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Optimality Conditions and Approximation of Optimization Problems

Ph.D. Thesis - Summary

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Contents

Introduction

Mathematical optimization (alternatively, mathematical programming or optimization) represents a vast area of research in mathematics, which consists to maximize or minimize an objective function by some conditions called constraints. Semi-infinite programming (SIP) is a subclasses of optimization and probably one of the oldest branches of mathematical programming. It contains convex and nonconvex optimization problems.

In 1924, we can found the first description of semi-infinite programming as Chebyshev approximation, but the name was coined in 1962 by Charnes, Cooper and Kortanek, in papers (see, [14], [15], [16]), about linear semi-infinite programming.

Semi-infinite programming is an optimization problem characterized by a finite number of variables and an infinite number of constraints, or an infinite number of variables and a finite number of constraints, so is the name semi-infinite. In last decades, the area of semi-infinite programming is very well developed in terms of theoretical and practical results. We mention for an introduction to (SIP) the paper by Hettich and Kortanek [36], for linear semi-infinite optimization and algorithms (see, [31], [65]), for numerical methods [37] and for standard and generalized semi-infinite programming (see, [34], [84]). In 1973, Kortanek and Gustafson, developed the first numerical methods for (SIP) in [35]. There were several authors that investigated applications to approximation problems and numerical methods: Hettich and Zencke [37], Fiacco and Kortanek [27], Tichatschke [86], Glashoff and Gustafson [30].

This area of semi-infinite programming has many applications in various fields of mathematics, engineering and economics, such as: air pollution control [88] solved by discretization methods (see, [38], [39], [78]), reverse Chebyshev approximation [42], portfolio problem [54], optimal layout of an assembly line [89], problems of manoeuvrability of robots [40], time minimal control (see, [50], [52]), statistics [19], system and control [31], design centering (see, [33], [43], [64], [66]), identification of regression models, dynamics of networks in the presence of uncertainty [90], robust optimization (see, [5], [87]), transportation problems, fuzzy sets, cooperative games (see, [36], [67]), gemstone cutting [51].

The notion of duality in semi-infinite programming, has its roots in the theory of uniform approximation of functions, in the classical theory of moments and in the theory of systems of linear inequalities. There exists an extensive literature on duality of convex (SIP) problems. We can mention here: an approach based on conjugate duality [82], a Mond-Weir dual problem for a nonlinear semi-infinite programming problem using the concepts of generalized semilocally type I-preinvex functions [46], augmented Lagrange multipliers [81] or other approaches (see, [3], [4], [7], [8], [10], [28]).

The reader can find other researches for semi-infinite optimization problems in (see, [13],

[21], [29], [44]).

This thesis consists of five chapters, which are briefly presented in the following lines.

Chapter 1, entitled Preliminary notions and results, contains the most important definitions and results from convex analysis and vector optimization.

Chapter 2, is dedicated to a convex-concave semi-infinite optimization problem whose objective function is convex while the constraint is convex-concave. In this chapter, to solve the optimization problem, we attach to it a dual problem which provide information about optimal solution of original problem. We consider four dual problems related to our semi-infinite optimization problem and we have established duality relations between them. In the case of weak duality, the optimal value of original problem is greater than or equal to each of the optimal values of considered dual problems. Strong duality, namely that the optimal value of the original problem is equal with the optimal value of the dual problems, achieved under different convexity assumptions and regularity conditions often called constraint qualifications.

In Section 2.1, we introduce a new dual type called (D_1). We establish relations between this dual and other three dual problems which are known in the literature. Weak duality is also established. To study strong duality, the dual problems (D_2), (D_3) are reformulated and we obtain three duals $(\widetilde{D_2})$, $(\widetilde{D_3})$, $(\widetilde{D_4})$.

In Section 2.2, we present some numerical results of our theoretical part, using Matlab program to find the optimal value of a problem.

The author's original contributions are presented below: Propositions: 2.1.1, 2.1.2, 2.1.5, 2.1.6, 2.1.9, 2.1.10, Examples: 2.2.1, 2.2.2, 2.2.3. Remarks: 2.1.7, 2.1.8.

The results of this area of research are included in the following paper: [76].

In **Chapter 3**, we consider a semi-infinite optimization problem and we propose to attach an η-approximated semi-infinite optimization problem, whose solutions will provide information about the optimal solutions of the original problem.

The idea of η-approximation a nonlinear mathematical programming problem appeared in papers (see, [1], [22]), in the case where index sets T and S are finite. This method was constructed by Antczak and was called η-approximated method. The novelty of results obtained in this chapter is that it is not require that the index sets T and S to be compact.

In Section 3.1, we constructed for a semi-infinite optimization problem, three first order η-approximated semi-infinite optimization problems. Then we establish connections between the feasible solutions of the original problem and the feasible solutions of $(0, 1) - \eta$ approximated semi-infinite optimization problem and the feasible solutions of the original problem and the feasible solutions of $(1, 1) - \eta$ approximated semi-infinite optimization problem. Some examples illustrating theoretical notions presented. In the next three subsections, the connections studied refers to the optimal solutions of $(1, 0) - \eta$ approximated semi-infinite optimization problem, $(0, 1) - \eta$ approximated semi-infinite optimization problem, $(1, 1) - \eta$ approximated semi-infinite optimization problem and the optimal solutions of original optimization problem. New results and examples are presented to establish the conditions when an optimal value of original problem is " $\leq$ " or " $\geq$ " then the optimal value of Problem $(P_{1,1})$. In the last part of this section, connections between optimal solutions of first order η-approximated semi-infinite optimization problems are formulated.

In Section 3.2, we deal with the same optimization problem and we constructed five second order η-approximated semi-infinite optimization problem. Other new connections between:

the feasible solutions of second order η-approximated semi-infinite optimization problems and the feasible solutions of the original problem, feasible solutions of first order η-approximated semi-infinite optimization problems and feasible solutions of second order η-approximated semi-infinite optimization problems are presented. Some theorems ensures, under different hypothesis, that any optimal solution of original optimization problem is an optimal solution of its η-approximated problem and vice-versa. Then we study the connections between the optimal solutions of second order η-approximated semi-infinite optimization problems. And finally, the last subsection provides connections between the optimal solutions of first and second order η-approximated semi-infinite optimization problems.

The author's original contributions are presented below: Theorems: 3.1.1, 3.1.3, 3.1.6, 3.1.7, 3.1.8, 3.1.9, 3.1.10, 3.1.11, 3.1.12, 3.1.14, 3.1.17, 3.1.18, 3.1.19, 3.1.20, 3.1.21, 3.1.22, 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.2.5, 3.2.6, 3.2.7, 3.2.8, 3.2.9, 3.2.10, 3.2.11, 3.2.12, 3.2.13, 3.2.14, 3.2.15, 3.2.16, 3.2.17, 3.2.18, 3.2.19, 3.2.20, 3.2.21, 3.2.22, 3.2.23, 3.2.24, 3.2.25, 3.2.26, 3.2.27, 3.2.28, 3.2.29, 3.2.30, 3.2.31, 3.2.32, 3.2.33, 3.2.34, 3.2.35, 3.2.36, 3.2.37, 3.2.38, 3.2.39, 3.2.40, 3.2.41, 3.2.42, 3.2.43, 3.2.44, 3.2.45, 3.2.46, 3.2.47, 3.2.48, 3.2.49, 3.2.50, 3.2.51, 3.2.52, 3.2.53, 3.2.54, 3.2.55, 3.2.56, 3.2.57, 3.2.58, 3.2.59, 3.2.60, 3.2.61, 3.2.62, 3.2.63, 3.2.64. Examples: 3.1.2, 3.1.4, 3.1.5, 3.1.13, 3.1.15, 3.1.16.

The results of this area of research are included in the following papers: [70], [71], [72], [73], [74].

Chapter 4, deals with the study of a vector optimization problem.

In Section 4.1, we study connections between the efficient solution of a vector optimization problem and the efficient solution of first order η-approximated problems.

Section 4.2, is devoted to applications for solving a vector optimization problem. We propose two methods: weighting methods and constraint methods. After a brief summary concerning the notions, conditions for a point to be an efficient solution and algorithms for these two methods, we give some numerical examples. We have shown that a minimal value set of a problem is not a singleton in general and we can have also a subset of minimal solutions or a representative part of it. For every example, we constructed a table with results and a graphical representation, using Matlab program and RStudio.

The author's original contributions are presented below: Theorems: 4.1.2, 4.1.3, 4.1.4, 4.1.5, 4.1.6, 4.1.7. Remark: 4.1.8. Examples: 4.2.3, 4.2.4, 4.2.5, 4.2.6, 4.2.7, 4.2.8, 4.2.10, 4.2.11, 4.2.12, 4.2.13, 4.2.14, 4.2.15.

Some of the results of this area of research are included in the following chapter: [56] and paper [75].

Chapter 5, is devoted to solve the optimization problems using Jensen's inequality, Radon's inequality, Hölder's inequality, Liapunov's inequality and some bounds for statistical indicators. This chapter is split in two sections.

In Section 5.1, motivated by recent theoretical results to inequalities and many applications of these, we solve different optimization problems using inequalities.

In Section 5.2, one solve other optimization problems for the following bounds used in statistics: dispersion (variance), standard deviation and coefficient of variation.

The author's original contributions are presented below: Theorems: 5.1.3, 5.1.4, 5.1.9, 5.1.10, 5.1.11, 5.1.13, 5.1.14, 5.2.1, 5.2.2, 5.2.3. Remark: 5.2.4.

The results of this area of research are included in the following papers: [60], [77].

Keywords

semi-infinite optimization problem, conjugate function, infimal convolution, efficient solution, η-approximated semi-infinite optimization problem, invex function, optimal solution, weighting method, constraint method, convex-concave function, vector optimization problem, Radon's inequality, Jensen's inequality, Hölder's inequality, Liapunov's inequality, variance, standard deviation, coefficient of variation.

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Chapter 1

Preliminary notions and results

In this chapter we present some basic concepts and conventional notations from convex analysis and vector optimization, which will be used in this work. We can find these notions in publications such as (see, [11], [23], [41], [47], [57], [61], [79]).

1.1 Basic concepts

In this section we recall the notions which appear in convex analysis or in vector optimization theory (invex function, conjugate function, infimal convolution, efficient point).

1.2 Results

In this part of chapter are given necessary and sufficient conditions for a point to be solution for a semi-infinite optimization problem.

Chapter 2

Convex-concave semi-infinite optimization problem

The aim of this chapter is to study dual problems for a convex-concave semi-infinite optimization problem. In order to do this we consider four dual problems related to a given semi-infinite optimization problem and we establish the relations between the optimal objective values of these dual problems. Moreover, under some sufficient conditions, we study strong duality between the primal and the dual problems, respectively weak duality.

There is an extensive literature on duality and optimality conditions in convex optimization (see, [31], [36], [54]). In [83] Shapiro and in [26] Fang, Li, Ng, gave results on Lagrangian dualities in convex semi-infinite programming (SIP). Mishra, Jaiswal and Thi Hoai An formulated in [62] for a nonsmooth (SIP) problem Wolfe and Mond-Weir duals and established weak, strong and stricte converse duality.

The semi-infinite optimization problem was studied in various forms of restriction imposed on function g_t, $(t \in T)$. For example: f (the objective function), g_t are lower semicontinuous in papers (see, [20], [32]) or continuous in paper [9]. In our case the objective function f is convex, while the constraint g_t, $(t \in T)$ is convex-concave.

This chapter consists of two sections. In the first section we consider four dual problems, where the dual (D_1) is newly constructed, while others are known in the literature. We prove weak duality and then some relations between the optimal objective values of dual problems are given. To prove strong duality, we rewrite duals (D_2) and (D_3) and we obtain three dual problems. For strong duality we use also Sion's theorem (see, [85]).

In the second section we consider three numerical examples in order to justify the theoretical part of first section.

2.1 Theoretical results

In this section, we assume that C is a nonempty convex subset of a locally convex Hausdorff topological vector space X, T is a nonempty (possibly infinite) index set and f, $g_t : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $(t \in T)$ are proper convex functions.

We consider the optimization problem

$$\begin{array}{lll} \inf & f(x) & \\ \text{subject to} & x \in C & \\ & g_t(x) \leqq 0, \ (t \in T). & \end{array} \tag{P}$$

We propose to attach to Problem (P), the following four dual problems:
The first dual is

$$\sup\{\inf\{f(x) + \sum_{t \in \operatorname{supp}(\lambda)} \lambda_t g_t(x) : x \in C\} : \lambda \in \mathbb{R}_+^{(T)}\}. \tag{D}$$

The second dual is

$$\sup\{\inf\{f_C(x) + \lambda g_t(x) : x \in X\} : \lambda \in \mathbb{R}_+, \ t \in T\}, \tag{D$_1$}$$

where $f_C : X \to \overline{\mathbb{R}}$ is defined as

$$f_C(x) := f(x) + \delta_C(x), \text{ for all } x \in X.$$

The third dual is

$$\sup\{-(f_C^*(-x^*) + (\lambda g_t)^*(x^*)) : \lambda \in \mathbb{R}_+, \ x^* \in X^*, \ t \in T\}. \tag{D$_2$}$$

The four dual is

$$\sup\{-(f^*(-x^* - u^*) + \delta_C^*(u^*) + (\lambda g_t)^*(x^*)) : \lambda \in \mathbb{R}_+, \ x^* \in X^*, \ u^* \in X^*, \ t \in T\}. \tag{D$_3$}$$

Proposition 2.1.1 (**A. Raţiu** [76]) The following inequality

$$\inf(P) \geqq \max\{\sup(D), \ \sup(D_1), \ \sup(D_2), \ \sup(D_3)\},$$

holds.

In the following, we are going to give some relations between the optimal objective values of different dual problems we introduced above.

Proposition 2.1.2 (**A. Raţiu** [76]) The following inequalities hold:
(i) $\sup(D_1) \geqq \max\{\sup(D_2), \ \sup(D_3)\}$,
(ii) $\sup(D_2) \geqq \sup(D_3)$.

Next, to prove strong duality, we rewrite duals (D_2) and (D_3) and we obtain the following dual problems:

$$\sup\{-(f_C + \lambda g_t)^*(0) : \lambda \in \mathbb{R}_+, \ t \in T\}, \tag{$\widetilde{D_2}$}$$

$$\sup\{-(f_C^* \square (\lambda g_t)^*)(0) : \lambda \in \mathbb{R}_+, \ t \in T\}, \tag{$\widetilde{D_3}$}$$

$$\sup\{-(f^* \square \delta_C^* \square (\lambda g_t)^*)(0) : \lambda \in \mathbb{R}_+, \ t \in T\}. \tag{$\widetilde{D_4}$}$$

Proposition 2.1.5 (**A. Raţiu** [76]) The following inequality

$$\inf(P) \geqq \max\{\sup(D),\ \sup(D_1),\ \sup(\widetilde{D_2})\},$$

holds.

Proposition 2.1.6 (**A. Raţiu** [76]) If $\square$ is exact, then the following inequality

$$\inf(P) \geqq \max\{\sup(\widetilde{D_3}),\ \sup(\widetilde{D_4})\},$$

holds.

Remark 2.1.7 (**A. Raţiu** [76]) From above we have that the dual $(\widetilde{D_4})$ can be considered like another way to write the dual $(\widetilde{D_3})$.

Remark 2.1.8 (**A. Raţiu** [76]) If $\square$ is exact then the following inequality

$$\inf(P) \geqq \max\{\sup(D),\ \sup(D_1),\ \sup(\widetilde{D_2}),\ \sup(\widetilde{D_3}),\ \sup(\widetilde{D_4})\},$$

holds.

Proposition 2.1.9 (**A. Raţiu** [76]) If $\square$ is exact then the following relation

$$\sup(D_1) = \sup(\widetilde{D_2}) = \sup(\widetilde{D_3}) = \sup(\widetilde{D_4}),$$

holds.

Proposition 2.1.10 (**A. Raţiu** [76]) If the set T is convex, compact, and the functions $f,\ g_t : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $(t \in T)$ are convex, continuous, and $g_t(x)$ are concave in t for every fixed point $x \in X$, then we have

$$\inf(P) = \max\{\sup(D),\ \sup(D_1),\ \sup(\widetilde{D_2}),\ \sup(\widetilde{D_3}),\ \sup(\widetilde{D_4})\}.$$

2.2 Numerical results

To illustrate relations

$$\inf(P) \geqq \sup(D),$$

and

$$\inf(P) \geqq \sup(D_1) \geqq \sup(D_2) \geqq \sup(D_3),$$

several test problems are listed in the following and solved in Matlab program. Throughout the computational experiments, the function g_t is convex in x and concave in t and T is a convex compact set. To find the optimal value for Problem (P) and dual problems, we use functions defined in Matlab Optimization Toolbox, starting at an initial estimate.

Chapter 3

η-Approximated semi-infinite optimization problems

In this chapter, we consider a semi-infinite optimization problem (P) with inequality and equality constraints. An approach to obtain sufficient optimality conditions for an optimization problem and its dual has been introduced by Antczak in [1]. He constructed an η-approximated problem equivalent to the original problem and studied the connections between optimal solutions of these two problems. Other authors who have used the approximation method are: Duca and Duca [22], Boncea and Duca [6], Cioban and Duca [17], Pop and Duca [68].

To determine the nature of Problem (P) we have the following cases: if the index sets T and S are finite, then Problem (P) is a classic optimization problem and if the index sets T and\or S are infinite we have a semi-infinite optimization problem. To obtain the optimal solutions of a semi-infinite optimization problem, we attach an approximate optimization problem which is constructed by a first and second order η-approximation. We will analyze the relationships between Problem (P) with set constraint $X \subset \mathbb{R}^n$ and eight related approximation problems with the same set constraint and functional constraints depending on a point $x^0 \in intX$ and a function $\eta : X \times X \to X$. Under different assumptions on the objective and the constraint functions it is shown that any optimal solution of Problem (P) is an optimal solution of its approximations and vice-versa. Then we will study the connections between the optimal solutions of original optimization problem and optimal solutions of first and second order η-approximated semi-infinite optimization problems. Connections between the feasible solutions of Problem (P) and feasible solutions of first and second order η-approximated semi-infinite optimization problems are established. In the last section of this chapter we will study the connections between the optimal solutions of η-approximated semi-infinite optimization problems.

3.1 First order η-approximated semi-infinite optimization problems

In this section, we assume that X is a nonempty subset of $\mathbb{R}^n$, T and S are index sets, $\eta : X \times X \to X$ a function, x^0 be an interior point of X and $f : X \to \mathbb{R}$, $g_t : X \to \mathbb{R}$, $(t \in T)$ and $h_s : X \to \mathbb{R}$, $(s \in S)$ are differentiable at x^0.

We consider the optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & \\ & x \in X \\ & g_t(x) \leqq 0,\ (t \in T) \\ & h_s(x) = 0,\ (s \in S). \end{array} \tag{P}$$

Let

$$\mathcal{F}(P) := \{x \in X :\ g_t(x) \leqq 0,\ \ (t \in T)\,,\ h_s(x) = 0,\ \ (s \in S)\}\,,$$

denote the set of all feasible solutions for Problem (P) and

$$v\,(P) := \inf\{f(x) : x \in \mathcal{F}\,(P)\},$$

is the optimal value for Problem (P).

One of the manners to solve Problem (P) is to attach another optimization problem, whose solutions provide information about the optimal solutions of the initial Problem (P).

We propose to attach to Problem (P), the problems $(P_{j,k})$, $((j,k) \in \{(1,0),(0,1),(1,1)\})$,

$$\begin{array}{ll} \min & F^{\langle j \rangle}\,(x) \\ \text{subject to} & \\ & x \in X \\ & G_t^{\langle k \rangle}\,(x) \leqq 0,\ (t \in T) \\ & H_s^{\langle k \rangle}\,(x) = 0,\ (s \in S), \end{array} \tag{P$_{j,k}$}$$

called (j,k)-η approximated semi-infinite optimization problem, where $F^{\langle 1 \rangle}, G_t^{\langle 1 \rangle}, H_s^{\langle 1 \rangle} : X \to \mathbb{R}$ $(t \in T,\ s \in S)$ are defined by:

$$F^{\langle 1 \rangle}\,(x) := f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right),$$

$$G_t^{\langle 1 \rangle}\,(x) := g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x,x^0)\right),\ \ (t \in T)\,,$$

$$H_s^{\langle 1 \rangle}\,(x) := h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x,x^0)\right),\ \ (s \in S)\,,$$

for all $x \in X$, and $F^{\langle 0 \rangle} = f$, $G_t^{\langle 0 \rangle} = g_t$, $H_s^{\langle 0 \rangle} = h_s$, $(t \in T,\ s \in S)$.

In what follows, we denote by:

$$\mathcal{F}_0 := \left\{x \in X : G_t^{\langle 0 \rangle}(x) \leqq 0,\ \ (t \in T)\,,\ H_s^{\langle 0 \rangle}(x) = 0,\ \ (s \in S)\right\},$$

and

$$\mathcal{F}_1 := \left\{x \in X :\ G_t^{\langle 1 \rangle}\,(x) \leqq 0,\ (t \in T),\ H_s^{\langle 1 \rangle}\,(x) = 0,\ (s \in S)\right\}.$$

Let's remark that if $\mathcal{F}(P)$ denote the set of all feasible solutions for Problem (P), then $\mathcal{F}_0 = \mathcal{F}(P) = \mathcal{F}(P_{1,0})$, $\mathcal{F}_1 = \mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1})$.

3.1.1 Connections between the sets of feasible solutions of Problem (P) and first order η-approximated problems

In the following two theorems establish connections between the sets of feasible solutions of Problem (P) and the problems $(P_{0,1})$, $(P_{1,1})$.

Theorem 3.1.1 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

then

$$\mathcal{F}_0 \subseteq \mathcal{F}_1.$$

Following example shows that the inclusion $\mathcal{F}_0 \subseteq \mathcal{F}_1$ can be strict.

Example 3.1.2 (**A. Raţiu**, D.I. Duca [72]) Let $g_k : \mathbb{R}^2 \to \mathbb{R}$, $(k \in \mathbb{N})$, the function defined by

$$g_k(x) = \begin{cases} x_1^2 - x_2, & k = 1 \\ x_1 + x_2 - 2, & k = 2 \\ (x_1 - 2)^2 + (x_2 - 5)^2 - 17 - \frac{1}{k}, & k \in \mathbb{N}, k \geqq 3, \end{cases}$$

We observe that

$$\mathcal{F}_0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2 \leqq 0,\ x_1 + x_2 \leqq 2, (x_1 - 2)^2 + (x_2 - 5)^2 - 17 \leqq 0\}.$$

For the point $x^0 = (-2, 4)$ and the function $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\eta(x, y) = x - y$, for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have

$$G_1^{\langle 1 \rangle}(x) := g_1(x^0) + \left[\nabla g_1(x^0)\right](\eta(x, x^0)) = -4x_1 - x_2 - 4,$$

$$G_2^{\langle 1 \rangle}(x) := g_2(x^0) + \left[\nabla g_2(x^0)\right](\eta(x, x^0)) = x_1 + x_2 - 2,$$

$$G_k^{\langle 1 \rangle}(x) := g_k(x^0) + \left[\nabla g_k(x^0)\right](\eta(x, x^0)) = -8x_1 - 2x_2 - 8 - \frac{1}{k},\ k \in \mathbb{N}, k \geqq 3,$$

for all $x \in X$.

Then,

$$\begin{aligned} \mathcal{F}_1 = \ & \{(x_1, x_2) \in \mathbb{R}^2 : -4x_1 - x_2 \leqq 4, \\ & x_1 + x_2 \leqq 2,\ -8x_1 - 2x_2 \leqq 8 + \tfrac{1}{k}; k \in \mathbb{N}, k \geqq 3\}, \end{aligned}$$

hence,

$$\mathcal{F}_0 \subseteq \mathcal{F}_1,$$

and

$$\mathcal{F}_1 \neq \mathcal{F}_0,$$

because,

$$(0, 0) \in \mathcal{F}_1 \setminus \mathcal{F}_0.$$

Theorem 3.1.3 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. If

(a) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

then

$$\mathcal{F}_1 \subseteq \mathcal{F}_0.$$

Following two examples shows that the inclusion $\mathcal{F}_1 \subseteq \mathcal{F}_0$ can be strict.

Example 3.1.4 (**A. Raţiu**, D.I. Duca [70]) Let $g_k : \mathbb{R}^2 \to \mathbb{R}$, $(k \in \mathbb{N})$ the function defined by

$$g_k(x) = \begin{cases} -x_1, & k = 1 \\ -x_2, & k = 2 \\ x_1 x_2 - \frac{1}{k}, & k \in \mathbb{N}, k \geqq 3. \end{cases}$$

We observe that

$$\mathcal{F}_0 = (\{0\} \times [0, +\infty[) \cup ([0, +\infty[\times \{0\}).$$

For the point $x^0 = (1, 0)$ and the function $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\eta(x, y) = x - y$, for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have

$$G_1^{\langle 1 \rangle}(x) := g_1(x^0) + \left[\nabla g_1(x^0)\right] \left(\eta(x, x^0)\right) = -x_1,$$

$$G_2^{\langle 1 \rangle}(x) := g_2(x^0) + \left[\nabla g_2(x^0)\right] \left(\eta(x, x^0)\right) = -x_2,$$

$$G_k^{\langle 1 \rangle}(x) := g_k(x^0) + \left[\nabla g_k(x^0)\right] \left(\eta(x, x^0)\right) = x_2 - \frac{1}{k},\ k \in \mathbb{N}, k \geqq 3,$$

for all $x \in X$.

Then,

$$\mathcal{F}_1 = [0, +\infty[\times \{0\}.$$

We remark that

$$\mathcal{F}_1 \subseteq \mathcal{F}_0.$$

Moreover,

$$\mathcal{F}_1 \neq \mathcal{F}_0,$$

because $(0, 1) \in \mathcal{F}_0 \setminus \mathcal{F}_1$.

Example 3.1.5 (**A. Raţiu**, D.I. Duca [70]) Let $g_k : \mathbb{R}^2 \to \mathbb{R}$, $(k \in \mathbb{N})$, the function defined by

$$g_k(x) = \begin{cases} -x_2, & k = 1 \\ -(x_1 - 2)^2 - (x_2 - 1)^2 + 1, & k = 2 \\ -x_1 - x_2 + \frac{k+1}{k+2}, & k \in \mathbb{N}, k \geqq 3 \end{cases}$$

For the point $x^0 = (1, 0)$ and the function $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\eta(x, y) = x - y$, for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have:

$$G_1^{\langle 1 \rangle}(x) := g_1(x^0) + \left[\nabla g_1(x^0)\right] \left(\eta(x, x^0)\right) = -x_2,$$

$$G_2^{\langle 1 \rangle}(x) := g_2(x^0) + \left[\nabla g_2(x^0)\right] \left(\eta(x, x^0)\right) = 2x_1 + 2x_2 - 3,$$

$$G_k^{\langle 1\rangle}(x) := g_k(x^0) + \left[\nabla g_k(x^0)\right]\left(\eta(x,x^0)\right) = -x_1 - x_2 + \frac{k+1}{k+2},\ k \in \mathbb{N}, k \geqq 3,$$

for all $x \in X$.

Obviously, we have

$$\mathcal{F}_1 \subseteq \mathcal{F}_0,$$

and

$$\mathcal{F}_1 \neq \mathcal{F}_0,$$

because $(2,2) \in \mathcal{F}_0 \setminus \mathcal{F}_1$.

3.1.2 Approximate problem $(P_{1,0})$

The goal of this section is to establish the connections between the optimal solutions of Problem (P) and the optimal solutions of approximated problem $(P_{1,0})$.

Following theorem shows that in some hypothesis, any optimal solution for Problem $(P_{1,0})$ is an optimal solution for Problem (P).

Theorem 3.1.6 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,

(b) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,0})$, then x^0 is an optimal solution for Problem (P).

Following theorem tells us when an optimal solution for Problem (P) is an optimal solution for approximated Problem $(P_{1,0})$.

Theorem 3.1.7 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and quasiincave at x^0 w.r.t. η,

(b) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{1,0})$.

3.1.3 Approximate problem $(P_{0,1})$

The aim of this section is to establish the connections between the optimal solutions for Problem (P) and the optimal solutions for approximated problem $(P_{0,1})$.

Following theorem shows when an optimal solution for Problem $(P_{0,1})$ is an optimal solution for Problem (P).

Theorem 3.1.8 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(P)$.

If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem (P).

In what follows next theorem shows when an optimal solution for Problem (P) is an optimal solution for Problem $(P_{0,1})$.

Theorem 3.1.9 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(P_{0,1})$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{0,1})$.

3.1.4 Approximate problem $(P_{1,1})$

The goal of this section is to establish the connections between the optimal solution of Problem (P) and the optimal solution of approximated problem $(P_{1,1})$.

Following theorem shows when an optimal solution for Problem $(P_{1,1})$ is an optimal solution for Problem (P).

Theorem 3.1.10 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P)$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,1})$, then x^0 is an optimal solution for Problem (P).

In what follows next theorem shows when an optimal solution for Problem (P) is an optimal solution for Problem $(P_{1,1})$.

Theorem 3.1.11 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{1,1})$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{1,1})$.

The following theorem, gives sufficient conditions under which optimal value for Problem $(P_{1,1})$ is less or equal than optimal value for Problem (P).

Theorem 3.1.12 (**A. Raţiu**, D.I. Duca [72]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T, s \in S)$. If

(a) the function f is differentiable at x^0 and invex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

then

$$\inf(P_{1,1}) \leqq \inf(P).$$

The following example illustrate the relation $\inf(P_{1,1}) < \inf(P)$.

Example 3.1.13 (**A. Raţiu**, D.I. Duca [72]) Consider the problem

$$\begin{array}{ll} \min & f(x) := (x_1+3)^2 + x_2^2 \\ \text{subject to} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & g_1(x) := x_1^2 - x_2 \leqq 0 \\ & g_2(x) = x_1 + x_2 - 2 \leqq 0 \\ & g_k(x) = (x_1-2)^2 + (x_2-5)^2 - 17 - \frac{1}{k} \leqq 0,\ k \in \mathbb{N}, k \geqq 3, \end{array} \qquad (\widetilde{P})$$

and the function $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by $\eta(x, y) = x - y$, for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$.

For the point $x^0 = (-2, 4)$, we have

$$F^{\langle 1 \rangle}(x) := f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) = 2x_1 + 8x_2 - 11,$$

for all $(x_1, x_2) \in \mathbb{R}^2$.

Then, $\eta-$approximated optimization problem is

$$\begin{array}{ll} \min & 2x_1 + 8x_2 - 11 \\ \text{subject to} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & -4x_1 - x_2 \leqq 4 \\ & x_1 + x_2 \leqq 2 \\ & -8x_1 - 2x_2 \leqq 8 + \frac{1}{k},\ k \in \mathbb{N},\ k \geqq 3, \end{array} \qquad (\widetilde{P_{1,1}})$$

and hence

$$\mathcal{F}_0 \subseteq \mathcal{F}_1.$$

Moreover,

$$\inf(\widetilde{P_{1,1}}) = -\infty < 17 = \inf(\widetilde{P}).$$

The following theorem, gives sufficient conditions under which optimal value for Problem $(P_{1,1})$ is great or equal than optimal value for Problem (P).

Theorem 3.1.14 (**A. Raţiu**, D.I. Duca [70]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T, s \in S)$. If

(a) the function f is differentiable at x^0 and incave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η, then

$$\inf(P_{1,1}) \geqq \inf(P).$$

The following two examples show that the relation $v(P) = v(P_{1,1})$ can be take place.

Example 3.1.15 (**A. Raţiu**, D.I. Duca [70]) Consider the problem

$$\begin{array}{ll} \min & f(x) := -(x_1-9)^2 - (x_2-10)^2 \\ \text{subject to} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & g_1(x) := -x_1 \leqq 0 \\ & g_2(x) := -x_2 \leqq 0 \\ & g_3(x) := -(x_1-2)^2 - (x_2-1)^2 + 1 \leqq 0 \\ & g_4(x) := x_1 + x_2 - 5 \leqq 0 \\ & g_k(x) := -x_1 - x_2 + \frac{k+1}{k+2} \leqq 0,\ k \in \mathbb{N}, k \geqq 5. \end{array} \qquad (\widehat{P})$$

For the point $x^0 = (1, 0)$ and the function $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\eta(x, y) = x - y$, for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$, we have

$$F^{\langle 1 \rangle}(x) := f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) = 16x_1 + 20x_2 - 180,$$

for all $x \in \mathbb{R}^2$.

Then η-approximated optimization problem is

$$\begin{array}{ll} \min & 16x_1 + 20x_2 - 180 \\ \text{subject to} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & -x_1 \leqq 0 \\ & -x_2 \leqq 0 \\ & 2x_1 + 2x_2 - 3 \leqq 0 \\ & x_1 + x_2 - 5 \leqq 0 \\ & -x_1 - x_2 + \frac{k+1}{k+2} \leqq 0,\ k \in \mathbb{N},\ k \geqq 5. \end{array} \qquad (\widehat{P_{1,1}})$$

We have

$$v(\widehat{P}) = v(\widehat{P_{1,1}}).$$

Example 3.1.16 (**A. Raţiu**, D.I. Duca [72]) Consider the problem

$$\begin{array}{ll} \min & f(x) := (x_1+6)^2 + (x_2-5)^2 \\ \text{subject to} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & g_1(x) := x_1^2 - x_2 \leqq 0 \\ & g_2(x) = x_1 + x_2 - 2 \leqq 0 \\ & g_k(x) = (x_1-2)^2 + (x_2-5)^2 - 17 - \frac{1}{k} \leqq 0,\ k \in \mathbb{N}, k \geqq 3, \end{array} \qquad (\widehat{P})$$

and the function $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\eta(x, y) = x - y$, for all $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$.

For the point $x^0 = (-2, 4)$, we have

$$F^{\langle 1\rangle}(x) := f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) = 8x_1 - 2x_2 + 41,$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$.

Then $\eta-$approximated optimization problem is

$$\begin{array}{ll} \min & 8x_1 - 2x_2 + 41 \\ \text{subject to} & \\ & x := (x_1, x_2) \in \mathbb{R}^2 \\ & -4x_1 - x_2 \leqq 4 \\ & x_1 + x_2 \leqq 2 \\ & -8x_1 - 2x_2 \leqq 8 + \frac{1}{k},\ k \in \mathbb{N},\ k \geqq 3. \end{array} \qquad (\widehat{P_{1,1}})$$

and hence

$$v(\widehat{P}) = 17 = v(\widehat{P_{1,1}}).$$

3.1.5 Connections between optimal solutions for problems $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$

The aim of this section is to establish connections between optimal solutions of first order η-approximated semi-infinite optimization problems.

Following theorem shows that in some hypothesis, an optimal solution for Problem $(P_{0,1})$ is an optimal solution for Problem $(P_{1,1})$.

Theorem 3.1.17 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0,

(c) for each $s \in S$, the function h_s is differentiable at x^0,

(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem $(P_{1,1})$.

In what follows next theorem shows when an optimal solution for Problem $(P_{1,1})$ is an optimal solution for Problem $(P_{0,1})$.

Theorem 3.1.18 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0,

(c) for each $s \in S$, the function h_s is differentiable at x^0,

(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,1})$, then x^0 is an optimal solution for Problem $(P_{0,1})$.

Following theorem shows when an optimal solution for Problem $(P_{1,1})$ is an optimal solution for Problem $(P_{1,0})$.

Theorem 3.1.19 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{1,0})$.

If x^0 is an optimal solution for Problem $(P_{1,1})$, then x^0 is an optimal solution for Problem $(P_{1,0})$.

Next result shows when an optimal solution for Problem $(P_{1,0})$ is an optimal solution for Problem $(P_{1,1})$.

Theorem 3.1.20 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η.

(d) $x^0 \in \mathcal{F}(P_{1,1})$.

If x^0 is an optimal solution for Problem $(P_{1,0})$, then x^0 is an optimal solution for Problem $(P_{1,1})$

In what follows next theorem shows when an optimal solution for Problem $(P_{1,0})$ is an optimal solution for Problem $(P_{0,1})$.

Theorem 3.1.21 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{0,1})$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,0})$, then x^0 is an optimal solution for Problem $(P_{0,1})$.

Following theorem shows that in some hypothesis, any optimal solution for Problem $(P_{0,1})$ is an optimal solution for Problem $(P_{1,0})$.

Theorem 3.1.22 (**A. Raţiu**, D.I. Duca [71]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{1,0})$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem $(P_{1,0})$

3.2 Second order η-approximated semi-infinite optimization problems

In this section, we assume that X is a nonempty subset of $\mathbb{R}^n$, T and S are index sets, $\eta : X \times X \to X$ a function, x^0 be an interior point of X and $f : X \to \mathbb{R}$, $g_t : X \to \mathbb{R}$, $(t \in T)$ and $h_s : X \to \mathbb{R}$, $(s \in S)$ are twice differentiable at x^0.

We consider the optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & \\ & x \in X \\ & g_t(x) \leqq 0,\ (t \in T) \\ & h_s(x) = 0,\ (s \in S). \end{array} \tag{P}$$

We denote by $F^{\langle 1 \rangle}, G_t^{\langle 1 \rangle}, H_s^{\langle 1 \rangle}, F^{\langle 2 \rangle}, G_t^{\langle 2 \rangle}, H_s^{\langle 2 \rangle} : X \to \mathbb{R}$ $(t \in T,\ s \in S)$ the functions defined by:

$$F^{\langle 1 \rangle}(x) := f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right),$$

$$G_t^{\langle 1 \rangle}(x) := g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right),\ \ (t \in T),$$

$$H_s^{\langle 1 \rangle}(x) := h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right),\ \ (s \in S),$$

$$\begin{aligned} F^{\langle 2 \rangle}(x) \ &:= f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) + \\ &+ \tfrac{1}{2}\left\langle \eta(x, x^0), \left[\nabla^2 f(x^0)\right]\left(\eta(x, x^0)\right)\right\rangle, \end{aligned}$$

$$\begin{aligned} G_t^{\langle 2 \rangle}(x) \ &:= g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) + \\ &+ \tfrac{1}{2}\left\langle \eta(x, x^0), \left[\nabla^2 g_t(x^0)\right]\left(\eta(x, x^0)\right)\right\rangle,\ (t \in T), \end{aligned}$$

$$\begin{aligned} H_s^{\langle 2 \rangle}(x) \ &:= h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right) + \\ &+ \tfrac{1}{2}\left\langle \eta(x, x^0), [\nabla^2 h_s(x^0)]\left(\eta(x, x^0)\right)\right\rangle,\ (s \in S), \end{aligned}$$

for all $x \in X$, and $F^{\langle 0 \rangle} = f$, $G_t^{\langle 0 \rangle} = g_t$, $H_s^{\langle 0 \rangle} = h_s$, $(t \in T,\ s \in S)$.

In what follows, we attach to Problem (P), the problems: $(P_{j,k})$, $((j,k) \in \{(2,0), (0,2), (1,2), (2,1), (2,2), (1,0), (0,1), (1,1)\})$,

$$\begin{array}{ll} \min & F^{\langle j \rangle}(x) \\ \text{subject to} & \\ & x \in X \\ & G_t^{\langle k \rangle}(x) \leqq 0,\ (t \in T) \\ & H_s^{\langle k \rangle}(x) = 0,\ (s \in S), \end{array} \tag{P$_{j,k}$}$$

called (j,k)-η approximated semi-infinite optimization problem.

We denote by:

$$\mathcal{F}_2 := \{x \in X : G_t^{\langle 2\rangle}(x) \leqq 0,\ \ (t \in T)\,,\ H_s^{\langle 2\rangle}(x) = 0,\ (s \in S)\}.$$

Let's remark that if $\mathcal{F}(P)$ denote the set of all feasible solutions for Problem (P), then $\mathcal{F}_0 = \mathcal{F}(P) = \mathcal{F}(P_{1,0}) = \mathcal{F}(P_{2,0})$, $\mathcal{F}_1 = \mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1}) = \mathcal{F}(P_{2,1})$ and $\mathcal{F}_2 = \mathcal{F}(P_{0,2}) = \mathcal{F}(P_{1,2}) = \mathcal{F}(P_{2,2})$.

3.2.1 Connections between the sets of feasible solutions of Problem (P) and second order η-approximated problems

In this section, we establish connections between the set of feasible solutions for Problem (P) and the set of feasible solutions of second order η-approximated semi-infinite optimization problems.

Next theorem tells us under what conditions the set of feasible solutions for Problem (P) is included in the set of feasible solutions for second order η-approximated semi-infinite optimization problems.

Theorem 3.2.1 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ are functions. If

(a) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order invex at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

then

$$\mathcal{F}_0 \subseteq \mathcal{F}_2.$$

Following theorem shows under what conditions the set of feasible solutions for second order η-approximated semi-infinite optimization problems is included in the set of feasible solutions for Problem (P).

Theorem 3.2.2 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ are functions. If

(a) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order incave at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

then

$$\mathcal{F}_2 \subseteq \mathcal{F}_0.$$

3.2.2 Connections between the sets of feasible solutions of first order η-approximated problems and second order η- approximated problems

The goal of this section is to establish connections between the set of feasible solutions of first order η-approximated semi-infinite optimization problems and the set of feasible solutions of second order η-approximated semi-infinite optimization problems.

In what follows next theorem shows when the set of feasible solutions of first order η-approximated semi-infinite optimization problems is included in the set of feasible solutions of second order η-approximated semi-infinite optimization problems .

Theorem 3.2.3 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ are functions. If

(a) for each $t \in T$, the function g_t is second order differentiable at x^0 and

$$\left\langle \eta(x, x^0), \left[\nabla^2 g_t(x^0)\right] \left(\eta(x, x^0)\right) \right\rangle \leqq 0, \text{ for all } x \in X,$$

(b) for each $s \in S$, the function h_s is second order differentiable at x^0 and

$$\left\langle \eta(x, x^0), [\nabla^2 h_s(x^0)] \left(\eta(x, x^0)\right) \right\rangle = 0, \text{ for all } x \in X,$$

then

$$\mathcal{F}_1 \subseteq \mathcal{F}_2.$$

Next theorem shows when the set of feasible solutions of second order η-approximated semi-infinite optimization problems is included in the set of feasible solutions of first order η-approximated semi-infinite optimization problems .

Theorem 3.2.4 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ are functions. If

(a) for each $t \in T$, the function g_t is second order differentiable at x^0 and

$$\left\langle \eta(x, x^0), \left[\nabla^2 g_t(x^0)\right] \left(\eta(x, x^0)\right) \right\rangle \geqq 0, \text{ for all } x \in X,$$

(b) for each $s \in S$, the function h_s is second order differentiable at x^0 and

$$\left\langle \eta(x, x^0), [\nabla^2 h_s(x^0)] \left(\eta(x, x^0)\right) \right\rangle = 0, \text{ for all } x \in X,$$

then

$$\mathcal{F}_2 \subseteq \mathcal{F}_1.$$

3.2.3 Approximate problem $(P_{0,2})$

The goal of this section is to establish the connections between the optimal solutions for Problem (P) and the optimal solutions for approximated problem $(P_{0,2})$.

In the following theorem is established when an optimal solution for Problem $(P_{0,2})$, is an optimal solution for Problem (P).

Theorem 3.2.5 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order invex at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(P)$.

If x^0 is an optimal solution for Problem $(P_{0,2})$, then x^0 is an optimal solution for Problem (P).

Following theorem shows that under some assumptions, any optimal solution for Problem (P) is an optimal solution for Problem $(P_{0,2})$.

Theorem 3.2.6 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order incave at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(P_{0,2})$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{0,2})$.

3.2.4 Approximate problem $(P_{2,0})$

The aim of this section is to establish connections between the optimal solutions for Problem (P) and the optimal solutions for approximated problem $(P_{2,0})$.

Following theorem shows when an optimal solution for Problem $(P_{2,0})$ is an optimal solution for Problem (P).

Theorem 3.2.7 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0 and second order pseudoinvex at x^0 w.r.t. η,

(b) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{2,0})$, then x^0 is an optimal solution for Problem (P).

In what follows next theorem shows when an optimal solution for Problem (P) is an optimal solution for Problem $(P_{2,0})$.

Theorem 3.2.8 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0 and second order quasiincave at x^0 w.r.t. η,

(b) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{2,0})$.

3.2.5 Approximate problem $(P_{1,2})$

The aim of this section is to establish the connections between the optimal solutions for Problem (P) and the optimal solutions for approximated problem $(P_{1,2})$.

In Theorem 3.2.9, we established when an optimal solution for Problem $(P_{1,2})$ is an optimal solution for Problem (P).

Theorem 3.2.9 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P)$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,2})$, then x^0 is an optimal solution for Problem (P).

In what follows next theorem shows when an optimal solution for Problem (P) is an optimal solution for Problem $(P_{1,2})$.

Theorem 3.2.10 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is differentiable at x^0 and quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{1,2})$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{1,2})$.

3.2.6 Approximate problem $(P_{2,1})$

The goal of this section is to establish the connections between the optimal solutions for Problem (P) and the optimal solutions for approximated problem $(P_{2,1})$.

Following theorem shows that in some hypothesis, any optimal solution for Problem $(P_{2,1})$ is an optimal solution for Problem (P).

Theorem 3.2.11 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0 and second order pseudoinvex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P)$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{2,1})$, then x^0 is an optimal solution for Problem (P).

In what follows next theorem shows when an optimal solution for Problem (P) is an optimal solution for Problem $(P_{2,1})$.

Theorem 3.2.12 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0 and second order quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{2,1})$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{2,1})$.

3.2.7 Approximate problem $(P_{2,2})$

The aim of this section is to establish connections between the optimal solutions for Problem (P) and the optimal solutions for approximated problem $(P_{2,2})$.

Relation between the optimal solutions for Problem $(P_{2,2})$ and optimal solutions for Problem (P) is established under following assumptions.

Theorem 3.2.13 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0 and second order pseudoinvex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(d) $\eta(x^0, x^0) = 0$,

(e) $x^0 \in \mathcal{F}(P)$.

If x^0 is an optimal solution for Problem $(P_{2,2})$, then x^0 is an optimal solution for Problem (P).

Next theorem shows when an optimal solution for Problem (P) is an optimal solution for Problem $(P_{2,2})$.

Theorem 3.2.14 (**A. Raţiu**, D.I. Duca [73]) Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0 and second order quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(d) $\eta(x^0, x^0) = 0$,

(e) $x^0 \in \mathcal{F}(P_{2,2})$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{2,2})$.

3.2.8 Connections between optimal solutions for problems $(P_{0,2})$, $(P_{2,0})$, $(P_{1,2})$, $(P_{2,1})$ and $(P_{2,2})$

The following results refer to connections between the optimal solutions for second order η-approximated semi-infinite optimization problems.

Following theorem shows that in some hypothesis, any optimal solution for Problem $(P_{2,0})$ is an optimal solution for Problem $(P_{1,2})$.

Theorem 3.2.15 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0, second order pseudoinvex at x^0 w.r.t. η and quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{1,2})$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{2,0})$, then x^0 is an optimal solution for Problem $(P_{1,2})$.

In what follows next theorem shows when an optimal solution for Problem $(P_{1,2})$ is an optimal solution for Problem $(P_{2,0})$.

Theorem 3.2.16 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0, second order quasiincave at x^0 w.r.t. η and pseudoinvex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P)$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,2})$, then x^0 is an optimal solution for Problem $(P_{2,0})$.

Relation between the optimal solutions for Problem $(P_{1,2})$ and optimal solutions for Problem $(P_{2,1})$ is established under following assumptions.

Theorem 3.2.17 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0, pseudoinvex at x^0 w.r.t. η and second order quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0, incave at x^0 w.r.t. η and second order invex at x^0 w.r.t. η

(c) for each $s \in S$, the function h_s is second order differentiable at x^0, incave at x^0 w.r.t. η and second order avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P) \cap \mathcal{F}(P_{2,1})$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,2})$, then x^0 is an optimal solution for Problem $(P_{2,1})$.

Next theorem establish when an optimal solution for Problem $(P_{2,1})$ is an optimal solution for Problem $(P_{1,2})$.

Theorem 3.2.18 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0, second order pseudoinvex at x^0 w.r.t. η and quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0, second order incave at x^0 w.r.t. η and invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0, second order avex at x^0 w.r.t. η and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P) \cap \mathcal{F}(P_{1,2})$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{2,1})$, then x^0 is an optimal solution for Problem $(P_{1,2})$.

Following theorem shows that under some assumptions, any optimal solution for Problem $(P_{2,1})$ is an optimal solution for Problem $(P_{2,0})$.

Theorem 3.2.19 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{2,0})$.

If x^0 is an optimal solution for Problem $(P_{2,1})$, then x^0 is an optimal solution for Problem $(P_{2,0})$.

Following theorem shows when an optimal solution for Problem $(P_{2,0})$ is an optimal solution for Problem $(P_{2,1})$.

Theorem 3.2.20 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{2,1})$,

If x^0 is an optimal solution for Problem $(P_{2,0})$, then x^0 is an optimal solution for Problem $(P_{2,1})$.

Next theorem shows when an optimal solution for Problem $(P_{0,2})$ is an optimal solution for Problem $(P_{1,2})$.

Theorem 3.2.21 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is differentiable at x^0 and quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0,

(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,2})$, then x^0 is an optimal solution for Problem $(P_{1,2})$.

Relation between the optimal solutions for Problem $(P_{1,2})$ and optimal solutions for Problem $(P_{0,2})$ is established under following assumptions.

Theorem 3.2.22 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0,

(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,2})$, then x^0 is an optimal solution for Problem $(P_{0,2})$.

In Theorem 3.2.23, we established that an optimal solution for Problem $(P_{0,2})$ is an optimal solution for Problem $(P_{2,0})$.

Theorem 3.2.23 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0 and second order quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η, (*d*) $x^0 \in \mathcal{F}(P)$,

(*e*) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,2})$, then x^0 is an optimal solution for Problem $(P_{2,0})$.

Following theorem shows that in some hypothesis, any optimal solution for Problem $(P_{2,0})$ is an optimal solution for Problem $(P_{0,2})$.

Theorem 3.2.24 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(*a*) the function f is second order differentiable at x^0 and second order pseudoinvex at x^0 w.r.t. η,

(*b*) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order incave at x^0 w.r.t. η,

(*c*) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(*d*) $x^0 \in \mathcal{F}(P_{0,2})$,

(*e*) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{2,0})$, then x^0 is an optimal solution for Problem $(P_{0,2})$.

Next theorem shows when an optimal solution for Problem $(P_{0,2})$ is an optimal solution for Problem $(P_{2,1})$.

Theorem 3.2.25 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(*a*) the function f is second order differentiable at x^0 and second order quasiincave at x^0 w.r.t. η,

(*b*) for each $t \in T$, the function g_t is second order differentiable at x^0, second order avex at x^0 w.r.t. η,

(*c*) for each $s \in S$, the function h_s is second order differentiable at x^0, second order avex at x^0 w.r.t. η,

(*d*) $x^0 \in \mathcal{F}(P) \cap \mathcal{F}(P_{2,1})$,

(*e*) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,2})$, then x^0 is an optimal solution for Problem $(P_{2,1})$.

Relation between the optimal solutions for Problem $(P_{2,1})$ and optimal solutions for Problem $(P_{0,2})$ is established under following assumptions.

Theorem 3.2.26 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(*a*) the function f is second order differentiable at x^0 and second order pseudoinvex at x^0 w.r.t. η,

(*b*) for each $t \in T$, the function g_t is second order differentiable at x^0, second order avex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0, second order avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P) \cap \mathcal{F}(P_{0,2})$,

(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{2,1})$, then x^0 is an optimal solution for Problem $(P_{0,2})$.

Following theorem shows when an optimal solution for Problem $(P_{0,2})$ is an optimal solution for Problem $(P_{2,2})$.

Theorem 3.2.27 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0 and second order incave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0,

(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,2})$, then x^0 is an optimal solution for Problem $(P_{2,2})$.

Following theorem shows that in some hypothesis, any optimal solution for Problem $(P_{2,2})$ is an optimal solution for Problem $(P_{0,2})$.

Theorem 3.2.28 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0 and second order invex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0,

(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{2,2})$, then x^0 is an optimal solution for Problem $(P_{0,2})$.

Next theorem shows when an optimal solution for Problem $(P_{1,2})$ is an optimal solution for Problem $(P_{2,2})$.

Theorem 3.2.29 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0,

(d) $\left\langle \eta(x, x^0), \left[\nabla^2 f(x^0)\right] \left(\eta(x, x^0)\right)\right\rangle \geqq \left\langle \eta(x^0, x^0), \left[\nabla^2 f(x^0)\right] \left(\eta(x^0, x^0)\right)\right\rangle$, for all $x \in \mathcal{F}(P_{1,2})$.

If x^0 is an optimal solution for Problem $(P_{1,2})$, then x^0 is an optimal solution for Problem $(P_{2,2})$.

In what follows next theorem shows when an optimal solution for Problem $(P_{2,2})$ is an optimal solution for Problem $(P_{1,2})$.

Theorem 3.2.30 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0,

(d) $\left\langle \eta(x,x^0), \left[\nabla^2 f(x^0)\right]\left(\eta(x,x^0)\right)\right\rangle \leqq \left\langle \eta(x^0,x^0), \left[\nabla^2 f(x^0)\right]\left(\eta(x^0,x^0)\right)\right\rangle$, for all $x \in \mathcal{F}(P_{1,2})$.

If x^0 is an optimal solution for Problem $(P_{2,2})$, then x^0 is an optimal solution for Problem $(P_{1,2})$.

In Theorem 3.2.31, we established that an optimal solution for Problem $(P_{2,2})$ is an optimal solution for Problem $(P_{2,0})$.

Theorem 3.2.31 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{2,0})$.

If x^0 is an optimal solution for Problem $(P_{2,2})$, then x^0 is an optimal solution for Problem $(P_{2,0})$.

Relation between the optimal solutions for Problem $(P_{2,0})$ and optimal solutions for Problem $(P_{2,2})$ is established under following assumptions.

Theorem 3.2.32 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and second order incave at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(d) $x^0 \in \mathcal{F}(P_{2,2})$.

If x^0 is an optimal solution for Problem $(P_{2,0})$, then x^0 is an optimal solution for Problem $(P_{2,2})$.

Following theorem shows that in some hypothesis, any optimal solution for Problem $(P_{2,1})$ is an optimal solution for Problem $(P_{2,2})$.

Theorem 3.2.33 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f,\ g_t,\ h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and

$$\left\langle \eta(x,x^0), \left[\nabla^2 g_t(x^0)\right]\left(\eta(x,x^0)\right)\right\rangle \geqq 0,$$

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and

$$\left\langle \eta(x, x^0), [\nabla^2 h_s(x^0)] \left(\eta(x, x^0)\right) \right\rangle = 0,$$

(d) $x^0 \in \mathcal{F}(P_{2,2})$.

If x^0 is an optimal solution for Problem $(P_{2,1})$, then x^0 is an optimal solution for Problem $(P_{2,2})$.

Under some assumptions, next theorem establish when optimal solution for Problem $(P_{2,2})$ is an optimal solution for Problem $(P_{2,1})$.

Theorem 3.2.34 (**A. Raţiu**, D.I. Duca [73]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and

$$\left\langle \eta(x, x^0), \left[\nabla^2 g_t(x^0)\right] \left(\eta(x, x^0)\right) \right\rangle \leqq 0,$$

(c) for each $s \in S$, the function h_s is second order differentiable at x^0 and

$$\left\langle \eta(x, x^0), [\nabla^2 h_s(x^0)] \left(\eta(x, x^0)\right) \right\rangle = 0, \text{ for all } x \in X.$$

(d) $x^0 \in \mathcal{F}(P_{2,1})$.

If x^0 is an optimal solution for Problem $(P_{2,2})$, then x^0 is an optimal solution for Problem $(P_{2,1})$.

3.2.9 Connections between optimal solutions for problems $(P_{1,0})$, $(P_{0,1})$, $(P_{1,1})$ and $(P_{0,2})$, $(P_{2,0})$, $(P_{1,2})$, $(P_{2,1})$, $(P_{2,2})$

In this section are establish connections between the optimal solutions for first and second order η-approximated semi-infinite optimization problems.

Following theorem shows that in some hypothesis, any optimal solution for Problem $(P_{0,1})$ is an optimal solution for Problem $(P_{0,2})$.

Theorem 3.2.35 (**A. Raţiu**, D.I. Duca) [74]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) for each $t \in T$, the function g_t is invex at x^0, second order differentiable at x^0 and second order incave at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(P) \cap \mathcal{F}(P_{0,2})$.

If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem $(P_{0,2})$.

In what follows next theorem shows when an optimal solution for Problem $(P_{0,2})$ is an optimal solution for Problem $(P_{0,1})$.

Theorem 3.2.36 (**A. Raţiu**, D.I. Duca) [74]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) for each $t \in T$, the function g_t is incave at x^0, second order differentiable at x^0 and second order invex at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is second order differentiable at x^0 and second order avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(P) \cap \mathcal{F}(P_{0,1})$.

If x^0 is an optimal solution for Problem $(P_{0,2})$, then x^0 is an optimal solution for Problem $(P_{0,1})$.

Relation between the optimal solutions for Problem $(P_{0,1})$ and optimal solutions for Problem $(P_{2,0})$ is established under following assumptions.

Theorem 3.2.37 (**A. Raţiu**, D.I. Duca) [74]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) for each $t \in T$, the function g_t is second order differentiable at x^0, second order quasiincave at x^0 and invex at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(P)$,

(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem $(P_{2,0})$.

Following theorem shows when an optimal solution for Problem $(P_{2,0})$ is an optimal solution for Problem $(P_{0,1})$.

Theorem 3.2.38 (**A. Raţiu**, D.I. Duca) [74]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$ such that:

(a) the function f is second order differentiable at x^0, incave at x^0 and second order pseudoinvex at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(P_{0,1})$,

(d) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{2,0})$, then x^0 is an optimal solution for Problem $(P_{0,1})$.

Next theorem shows when an optimal solution for Problem $(P_{0,1})$ is an optimal solution for Problem $(P_{1,2})$.

Theorem 3.2.39 (**A. Raţiu**, D.I. Duca) [74]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and quasiincave at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is second order differentiable at x^0 and invex at x^0 w.r.t. η,

(c) for each $s \in S$, the function h_s is second order differentiable at x^0, second order incave at x^0 and second order avex at x^0 w.r.t. η,

(*d*) $x^0 \in \mathcal{F}(P) \cap \mathcal{F}(P_{1,2})$,
(*e*) $\eta(x^0, x^0) = 0$.
If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem $(P_{1,2})$.

Other theorems that establish connections between optimal solutions for problems $(P_{1,0})$, $(P_{0,1})$, $(P_{1,1})$ and $(P_{0,2})$, $(P_{2,0})$, $(P_{1,2})$, $(P_{2,1})$, $(P_{2,2})$ are presented in paper [74].

Chapter 4

Vector optimization problems

Vector optimization (multi-objective programming or Pareto optimization) is an area of operations research that considers multiple criteria in decision. We find applications in various fields as: economics, logistics (optimal control) and engineering (optimal design), where optimal decisions must be taken not only for one criterion, but for more than, often in conflict with each other. In literature there are many approaches to solve a nonlinear constrained vector optimization problem (see, [2], [46], [48], [55]). One of them is using saddle points criteria (see, [18], [24], [53]). In [2], Antczak introduced η-approximated method for vector optimization problems with invex functions. He replace the original problem by another equivalent vector optimization problem modifying the objective and constraint functions in the original vector optimization problem at an arbitrary but fixed feasible point $\overline{x}$. In [80] we can find some scalarization for a vector optimization problem in infinite dimensional.

This chapter is split in two sections. First section contains a study of a vector optimization problem. To obtain information about the efficient solutions of this problem, we attach three η-approximated vector optimization problems. Some connections between the efficient solutions for original problem and approximated problem are presented. The last section deals with two methods: weighting method and constraint method for solving vector optimization problems. For each method is presented the algorithm and numerical examples. To illustrate a singleton or a subset of minimal solutions, we gave also the graphical representations. These are interesting from mathematical and practical points of view.

4.1 Approximations of semi-infinite vector optimization problems

In this section, we assume that X is a nonempty subset of $\mathbb{R}^n$, T and S are index sets, $\eta : X \times X \to X$ is a function, x^0 is an interior point of X and $f : X \to \mathbb{R}^k$, $g_t : X \to \mathbb{R}$, $(t \in T)$ and $h_s : X \to \mathbb{R}$, $(s \in S)$ are differentiable at x^0.

We consider the vector optimization problem:

$$
\begin{array}{lll}
\min & f(x) = (f_1, f_2, ..., f_k)(x) & \\
\text{subject to} & & \\
 & x = (x_1, x_2, ..., x_n) \in X & \text{(PV)} \\
 & g_t(x) \leqq 0,\ (t \in T) & \\
 & h_s(x) = 0,\ (s \in S). &
\end{array}
$$

We propose to attach to Problem (PV), the problems $(PV_{j,k})$, $((j,k) \in \{(1,0),(0,1),(1,1)\})$,

$$
\begin{array}{lll}
\min & F^{\langle j \rangle}(x) & \\
\text{subject to:} & & \\
 & x = (x_1, x_2, ..., x_n) \in X & (\text{PV}_{j,k}) \\
 & G_t^{\langle k \rangle}(x) \leqq 0,\ (t \in T) & \\
 & H_s^{\langle k \rangle}(x) = 0,\ (s \in S), &
\end{array}
$$

called (j,k)-η approximated vector optimization problem, where $F^{\langle 1 \rangle} : X \to \mathbb{R}^k$, $G_t^{\langle 1 \rangle}$, $H_s^{\langle 1 \rangle} : X \to \mathbb{R}$ $(t \in T,\ s \in S)$ are defined by

$$F^{\langle 1 \rangle}(x) := f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right),$$

$$G_t^{\langle 1 \rangle}(x) := g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right),\ \ (t \in T),$$

$$H_s^{\langle 1 \rangle}(x) := h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right),\ \ (s \in S),$$

for all $x = (x_1, x_2, ..., x_n) \in X$, and $F^{\langle 0 \rangle} = f$, $G_t^{\langle 0 \rangle} = g_t$, $(t \in T)$, $H_s^{\langle 0 \rangle} = h_s$, $(s \in S)$.

Definition *We say that $x^0 \in \mathcal{F}(PV)$ is an efficient solution for Problem (PV) if there is no $x \in \mathcal{F}(PV)$ such that*

$$f(x) \leq f(x^0).$$

or equivalently,

$$f(x^0) - f(x) \in \mathbb{R}^k_+ \setminus \{0\}.$$

4.1.1 Approximate problem $(PV_{1,0})$

The aim of this section is to establish the connections between the efficient solutions for Problem (PV) and the efficient solutions for approximated problem $(PV_{1,0})$.

Following theorem shows that in some hypothesis, an efficient solution for Problem $(PV_{1,0})$ is an efficient solution for Problem (PV).

Theorem 4.1.2 (**A. Raţiu**, D.I. Duca [75]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}^k$, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and invex at x^0 w.r.t. η,

(b) $\eta(x^0, x^0) = 0$.

If x^0 is an efficient solution for Problem $(PV_{1,0})$, then x^0 is an efficient solution for Problem (PV).

Theorem 4.1.3, shows when an efficient solution for Problem (PV) is an efficient solution for Problem $(PV_{1,0})$.

Theorem 4.1.3 (**A. Raţiu**, D.I. Duca [75]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}^k$, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and incave at x^0 w.r.t. η,

(b) $\eta(x^0, x^0) = 0$.

If x^0 is an efficient solution for Problem (PV), then x^0 is an efficient solution for Problem $(PV_{1,0})$.

4.1.2 Approximate problem $(PV_{0,1})$

The goal of this section is to establish connections between the efficient solutions for Problem (PV) and the efficient solutions for approximated problem $(PV_{0,1})$.

In what follows next theorem shows when an efficient solution for Problem $(PV_{0,1})$ is an efficient solution for Problem (PV).

Theorem 4.1.4 (**A. Raţiu**, D.I. Duca [75]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}^k$, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(PV)$.

If x^0 is an efficient solution for Problem $(PV_{0,1})$, then x^0 is an efficient solution for Problem (PV).

Relation between the efficient solution for Problem (PV) and efficient solution for Problem $(PV_{0,1})$ is established under following assumptions.

Theorem 4.1.5 (**A. Raţiu**, D.I. Duca [75]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}^k$, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,

(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,

(c) $x^0 \in \mathcal{F}(PV_{0,1})$.

If $x^0$is an efficient solution for Problem (PV), then x^0 is an efficient solution for Problem $(PV_{0,1})$.

4.1.3 Approximate problem $(PV_{1,1})$

The aim of this section is to establish connections between the efficient solutions for Problem (PV) and the efficient solutions for approximated problem $(PV_{1,1})$.

Following theorem shows that in some hypothesis, an efficient solution for Problem $(PV_{1,1})$ is an efficient solution for Problem (PV).

Theorem 4.1.6 (**A. Raţiu**, D.I. Duca [75]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}^k$, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:

(a) the function f is differentiable at x^0 and invex at x^0 w.r.t. η,

(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(PV)$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an efficient solution for Problem $(PV_{1,1})$, then x^0 is an efficient solution for Problem (PV).

Next theorem shows when an efficient solution for Problem (PV) is an efficient solution for Problem $(PV_{1,1})$.

Theorem 4.1.7 (**A. Raţiu**, D.I. Duca [75]) Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and $f : \mathbb{R}^n \to \mathbb{R}^k$, g_t, $h_s : X \to \mathbb{R}$, $(t \in T,\ s \in S)$. Assume that:
(a) the function f is differentiable at x^0 and incave at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(PV_{1,1})$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an efficient solution for Problem (PV), then x^0 is an efficient solution for Problem $(PV_{1,1})$.

Remark 4.1.8 Second order η-approximated vector optimization problems can be addressed in a similar manner to the problems presented in the previous chapter.

4.2 Methods for solving vector optimization problems

In this section we give two methods for solving the following vector optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & \\ & x = (x_1, x_2, ..., x_n) \in X, \end{array} \qquad (\widetilde{PV})$$

where X is a nonempty set in $\mathbb{R}^n$, $f = (f_1, f_2, ..., f_k) : X \to \mathbb{R}^k$.

4.2.1 Weighting method

In this method, we choose weighting vectors $p = (p_1, ..., p_k) \geq 0$, whose coordinates are not all zero and solve the corresponding scalar problem

$$\begin{array}{ll} \min & \sum\limits_{i=1}^{k} p_i f_i(x) \\ \text{subject to} & \\ & x \in X, \end{array} \qquad (PS_p)$$

which generates a set of minimal solutions and a set of minimal values for Problem $(\widetilde{PV})$.

The black points marked on graphical representation are efficient values for the problem.

Example 4.2.4 (**A. Raţiu** [56]) Consider the problem

$$\begin{array}{ll} \min & f(x) = (x_1^2 + x_2 - 2,\ 5x_1^2 - 2x_1x_2 + x_2^2 + 3) \\ \text{subject to} & x \in X, \end{array}$$

where

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 2\}.$$

For every $m \geq 1$, we obtain an efficient solution $x = (0, 0)$ with value $f(x) = (-2, 3)$.

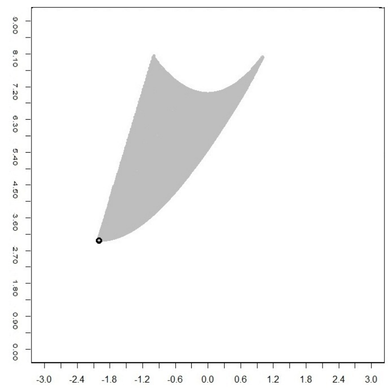

Figure for Example 4.2.4 , for $m \geq 1$

4.2.2 Constraint method

In this method, we choose $\ell \in \{1, ..., k\}, L_j \in \mathbb{R}, j \in \{1, ..., k\}, j \neq \ell$, and solve the corresponding scalar problem

$$\begin{array}{ll} \min & f_\ell(x) \\ \text{subject to} & \\ & f_j(x) \geq L_j, j = 1, ..., k, j \neq \ell \\ & x \in X, \end{array} \qquad (PS_{e^\ell})$$

where $e^\ell = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}^k$.

Example 4.2.12 (**A. Raţiu** [56]) Consider the problem

$$\begin{array}{ll} \min & f(x) = (-1 + x_1^2 + x_2^2,\ -300x_1 - 400x_2 + x_1^2 + 2x_1x_2 + 2x_2^2) \\ \text{subject to} & x \in X, \end{array}$$

where

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 2\}.$$

The efficient values obtained for $r = 4$, are represented in the following figure:

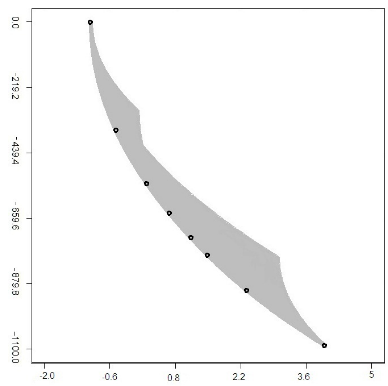

Figure for Example 4.2.12, for $r = 4$

Chapter 5

Solving some optimization problems. Applications in statistics

In this chapter we solve some optimization problems using the theory of inequalities. Jensen's inequality, Radon's inequality, Hölder's inequality, Liapunov's inequality are used and also new inequalities that have emerged in the paper [77] and belong to us. In the last section are obtained new bounds for dispersion, standard deviation and coefficient of variation.

5.1 Solving some optimization problems using inequalities

We consider the expression

$$\Delta^{[p]}(x;y) := \sum_{i=1}^{n} \frac{x_i^p}{y_i^{p-1}} - \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^p}{\left(\sum\limits_{i=1}^{n} y_i\right)^{p-1}}, \tag{5.1}$$

where $x = (x_1, ..., x_n) \geqq 0, y = (y_1, ..., y_n) > 0$, $p > 1$.

In [69], Radon formulated the following result:

Theorem 5.1.1 For all $x = (x_1, ..., x_n) \geqq 0, y = (y_1, ..., y_n) > 0$ and $p > 1$ the following inequality

$$\Delta^{[p]}(x;y) \geqq 0, \tag{5.2}$$

holds. If there exist a real number $\lambda \geqq 0$ such that $x = \lambda y$, then inequality (5.2) (known as Radon's inequality) becomes equality.

Theorem 5.1.3 (**A. Raţiu**, N. Minculete [77]) Let $p \geqq 1$. The optimization problem

$$\begin{array}{ll} \min & F(x,y) = p\left(\Delta^{[p]}(x;y) - \dfrac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n} y_i}\Delta^{[p-1]}(x;y)\right) - \Delta^{[p]}(x;y) \\ \text{subject to} & \\ & x = (x_1, ..., x_n) \geqq 0 \\ & y = (y_1, ..., y_n) > 0, \end{array}$$

has the optimal value 0. An optimal solution is any point $(x,y) \in \mathbb{R}_+^n \times \text{int}\mathbb{R}_+^n$, which has the property that exists $\lambda > 0$ such that $x = \lambda y$.

Theorem 5.1.4 (**A. Raţiu**, N. Minculete [77]) Let $p \geqq 1$ and M, m positive real numbers. The optimization problem

$$\begin{array}{ll} \min & F(x,y) = \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})\left(\sum\limits_{i=1}^{n} y_i\right) - \Delta^{[p]}(x;y) \\ \text{subject to} & \\ & x = (x_1, ..., x_n) \geqq 0 \\ & y = (y_1, ..., y_n) > 0 \\ & my_i \leqq x_i \leqq My_i,\ i \in \{1, ..., n\}, \end{array}$$

has $F(x,y) \geqq 0$, for all feasible solutions.

Theorem 5.1.9 (**A. Raţiu**, N. Minculete [77]) Let $p \geqq 1$ and M, m positive real numbers. The optimization problem

$$\begin{array}{ll} \min & F(x,y) = \Delta^{[p]}(x;y) - \max\limits_{1 \leqq i < j \leqq n}\left[\dfrac{x_i^p}{y_i^{p-1}} + \dfrac{x_j^p}{y_j^{p-1}} - \dfrac{(x_i+x_j)^p}{(y_i+y_j)^{p-1}}\right] \\ \text{subject to} & \\ & x = (x_1, ..., x_n) \geqq 0 \\ & y = (y_1, ..., y_n) > 0 \\ & my_i \leqq x_i \leqq My_i,\ i \in \{1, ..., n\}, \end{array}$$

has the optimal value 0. An optimal solution is any point $(x,y) \in \mathbb{R}_+^n \times \text{int}\mathbb{R}_+^n$, which has the property that exists $\lambda > 0$ such that $x = \lambda y$.

Theorem 5.1.10 (**A. Raţiu**, N. Minculete [77]) Let $p \geqq 1$ and M, m positive real numbers. The optimization problem

$$\begin{array}{ll} \min & F(x,y) = \left[M^p + m^p - \dfrac{(M+m)^p}{2^{p-1}}\right]\left(\sum\limits_{i=1}^{n} y_i\right) - \Delta^{[p]}(x;y) \\ \text{subject to} & \\ & x = (x_1, ..., x_n) \geqq 0 \\ & y = (y_1, ..., y_n) > 0 \\ & my_i \leqq x_i \leqq My_i,\ i \in \{1, ..., n\}, \end{array}$$

has the objective function $F(x,y) \geqq 0$, for all feasible solutions.

Theorem 5.1.11 (**A. Raţiu**, N. Minculete [77]) Let $p \geqq 1$ and M, m positive real numbers. The optimization problem

$$\min\{\frac{\left[(M+m)\sum\limits_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i\right]^p}{\left(\sum\limits_{i=1}^{n} y_i\right)^{p-1}} - \frac{(M+m)^p}{2^{p-1}}\left(\sum_{i=1}^{n} y_i\right) +$$

$$+\left(\sum_{i=1}^{n} \frac{x_i^p}{y_i^{p-1}}\right) - \Delta^{[p]}(x;y)\}$$

subject to

$$\begin{aligned} &x = (x_1, ..., x_n) \geqq 0 \\ &y = (y_1, ..., y_n) > 0 \\ &my_i \leqq x_i \leqq My_i, \ i \in \{1, ..., n\}, \end{aligned}$$

has nonnegative objective function.

Theorem 5.1.13 (**A. Raţiu**, N. Minculete [77]) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. The optimization problem

$$\min \quad \{\sum_{k=1}^{n} z_k v_k + \left(\sum_{k=1}^{n} v_k^q\right)^{\frac{1}{q}} F\left(\frac{\sum\limits_{k=1}^{n} z_k v_k}{\left(\sum\limits_{k=1}^{n} v_k^q\right)^{\frac{1}{q}}}, T^{[p]}(zv; v^q), p\right) -$$

$$-\left(\sum_{k=1}^{n} z_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} v_k^q\right)^{\frac{1}{q}}\}$$

subject to

$$\begin{aligned} &z = (z_1, ..., z_n) \geqq 0 \\ &v = (v_1, ..., v_n) > 0 \end{aligned}$$

where

$$T^{[p]}(x;y) := (p-1) \max_{1 \leqq i < j \leqq n} \frac{(x_i + x_j)^{p-2}(x_i y_j - x_j y_i)^2}{y_i y_j (y_i + y_j)^{p-1}},$$

has nonnegative objective function.

Theorem 5.1.14 (**A. Raţiu**, N. Minculete [77]) Let $r - 1 \geqq s > t > 0$. The optimization problem

$$\min \quad \{\left(\sum_{k=1}^{n} z_k^s\right)^{r-t} + \left(\sum_{k=1}^{n} z_k^r\right)^{s-t} \cdot \left[T^{[p]}(z^s; z^r)\right]^{r-s} -$$

$$-\left(\sum_{k=1}^{n} z_k^t\right)^{r-s} \left(\sum_{k=1}^{n} z_k^r\right)^{s-t}\}$$

subject to

$$z = (z_1, ..., z_n) > 0,$$

has nonnegative objective function.

5.2 New bounds to indicators in statistics

In this section we consider the following indicators: dispersion, standard deviation and coefficient of variation.

Theorem 5.2.1 (N. Minculete, N.B. Pipu, **A. Raţiu** [60]) The optimization problems

$$\min \quad \left\{ \frac{\left(x_1 - \frac{\sum_{i=1}^n x_i}{n}\right)^2 + ... + \left(x_n - \frac{\sum_{i=1}^n x_i}{n}\right)^2}{n} - \right.$$

$$\left. -2 \cdot min\{x_1, ..., x_n\} \cdot \left(\frac{\sum_{i=1}^n x_i}{n} - \sqrt[n]{x_1 \cdot ... \cdot x_n} \right) \right\}$$

subject to

$$x = (x_1, ..., x_n) \in \mathbb{R}_+^n,$$

and

$$\min \quad \left\{ 2 \cdot max\{x_1, ..., x_n\} \cdot \left(\frac{\sum_{i=1}^n x_i}{n} - \sqrt[n]{x_1 \cdot ... \cdot x_n} \right) - \right.$$

$$\left. - \frac{\left(x_1 - \frac{\sum_{i=1}^n x_i}{n}\right)^2 + ... + \left(x_n - \frac{\sum_{i=1}^n x_i}{n}\right)^2}{n} \right\}$$

subject to

$$x = (x_1, ..., x_n) \in \mathbb{R}_+^n,$$

have nonnegative objective functions.

Theorem 5.2.2 (N. Minculete, N.B. Pipu, **A. Raţiu** [60]) For $x_1, ..., x_n \geqq 0$, there is the following equality:

$$\left(\max\{x_1, ..., x_n\} - \frac{\sum_{i=1}^n x_i}{n} \right) \cdot \left(\frac{\sum_{i=1}^n x_i}{n} - \min\{x_1, ..., x_n\} \right) - \sigma_{\overline{X}}^2 =$$

$$= \frac{1}{n} \sum_{i=1}^n \left(\max\{x_1, ..., x_n\} - x_i \right) \left(x_i - \min\{x_1, ..., x_n\} \right).$$

Theorem 5.2.3 (N. Minculete, N.B. Pipu, **A. Raţiu** [60]) If $M' = \max\{x_i | x_i \neq \max\{x_1, ..., x_n\}\}$ and $m' = \min\{x_i | x_i \neq \min\{x_1, ..., x_n\}\}$, $i \in \{1, ..., n\}$, then the optimization problem

$$\begin{gathered}\min\ \{\sigma^2_{\overline{X}} - \left(\max\{x_1,...,x_n\} - \frac{\sum\limits_{i=1}^{n} x_i}{n}\right) \cdot \left(\frac{\sum\limits_{i=1}^{n} x_i}{n} - \min\{x_1,...,x_n\}\right) - \\ - \max\{(m' - \min\{x_1,...,x_n\}) \cdot \left(\max\{x_1,...,x_n\} - \frac{\sum\limits_{i=1}^{n} x_i}{n}\right) \\ (\max\{x_1,...,x_n\} - M') \cdot \left(\frac{\sum\limits_{i=1}^{n} x_i}{n} - \min\{x_1,...,x_n\}\right)\}\end{gathered} \tag{5.2.9}$$

subject to

$$x = (x_1, ..., x_n) \in \mathbb{R}^n_+,$$

has nonnegative objective function.

Remark 5.2.4 (N. Minculete, N.B. Pipu, **A. Raţiu** [60]) Nonnegativity of the objective function of problem (5.2.9) leads to other bounds for variance, standard deviation and coefficient of variation:

$$\begin{aligned}2m\left(\overline{X} - \overline{X}_g\right) &\leqq \sigma^2_{\overline{X}} \leqq \\ &\leqq \left(M - \overline{X}\right)\left(\overline{X} - m\right) - \\ &- \max\left\{(m' - m)\left(M - \overline{X}\right),\ (M - M')\left(\overline{X} - m\right)\right\},\end{aligned}$$

$$\begin{aligned}\sqrt{2m\left(\overline{X} - \overline{X}_g\right)} &\leqq \sigma_{\overline{X}} \leqq \\ &\leqq \sqrt{\left(M - \overline{X}\right)\left(\overline{X} - m\right) - \max\left\{\begin{array}{c}(m' - m)\left(M - \overline{X}\right), \\ (M - M')\left(\overline{X} - m\right)\end{array}\right\}},\end{aligned}$$

and

$$\begin{aligned}\frac{\sqrt{2m\left(\overline{X} - \overline{X}_g\right)}}{\overline{X}} &\leqq C_V \leqq \\ &\leqq \frac{\sqrt{\left(M - \overline{X}\right)\left(\overline{X} - m\right) - \max\left\{(m' - m)\left(M - \overline{X}\right),\ (M - M')\left(\overline{X} - m\right)\right\}}}{\overline{X}},\end{aligned}$$

where $M = \max\{x_1, ..., x_2\}$, $m = \min\{x_1, ..., x_2\}$.

In Remark 5.2.4, we have lower and upper bounds for statistical indicators: variance, standard deviation and coefficient of variation.

Bibliography

[1] T. Antczak: *An η-approximation approach to nonlinear mathematical programming involving invex functions*, Numerical Functional Analysis and Optimization, **25**, 423-438, 2004.

[2] T. Antczak: *An η-approximation method in vector optimization*, Nonlinear Analysis, Theory, Methods and Applications, **63**, 225-236, 2005.

[3] A. Auslender: *Existence of optimal solutions and duality results under weak conditions*, Mathematical Programming, Seria A, **88**, 45-59, 2000.

[4] A. Ben-Tal, E.E. Rosinger, A. Ben-Israel: *A Helly type theorem and semi-infinite programming*, Constructive Approaches to Mathematical Models, 127-135, 1979.

[5] A. Ben-Tal, A. Nemirovski: *Robust solutions of uncertain linear programs*, Operations Research Letters, **25**, 1-13, 1999.

[6] H.V. Boncea, D. Duca: *On the η-(1,2) approximated optimization problems*, Carpathian Journal of Mathematics, **28(1)**, 17-24, 2012.

[7] J.M. Borwein: *Direct theorems in semi-infinite convex programming*, Mathematical Programming, **21**, 301-308, 1981.

[8] R.I. Bot, G. Wanka: *An alternative formulation for a new closed cone constraint qualification*, Nonlinear Analysis, **64**, 1367-1381, 2006.

[9] R.I. Bot, G. Wanka: *Farkas-type results with conjugate functions*, SIAM Journal Optimization, **15**, 540-554, 2005.

[10] R.I. Bot, E.R. Csetnek, G. Wanka: *Regularity conditions via quasi-relative interior in convex programming*, SIAM Journal Optimization, **19**, 217-233, 2008.

[11] R.I. Bot, S.M. Grad, G. Wanka: *Duality in vector optimization*, Springer, 2009.

[12] P.S. Bullen: *Handbook of means and their inequalities*, Kluwer Academic Publishers, Dordrecht, 2003.

[13] M.J. Canovas, M.A. López, J. Parra: *Stability in the discretization of a parametric semi-infinite convex inequality system*, Mathematics of Operations Research, **27**, 755-774, 2002.

[14] A. Charnes, W.W. Cooper, K.O. Kortanek: *Duality, Haar programs and finite sequence spaces*, Proceedings of the National Academy of Science, **48**, 783-786, 1962.

[15] A. Charnes, W.W. Cooper, K.O. Kortanek: *Duality in semi-infinite programs and some works of Haar and Caratheodory*, Management Sciences **9**, 209-228, 1963.

[16] A. Charnes, W.W. Cooper, K.O. Kortanek: *On the theory of semi-infinite programming and a generalization of the Kuhn-Tucker saddle point theorem for arbitrary convex functions*, Naval Research Logistics Quarterly **16**, 41-51, 1969.

[17] L. Cioban, D. Duca: *Optimization problems and* $(0, 2)$-η *approximated optimization problems*, Carpathian Journal of Mathematics, **28(1)**, 37-46, 2012.

[18] B.D. Craven: *Quasimin and quasisaddlepoint for vector optimization*, Numerical Functional Analysis and Optimization, **11**, 45-54, 1990.

[19] M. DallAglio: *On some applications of LSIP to probability and statistics,* in M.A. Goberna, M.A. Lopez (Eds.), Semi-Infinite Programming. Recent Advances, Nonconvex Optimization and Its Applications, Kluwer, **57**, 237-254, 2001.

[20] N. Dinh, M.A. Goberna, M.A. Lopez: *From linear to convex systems: consistency, Farkas lemma and applications*, Journal of Convex Analysis, **13**, 279-290, 2006.

[21] N. Dinh, B.S. Mordukhovich, T.T.A. Nghia: *Qualification and optimality conditionsfor convex and DC programs with infinite constraints*, Acta Mathematica Vietnamica, **34**, 123-153, 2009.

[22] D. Duca, E. Duca: *Optimization problems and* η*-approximated optimization problems*, Studia Universitatis Babes-Bolyai Mathematica, **54**, 49-62, 2009.

[23] D. Duca: *Multicriteria optimization in complex space*, Casa Cartii de Stiinta, Cluj-Napoca, 2005.

[24] R.R. Egudo, M.A. Hanson: *Multi-objective duality with invexity*, Journal of Mathematical Analysis and Applications, **126**, 469-477, 1987.

[25] J.R. Evans: *Statistics, data analysis and decision modeling*, Pearson Prentice Hall, New Jersey, 2007.

[26] D.H. Fang, C. Li, K.F. Ng: *Constraint qualifications for extended Farkas's lemmas and Lagrangian dualities in convex infinite programming*, SIAM Journal Optimization, **20**, 1311 -1332, 2009.

[27] A.V. Fiacco, K.O. Kortanek: *Semi-infinite programming and applications*, Lecture Notes in Economics and Mathematical Systems, 215, 1983.

[28] J.B.G. Frenk, G. Kassay: *On classes of generalized convex functions. Gordan-Farkas type theorems and Lagrangian duality*, Journal of Optimization Theory and Applications, **102**, 315-343, 1999.

[29] V. Gaya, M.A. López, V.N. Vera de Serio: *Stability in convex semi-infinite programming and rates of convergence of optimal solutions of discretized finite subproblems*, Optimization, **52**, 693-713, 2003.

[30] K. Glashoff, S.A. Gustafson: *Linear optimization and approximation*, Springer Verlag, Berlin, 1983.

[31] M.A. Goberna, M.A. Lopez: *Linear semi-infinite optimization*, Wiley, Chichester, 1998.

[32] M. A. Goberna, V. Jeyakumar, M.A. Lopez: *Necessary and sufficient conditions for solvability of systems of infinite convex inequalities*, Nonlinear Analysis, **68**, 1184-1194, 2008.

[33] P. Gritzmann, V. Klee: *On the complexity of some basic problems in computational convexity. I. Containment problems*, Discrete Mathematics, **136**, 129-174, 1994.

[34] F. Guerra Vázquez, J.J. Rückmann, O. Stein, G. Still: *Generalized semi-infinite programming: a tutorial*, Journal of Computational and Applied Mathematics, **217**, 394-419, 2008.

[35] S.A. Gustafson, K.O. Kortanek: *Numerical treatment of a class of semi-infinite programming problems*, Naval Research Logistics Quarterly, **20**, 477-504, 1973.

[36] R. Hettich, K.O. Kortanek: *Semi-infinite programming: theory, methods, and applications*, SIAM Review, **35**, 380-429, 1993.

[37] R. Hettich, P. Zencke: *Numerische methoden der approximation und semi-infiniten pptimierung*, Teubner, Stuttgart, 1982.

[38] R. Hettich: *An implementation of a discretization method for semi-infinite programming*, Mathematical Programming, **34(3)**, 354-361, 1986.

[39] R. Hettich, G. Gramlich: *A note on an implementation of a method for quadratic semi-infinite programming*, Mathematical Programming, **46**, 249-254, 1990.

[40] R. Hettich, G. Still: *Semi-infinite programming models in robotics*, in: J. Guddat, H.Th. Jongen, B. Kummer, F. Nožička (eds): Parametric Optimization and Related Topics II, Akademie Verlag, Berlin, 112-118, 1991.

[41] J.B. Hiriart-Urruty, C. Lemaréchal: *Convex analysis and minimization algorithms I: Fundamentals*, Springer-Verlang, Berlin, 1999.

[42] A. Hoffmann, R. Reinhardt: *On reverse Chebyshev approximation problems*, Preprint M80/94, Technical University of Ilmenau, 1994.

[43] R. Horst, H. Tuy: *Global Optimization*, Springer, Berlin, 1990.

[44] N.Q. Huy, D.S. Kim: *Lipschitz behavior of solutions to nonconvex semi-infinite vector optimization problems*, Journal of Global Optimization, **56(2)**, 431-448, 2013.

[45] M. Jaiswal, S.K. Mishra, B.A. Shamary: *Optimality conditions and duality for semi-infinite programming involving semilocally type I-preinvex and related functions*, Communication Korean Mathematical Society, **27(2)**, 441-423, 2012.

[46] J. Jahn: *Mathematical vector optimization in partially ordered linear spaces*, Verlag Peter Lang, Frankfurt, 1986.

[47] J. Jahn: *Vector optimization- theory, applications and extensions*, Springer-Verlang, Berlin, 2004.

[48] V. Jeyakumar, B. Mond: *On generalized convex mathematical programming*, Journal of Australian Mathematical Society, Ser.B, **34**, 43-53, 1992.

[49] N. Kanzi: *Lagrange multiplier rules for non-differentiable DC generalized semi-infinite programming problems*, Journal of Global Optimization, **56(2)**, 417-430, 2013.

[50] A. Kaplan, R. Tichatschke: *On a class of terminal variational problems*, in: J. Guddat, H.Th. Jongen, F. Nožička, G. Still, F. Twilt (eds): Parametric Optimization and Related Topics IV, Peter Lang, Frankfurt, 185-199, 1997.

[51] K.H. Kfer, O. Stein, A. Winterfeld: *Semi-infinite optimization meets industry: A deterministic approach to gemstone cutting*, SIAM News, **41(8)**, 2008.

[52] W. Krabs: *On time-minimal heating or cooling of a ball*, International Series of Numerical Mathematics, Birkhäuser, Basel, **81**, 121-131, 1987.

[53] T. Li, Y. Wang, Z. Liang, P.M. Pardalos: *Local saddle point and a class of convexification methods for nonconvex optimization problems*, Journal of Global Optimization, **38**, 405-419, 2007.

[54] M. Lopez, G. Still: *Semi-Infinite Programming*, European Journal of Operational Research, **180(2)**, 491-518, 2007.

[55] D.T. Luc: *Scalarization of vector optimization problems*, Journal of Optimization Theory and Applications, **55**, 85-102, 1987.

[56] D.T. Luc, **A. Raţiu**: *Fixed Point Theory, Variational Analysis and Optimization, Chapter Vector Optimization: Basic Concepts and Solution Methods*, to appear at Taylor and Francis on March 25, 2014.

[57] D.T. Luc: *Theory of vector optimization*, Springer-Verlang, Berlin, 1989.

[58] A.McD. Mercer: *Bounds for A-G, A-H, G-H, and a family of inequalities of Ky Fan's type, using a general method*, Journal of Mathematical Analysis and Applications, **243(1)**, 163-173, 2000.

[59] D. Mărghidanu: *Generalizations and refinements for Bergstrőm and Radon's inequalities*, Journal of Science and Arts, **8(1)**, 57-62, 2008.

[60] N. Minculete, N.B. Pipu, **A. Raţiu**: *Characterization of some bounds for several statistical indicators*, Jökull Journal, **63(7)**, 271-276, 2013.

[61] S.K. Mishra, G. Giorgi: *Invexity and optimization*, Nonconvex optimization and Its Applications, Springer-Verlag, Berlin, **88**, 2008.

[62] S.K. Mishra, M. Jaiswal, Le Thi Hoai An: *Duality for nonsmooth semi-infinite programming problems*, Optimization Letters, **6(2)**, 261-271, 2012.

[63] C.P. Niculescu, L.E. Persson: *Convex functions and their applications. A contemporary approach*, CMS Books in Mathematics, Springer-Verlang, New York, 23, 2006.

[64] V.H. Nguyen, J.J. Strodiot: *Computing a global optimal solution to a design centering problem*, Mathematical Programming, **53**, 111-123, 1992.

[65] E. Polak: *Optimization. Algorithms and consistent approximations*, Springer, Berlin, 1997.

[66] E. Polak: *An implementable algorithm for the optimal design centering, tolerancing and tuning problem*, Journal of Optimization Theory and Applications, **37**, 45-67, 1982.

[67] E. Polak: *On the mathematical foundation of nondifferentiable optimization in engineering design*, SIAM Review, **29**, 21-89, 1987.

[68] E.L. Pop, D. Duca: *Optimization problems, first order approximated optimization problems and their connections*, Carpathian Journal of Mathematics, **28(1)**, 133-141, 2012.

[69] J. Radon: *Theorie und anwendungen der absolut additiven mengenfunktionen*, Wiener Sitzungsber, **122**, 1295-1438, 1913.

[70] **A. Raţiu**, D.I. Duca: *First order approximated semi-infinite optimization problems*, General Mathematics, Special Issue, **20(5)**, 99-106, 2012.

[71] **A. Raţiu**, D.I. Duca: *Semi-infinite optimization problems and their approximations*, Studia Universitatis Babes-Bolyai, Mathematica, **58(3)**, 407-417, 2013.

[72] **A. Raţiu**, D.I. Duca: *Semi-infinite optimization problems and their first order approximated*, Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, **11**, 2013.

[73] **A. Raţiu**, D.I. Duca: *Semi-infinite optimization problems-second order approximated optimization problems*, Jökull Journal, **63(9)**, 2013.

[74] **A. Raţiu**, D.I. Duca: *Connections between the optimal solutions of semi-infinite optimization problem and her approximate*, submitted.

[75] **A. Raţiu**, D.I. Duca: *Approximations of semi-infinite vector optimization problems*, submitted.

[76] **A. Raţiu**: *Convex-concave infinite minimization*, submitted.

[77] **A. Raţiu**, N. Minculete: *Several refinements and counterparts of Radon's inequality*, to appear in Mathematica Bohemica.

[78] R. Reemtsen: *Discretization methods for the solution of semi-infinite programming problems*, Journal of Optimization Theory and Applications, **71(1)**, 5-103, 1991.

[79] R.T. Rockafellar: *Convex analysis*, Princeton University Press, Princeton, 1970.

[80] S. Rolewicz: *Suficient condition for Pareto optimization in Banach spaces*, Studia Mathematica, **77**, 111-114, 1981.

[81] J. J. Rückmann, A. Shapiro: *Augmented Lagrangians in semi-infinite programming*, Mathematical Programming, Ser. B, 116:499-512, 2009.

[82] A. Shapiro: *Semi-infinite programming, duality, discretization and optimality conditions*, Optimization, **58(2)**, 133-161, 2009.

[83] A. Shapiro: *On duality theory of convex semi-infinite programming*, Optimization, **54**, 535-543, 2005.

[84] O. Stein: *Bi-level strategies in semi-infinite programming*, Kluwer, Boston, 2003.

[85] F. Terkelsen: *Some minimax theorems*, Mathematica Scandinavica, **31**, 405-413, 1972.

[86] R. Tichatschke: *Lineare semi-infinite optimierungsaufgaben und ihre anwendungen in der approximations theorie*, Karl-Marx-Stadt: Wissenschaftliche Schriftenreihe der Technischen Hochschule, 1981.

[87] R.H. Tütüncü, M. Koenig: *Robust asset allocation*, Annals of Operations Research, **132**, 157-187, 2004.

[88] A.I.F. Vaz, E.C. Ferreira: *Air pollution control with semi-infinite programming*, Applied Mathematical Modelling, **33**, 1957-1969, 2009.

[89] G.W. Weber: *Generalized Semi-Infinite Optimization and Related Topics*, Heldermann Verlag, Lemgo, 2003.

[90] G.W. Weber, A. Tezel: *On generalized semi-infinite optimization of genetic networks*, TOP, **15**, 65-77, 2007.