics*, Cambridge University Press, Cambridge, 2001.

[129] Yu. I. Volkov, Certain positive linear operators, *Translated from Mathematicheskie Zametki*, **23**(5), 1978, 659–669.

[130] E. V. Voronovskaja, Détermination de la forme asymptotique de l'approximation des functions par les polynômes de S. Bernstein, *Dokl. Akad. Nauk. SSSR*, **A** (1832), 79–85.

[131] Z. Walczak, Approximation of functions of two variables by some linear positive operators, *Acta. Math. Univ. Comenianae* **LXXIV**(1) (2005) 37-48.

[132] P. Wriggers, *Computational Contact Mechanics*, Wiley, Chichester, 2002.

[133] P. Wriggers and U. Nackenhorst, eds., *Analysis and Simulation of Contact Problems*, Lecture Notes in Applied and Computational Mechanics **27**, Springer, Berlin, 2006.

[134] P. Wriggers and P.D. Panagiotopoulos, eds., *New Developments in Contact Problems*, Springer-Verlag, Wien, New York, 1999.

[135] E. Zeidler, *Nonlinear Functional Analysis and its Applications.* I*: Fixed-point Theorems*, Springer-Verlag, New York, 1986.

[136] E. Zeidler, *Nonlinear Functional Analysis and its Applications.* III*: Variational Methods and Optimization*, Springer-Verlag, New York, 1986.

[137] E. Zeidler, *Nonlinear Functional Analysis and its Applications.* IV*: Applications to Mathematical Physics*, Springer-Verlag, New York, 1988.

[138] E. Zeidler, *Nonlinear Functional Analysis and its Applications,* II/A*: Linear Monotone Operators*, Springer-Verlag, New York, 1990.

[139] E. Zeidler, *Nonlinear Functional Analysis and its Applications,* II/B*: Nonlinear Monotone Operators*, Springer-Verlag, New York, 1990.

[140] E. Zeidler, *Applied Functional Analysis: Main Principles and Their Applications*, Springer-Verlag, New York, 1995.