BABEŞ-BOLYAI UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

Oana Crişan

CONTRIBUTIONS TO GEOMETRIC FUNCTION THEORY OF ONE COMPLEX VARIABLE

Ph.D. Thesis Summary

Scientific Advisors:

Prof. Univ. Dr. Grigore Ştefan Sălăgean
Prof. Univ. Dr. Hab. Stanisława Kanas

Cluj-Napoca
2013

Contents

Introduction

With a history dating back to the 18th century, complex analysis is a broad and textured subject, with applications not only to other parts of analysis, but also to many areas of mathematics and science in general.

Two important branches of complex analysis are the theory of conformal representation and the geometric function theory of analytic functions. The latter, which deals with the geometric properties of analytic functions, was born around the turn of the 20th century, yet it remains an active field of current research. Among the first important papers which discuss topics from this domain are the works of P. Koebe [37], I.W. Alexander [3] or L. Bieberbach [8]. Koebe initiated in 1907 the univalent functions study, while Bieberbach presented in 1916 what would soon become a famous conjecture. One of the major problems and a cornerstone for the subsequent development of this field, the Bieberbach conjecture asserts that the coefficients in the Taylor series expansion of every function from the class $\mathcal{S}$ of normalized univalent functions in the unit disc satisfy the inequality $|a_n| \leq n$. This problem stood for many years as a challenge, inspiring the development of new and elaborate research methods, such as Löwner's parametric method, the variational methods introduced by M. Schiffer and G.M. Goluzin, the extreme points method owed to L. Brickman, etc..

Although almost 70 years had passed before the Biberbach conjecture was finally proved, bounds for the Taylor series coefficients were obtained in the meantime much more easily for some subclasses of univalent functions than for the full class $\mathcal{S}$, and the study of different classes of analytic, univalent or meromorphic functions began to take shape, remaining today a subject of vast interest.

Another field with many applications in geometric function theory is that of differential subordinations, which traces its origins to an article from 1981 by P.T. Mocanu and S.S. Miller [53], a paper which laid a foundation for the following development of hundreds of articles related to this subject.

There are many books and monographs nowadays dedicated to geometric function theory or the study of univalent functions, of which we mention those of L.V. Ahlfors [2], C. Pommerenke [68], J.B. Conway [13], A.W. Goodman [27], P.L. Duren [23], , D.J. Hallenbeck and T.H MacGregor [32], S.S. Miller and P.T. Mocanu [52].

The purpose of this thesis is to present some new classes of both analytic functions and meromorphic functions and study their properties, as well as to establish some new results on differential subordinations.

The thesis consists of four chapters. The first chapter starts with some basic notations and notions and a summary of the fundamental results related to the aforementioned class $\mathcal{S}$. Various subclasses of both analytic and meromorphic functions are considered next. Another topic treated in this chapter is the basic theory of the field of differential subordinations, while the final section in Chapter 1 gives a brief presentation of some differential and integral operators.

The second chapter focuses on some original results on analytic functions: in Section 2.1 and Section 2.2 are presented results concerning a differential operator and an integral operator, respectively, results which are obtained by means of the differential subordinations method. Section 2.3 studies several coefficient estimates problems for different subclasses of bi-univalent functions in the unit disc.

Chapter 3 is dedicated to the study of meromorphic functions. In Sections 3.1 and 3.2 are introduced, through subordination, and investigated new classes of meromorphic functions for which are given, among other results, coefficient bounds, inclusion relations, integral-preserving properties and convolution properties. Section 3.3 contains a series of sufficient conditions for meromorphic multivalent close-to-convex functions, obtained by using the admissible functions method.

Finally, Chapter 4 deals with subordinations results involving expressions using the combined arithmetic and geometric means. Some special cases which provide interesting applications are also considered. With the exception of Lemma 4.1.1, all the results from this chapter are also original.

This work was possible with the financial support of the Sectoral Operation Programme for Human Resources Development 2007-2013, co-financed by the European Social Fund, under the project number POSDRU/107/1.5/ S/76841 with the title "Modern Doctoral Studies: Internationalization and Interdisciplinarity".

I express my sincere thanks to Prof. Grigore Ştefan Sălăgean and Prof. Stanisława Kanas, for their advice, valuable criticism and encouragement throughout the preparation of this work.

In the following I selected the most relevant results, with emphasis on my original contributions. The full bibliography is also included.

Keywords: univalence, starlikeness, convexity, close-to-convexity, spirallikeness, functions with positive real part, differential subordinations, Ruscheweyh differential operator, Al-Oboudi differential operator, Sălăgean integral operator, Bernardi integral operator, bi-univalent functions, meromorphic functions.

Chapter 1

Preliminary results

We begin with the presentation of some notions and fundamental results from the geometric function theory of one complex variable. We present first the basic properties of the class $\mathcal{S}$ of normalized univalent functions on the unit disc, then we consider various subclasses of univalent functions, such as the well-known starlike and convex functions, but also, among others, $\alpha-$convex, close-to-convex, spirallike functions or functions which are starlike or convex with respect to symmetric points. Some subclasses of meromorphic functions are also presented. One of the sections in this chapter is dedicated to the method of differential subordinations. The final section includes the definitions and some basic properties of some differential and integral operators.

1.1 Notations and elementary results from the theory of univalent functions

Let $\mathbb{C}$ be the complex plane and let $\mathcal{U}(z_0, r)$ be the open disc of radius $r > 0$ centered at $z_0 \in \mathbb{C}$,

$$\mathcal{U}(z_0, r) = \{z \in \mathbb{C} \, : \, |z - z_0| < r\}.$$

The open disc $\mathcal{U}(0, r)$ will be denoted by $\mathcal{U}_r$, and the unit disc $\mathcal{U}_1$ will be denoted by $\mathcal{U}$. We shall also use the notations $\mathcal{U}^* := \mathcal{U}\backslash\{0\}$ and $\overline{\mathcal{U}}(z_0, r)$ for the closure of $\mathcal{U}(z_0, r)$. The boundary of a set G will be denoted by ∂G.

If G is an open subset of $\mathbb{C}$, we will denote by $\mathcal{H}(G)$ the set of all analytic functions on G with values in $\mathbb{C}$. Endowed with the topology of local uniform convergence (or uniform convergence on compact subsets), the set $\mathcal{H}(G)$ is a linear topological space. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathcal{U}) \ : \ f(z) = a + a_n z^n + \cdots , \ z \in \mathcal{U}\}$$

and

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(\mathcal{U}) \ : \ f(z) = z + a_{n+1} z^{n+1} + \cdots , \ z \in \mathcal{U}\right\},$$

with

$$\mathcal{A} := \mathcal{A}_1.$$

Definition 1.1.1. Let D be a domain in $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is called *univalent* if $f \in \mathcal{H}(D)$ and f is one-to-one on D. We denote by $\mathcal{H}_u(D)$ the class of univalent functions in D and by $\mathcal{S}$ the class of functions $f \in \mathcal{H}_u(\mathcal{U})$ which are normalized by the condition $f(0) = f'(0) - 1 = 0$.

Bieberbach's conjecture. If the function $f(z) = z + a_2 z^2 + \cdots$ belongs to $\mathcal{S}$, then

$$|a_n| \leq n, \quad n = 2, 3, \ldots. \tag{1.1}$$

Equality occurs in (1.1) for a given $n \geq 2$ if and only if f is a rotation of the Koebe function.

The above conjecture remained unsolved until 1985, when it was proved by L. de Branges [22], by means of the method of Löwner chains.

1.2 The Carathéodory class. Subordination. The principle of subordination

We give in this section the basic properties of functions with positive real part in the unit disc $\mathcal{U}$. We also present the concept of subordination in the complex plane.

Definition 1.2.1. The Carathéodory class, denoted by $\mathcal{P}$, is the class of functions p analytic in $\mathcal{U}$ with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, $z \in \mathcal{U}$.

Definition 1.2.2 ([62], [68])**.** Let f and g be two analytic functions in $\mathcal{U}$. We say that f *is subordinate to* g, written as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathcal{U}$ such that

$$f(z) = g(w(z)), \quad z \in \mathcal{U}.$$

Property 1.2.3 ([62])**.** *Let $f, g \in \mathcal{H}(\mathcal{U})$. If $f \prec g$ then the following are true:*

(i) $f(\overline{\mathcal{U}_r}) \subseteq g(\overline{\mathcal{U}_r})$, *for any* $0 < r < 1$;

(ii) $\max\{|f(z)| : |z| < r\} \leq \max\{|g(z)| : |z| < r\}$, *for any* $0 < r < 1$;

(iii) $|f'(0)| \leq |g'(0)|$.

In all the above inequalities, equality is attained when $f(z) = g(\lambda z)$, where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

For the case when g is a univalent function, we have the following characterization for subordination:

Theorem 1.2.4 ([62], [68])**.** *Let f be an analytic function and g be analytic and univalent in $\mathcal{U}$. Then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.*

1.3 Starlike and convex functions

We introduce now some special classes of univalent functions defined by geometric properties and which can be completely characterized by simple inequalities. These classes are closely related with functions of positive real part and with subordination.

Definition 1.3.1. Let $f \in \mathcal{H}(\mathcal{U})$ such that $f(0) = 0$. We say that f is *starlike with respect to the origin* (or simply, *starlike*) if f is univalent and the image $f(\mathcal{U})$ is a starlike domain with respect to the origin.

Theorem 1.3.2 ([62])**.** *Let $f \in \mathcal{H}(U)$ with $f(0) = 0$. The function f is starlike if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0\,, \quad z \in \mathcal{U}.$$

Definition 1.3.3 ([62])**.** The class $\mathcal{S}^*$ is defined as the set of all starlike and normalized functions in $\mathcal{U}$, i.e.

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} \,:\, \operatorname{Re} \frac{zf'(z)}{f(z)} > 0\,,\ z \in \mathcal{U} \right\}.$$

Definition 1.3.4. Let $f \in \mathcal{H}(\mathcal{U})$. We say that f is *convex* in $\mathcal{U}$ if f is univalent in $\mathcal{U}$ and the image $f(\mathcal{U})$ is a convex domain in $\mathbb{C}$.

Theorem 1.3.5 ([62])**.** *A function $f \in \mathcal{H}(\mathcal{U})$ is convex if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0\,, \quad z \in \mathcal{U}.$$

The following result, due to Alexander [3], provides a connection between starlikeness and convexity.

Theorem 1.3.6 ([3], Alexander's duality theorem)**.** *A function f is convex in $\mathcal{U}$ if and only if the function g defined by $g(z) = zf'(z)$, $z \in \mathcal{U}$, is starlike in $\mathcal{U}$.*

Definition 1.3.7 ([62])**.** The class $\mathcal{K}$ is the set of all normalized and convex functions in $\mathcal{U}$, i.e.

$$\mathcal{K} = \left\{ f \in \mathcal{A} \,:\, \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0\,,\ z \in \mathcal{U} \right\}.$$

We have the inclusions $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$. Alexander's duality theorem can be rewritten using the classes $\mathcal{S}^*$ and $\mathcal{K}$ as

$$f(z) \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*.$$

1.4 Classes of functions related to starlikeness and convexity

We turn now to some subclasses of the normalized starlike and convex functions in the unit disc, and we give some of their basic properties.

Definition 1.4.1. Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{A}$ such that $\dfrac{f(z)f'(z)}{z} \neq 0$, $z \in \mathcal{U}$. Let also

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

We say that f is $\alpha-$*convex in* $\mathcal{U}$ if

$$\operatorname{Re} J(\alpha, f; z) > 0, \quad z \in \mathcal{U}.$$

We denote by $\mathcal{M}_\alpha$ the class of all $\alpha-$convex functions in $\mathcal{U}$. It is clear that $\mathcal{M}_0 = \mathcal{S}^*$ and $\mathcal{M}_1 = \mathcal{K}$.

Theorem 1.4.2 ([59], [57])**.** *If* $\alpha \in \mathbb{R}$, *then* $\mathcal{M}_\alpha \subseteq \mathcal{S}^*$. *Moreover, for all* $\alpha, \beta \in \mathbb{R}$ *with* $0 \leq \alpha/\beta < 1$, $\mathcal{M}_\beta \subset \mathcal{M}_\alpha$.

Using the geometric means, Lewandowski et al. [42] defined, in a similar manner as Mocanu, the class of $\gamma-$starlike functions:

Definition 1.4.3 ([42])**.** Let $\gamma \in \mathbb{R}$ and $f \in \mathcal{A}$. Let also

$$L(\gamma, f; z) = \left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}.$$

We say that f is $\gamma-$*starlike* in $\mathcal{U}$ if

$$L(\gamma, f; z) > 0, \quad z \in \mathcal{U}.$$

We denote by $\mathcal{L}_\gamma$ the class of all $\gamma-$starlike functions in $\mathcal{U}$. It is clear that $\mathcal{L}_0 = \mathcal{S}^*$ and $\mathcal{L}_1 = \mathcal{K}$.

Theorem 1.4.4 ([42], [43])**.** *Let* $\gamma \in \mathbb{R}$. *Then* $\mathcal{L}_\gamma \subset \mathcal{S}^*$.

The following notions were introduced by Robertson [72].

Definition 1.4.5. Let $\alpha \in [0,1)$ and let f be an analytic function on $\mathcal{U}$. We say that f is *starlike of order* α *in* $\mathcal{U}$ if $f(0) = 0$, $f'(0) \neq 0$ and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathcal{U}.$$

Also, we say that f is *convex of order* α in $\mathcal{U}$ if $f'(0) \neq 0$ and

$$\operatorname{Re}\frac{zf''(z)}{f'(z)} + 1 > \alpha, \quad z \in \mathcal{U}.$$

Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of normalized starlike functions of order α in $\mathcal{U}$ and convex functions of order α in $\mathcal{U}$, respectively.

Definition 1.4.6. Let $\gamma \in (0,1]$. A function $f \in \mathcal{A}$ is called *strongly starlike of order* γ if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\gamma, \quad z \in \mathcal{U}.$$

Also, $f \in \mathcal{A}$ is called *strongly convex of order* γ if

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\gamma, \quad z \in \mathcal{U}.$$

Let $\mathcal{SS}^*(\gamma)$ and $\mathcal{SK}(\gamma)$ denote the classes of all strongly starlike functions of order γ and strongly convex functions of order γ, respectively.

These two classes of functions have been extensevely studied by Mocanu and Nunokawa (the reader is referred, for example, to [60], [61], [66]).

Ma and Minda [49] unified various subclasses of starlike and convex functions, by introducing two new subclasses using the requirement that either of the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general function.

Definition 1.4.7 ([49]). Let ϕ be an analytic function with positive real part in $\mathcal{U}$, such that $\phi(0) = 1$, $\phi'(0) > 0$ and with the property that $\phi(\mathcal{U})$ is starlike with respect to 1 and symmetric with respect to the real axis. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\phi)$ if it satisfies the subordination

$$\frac{zf'(z)}{f(z)} \prec \phi(z).$$

The class $\mathcal{S}^*(\phi)$ is called the class of *Ma-Minda starlike functions*. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}(\phi)$ if it satisfies the subordination

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z).$$

The class $\mathcal{K}(\phi)$ is called the class of *Ma-Minda convex functions*.

We present next the class of starlike functions with respect to symmetric points, denoted by $\mathcal{S}_s^*$, which was introduced and investigated by Sakaguchi in [77], and the class of convex functions with respect to symmetric points, denoted by $\mathcal{K}_s$, introduced by Wang et al. in [88].

Definition 1.4.8 ([77]). A function $f \in \mathcal{A}$ is said to be *starlike with respect to symmetric points in* $\mathcal{U}$ if for every $r < 1$, sufficiently close to 1, and every z_0 on the circle $|z| = r$, the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction.

Remark 1.4.9 ([77]). The class of functions univalent and starlike with respect to symmetric points includes the classes of convex functions and odd starlike functions.

Theorem 1.4.10 ([77])**.** *Let $f \in \mathcal{A}$. A necessary and sufficient condition for a function f to be univalent and starlike with respect to symmetric points in $\mathcal{U}$ is*

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(-z)} > 0, \quad z \in \mathcal{U}.$$

Definition 1.4.11 ([88])**.** A function $f \in \mathcal{S}$ is said to be *convex with respect to symmetric points in $\mathcal{U}$* if it satisfies the inequality

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z) + f'(-z)} > 0, \quad z \in \mathcal{U}.$$

In the style of Ma and Minda, Ravichandran [69] generalized the classes $\mathcal{S}_s^*$ and $\mathcal{K}_s$ by means of subordination, defining the classes $\mathcal{S}_s^*(\phi)$ and $\mathcal{K}_s(\phi)$, with ϕ as in Definition 1.4.7:

Definition 1.4.12 ([69])**.** A function $f \in \mathcal{A}$ is said to be in $\mathcal{S}_s^*(\phi)$ if the following subordination holds

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z).$$

Definition 1.4.13 ([69])**.** A function $f \in \mathcal{A}$ is said to be in $\mathcal{K}_s(\phi)$ if it satisfies the subordination

$$\frac{2(zf'(z))'}{f'(z) + f'(-z)} \prec \phi(z).$$

At the end of this section we give the definition of the prestarlike functions, which were introduced by Ruscheweyh in [74].

Definition 1.4.14 ([74])**.** Let $f \in \mathcal{A}$ and $\gamma < 1$. We say that f is *prestarlike of order γ in $\mathcal{U}$* if

$$f(z) * \frac{z}{(1-z)^{2-2\gamma}} \in \mathcal{S}^*(\gamma).$$

We denote by $\mathcal{R}(\gamma)$ the class of all prestarlike functions of γ in $\mathcal{U}$. The class $\mathcal{R}(1)$ is consists of all functions $f \in \mathcal{A}$ for which the inequality $\operatorname{Re}(f(z)/z) > 1/2$, $z \in \mathcal{U}$, holds true.

1.5 Close-to-convexity, quasi-convexity, spirallikeness

The notion of close-to-convexity is due to Kaplan [36].

Definition 1.5.1. Let $f \in \mathcal{H}(\mathcal{U})$. We say that f is *close-to-convex in $\mathcal{U}$* (or simply *close-to-convex*) if there exists a convex function g in $\mathcal{U}$ such that

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in \mathcal{U}. \tag{1.2}$$

Remark 1.5.2. Using Theorem 1.3.6, the condition (1.2) can be replaced by the requirement that

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0$$

where g is a starlike function in $\mathcal{U}$. From here it follows that if f is starlike in $\mathcal{U}$, then f is also close-to-convex.

Definition 1.5.3. The class $\mathcal{C}$ is the set of all normalized and close-to-convex functions in $\mathcal{U}$, i.e.

$$\mathcal{C} = \left\{ f \in \mathcal{A} \,:\, \mathrm{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0,\ g \text{ convex},\ z \in \mathcal{U} \right\}.$$

Remark 1.5.4. We have the inclusions

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}.$$

Related to the class $\mathcal{C}$, we next give the definition of quasi-convex functions.

Definition 1.5.5 ([64])**.** A function $f \in \mathcal{A}$ is said to be *quasi-convex in* $\mathcal{U}$ (or simply, *quasi-convex*) if there exists a function $g \in \mathcal{K}$ such that

$$\mathrm{Re}\left(\frac{(zf'(z))'}{g'(z)}\right) > 0, \quad z \in \mathcal{U}. \tag{1.3}$$

We denote by $\mathcal{Q}$ the class of all quasi-convex functions in $\mathcal{U}$, i.e.

$$\mathcal{Q} = \left\{ f \in \mathcal{A} \,:\, \mathrm{Re}\left(\frac{(zf'(z))'}{g'(z)}\right) > 0,\ g \in \mathcal{K},\ z \in \mathcal{U} \right\}.$$

Definition 1.5.6. Let $f \in \mathcal{H}(\mathcal{U})$ such that $f(0) = 0$. We say that f is λ-*spirallike* if f is univalent in $\mathcal{U}$ and $f(\mathcal{U})$ is a $\lambda-$spirallike domain. We say that f is *spirallike* if there exists an $\lambda \in (-\pi/2, \pi/2)$ such that f is $\lambda-$spirallike.

We denote the class of normalized $\lambda-$spirallike functions by $\hat{\mathcal{S}}_\lambda$. We observe that $\hat{\mathcal{S}}_0 = \mathcal{S}^*$.

The class of spirallike functions was first studied by Spaček [81], who also gave the following theorem which provides a necessary and sufficient condition for $\lambda-$spirallikeness in $\mathcal{U}$.

Theorem 1.5.7 ([81])**.** *Let $f \in \mathcal{H}(U)$ such that $f(0) = 0$ and $f'(0) \neq 0$. Also let $\lambda \in (-\pi/2, \pi/2)$. Then f is $\lambda-$spirallike if and only if*

$$\mathrm{Re}\left(e^{i\lambda}\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathcal{U}.$$

The class of $\lambda-$*spirallike functions of order* α, defined below and denoted by $\mathcal{S}^\lambda(\alpha)$, was first considered by Libera in [45].

Definition 1.5.8 ([45])**.** Let $0 \le \alpha < 1$ and $\lambda \in (-\pi/2, \pi/2)$. A function $f \in \mathcal{A}$ is said to be *λ-spirallike of order α* in $\mathcal{U}$ if

$$\mathrm{Re}\left(e^{i\lambda}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\lambda\,, \quad z \in \mathcal{U}. \tag{1.4}$$

The next class of functions was introduced and first studied by Chichra in [12].

Definition 1.5.9 ([12])**.** Let $0 \le \alpha < 1$ and $\lambda \in (-\pi/2, \pi/2)$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{F}^\lambda(\alpha)$ if it satisfies the inequality

$$\mathrm{Re}\left[e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \alpha\cos\lambda\,, \quad z \in \mathcal{U}. \tag{1.5}$$

Remark 1.5.10. It follows from (1.4) and (1.5) that $f(z) \in \mathcal{F}^\lambda(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^\lambda(\alpha)$.

We also note that $\mathcal{S}^\lambda(0) = \hat{\mathcal{S}}_\lambda$, $\mathcal{S}^0(\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{F}^0(\alpha) = \mathcal{K}(\alpha)$.

1.6 Differential subordinations

The differential subordinations method (or admissible functions method) is one of the newest and most frequently used methods in the geometric theory of analytical functions. The bases of this theory where made by S.S. Miller and P.T. Mocanu in their papers [53] and [54].

Definition 1.6.1 ([53])**.** Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in $\mathcal{U}$. If $p \in \mathcal{H}[a, n]$ satisfies the *differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in U, \tag{1.6}$$

then we say that p is a (a, n)-*solution* (or, simply, *solution*) of the differential subordination (1.6).

A univalent function q is called a (a, n)-*dominant* (or, simply, *dominant*) of the differential subordination (1.6) if $p(z) \prec q(z)$, for all p solutions of (1.6).

A dominant $\widetilde{q}$ with the property $\widetilde{q} \prec q$ for all dominants q of (1.6) is said to be the *best* (a, n)-*dominant* (or, simply, *best dominant*) of the differential subordination (1.6).

The method of differential subordinations is based on the following fundamental lemmas.

Lemma 1.6.2 ([54])**.** *Let $z_0 = r_0e^{i\theta_0}$, $0 < r_0 < 1$, and let $f(z) = a_nz^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots$ be continuous on $\overline{\mathcal{U}}_{r_0}$ and analytic on $\mathcal{U}_{r_0} \cup \{z_0\}$ with $f(z) \not\equiv 0$ and $n \ge 1$. If*

$$|f(z_0)| = \max\left\{|f(z)| \,:\, z \in \overline{\mathcal{U}_{r_0}}\right\},$$

then there exists $m \ge n$ such that

(i) $\dfrac{z_0 f'(z_0)}{f(z_0)} = m$

and

(ii) $\operatorname{Re}\dfrac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq m.$

In order to extend the ideas of this lemma, by replacing the disc $|w| < r_0$ with a more general region Δ, the following class of functions was introduced:

Definition 1.6.3. We denote by Q the set of functions q which are univalent in $\overline{\mathcal{U}}\backslash E(q)$, where

$$E(q) = \left\{\zeta \in \partial\mathcal{U} \,:\, \lim_{z\to\zeta} q(z) = \infty\right\},$$

and with the property that $q'(\zeta) \neq 0$, $\zeta \in \partial\mathcal{U}\backslash E(q)$.

Lemma 1.6.4 ([53])**.** *Let $p(z) = a + a_n z^n + \cdots$ be analytic in $\mathcal{U}$ with $p(z) \not\equiv a$ and $n \geq 1$, and let $q \in Q$ with $q(0) = a$. If there exist two points $z_0 = r_0 e^{i\theta_0} \in \mathcal{U}$ and $\zeta_0 \in \partial\mathcal{U}\backslash E(q)$ such that $p(z_0) = q(\zeta_0)$ and $p(\mathcal{U}_{r_0}) \subset q(\mathcal{U})$, then there exists a number $m \geq n \geq 1$ such that*

(i) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$

and

(ii) $\operatorname{Re}\dfrac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq m\operatorname{Re}\left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right].$

The next lemma is a variation of the previous one, and it considers a case of subordination of two functions.

Lemma 1.6.5 ([52])**.** *Let $q \in Q$ with $q(0) = a$ and let $p = a + a_n z^n + \cdots$ be analytic in $\mathcal{U}$ with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist two the points $z_0 = r_0 e^{\theta_0} \in \mathcal{U}$ and $\zeta_0 \in \partial\mathcal{U} \setminus E(q)$, and a number $m \geq n \geq 1$ for which $p(\mathcal{U}_{|z_0|}) \subset q(\mathcal{U})$ and*

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$

and

(iii) $\operatorname{Re}\dfrac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq m\operatorname{Re}\left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right].$

Definition 1.6.6 ([53])**.** Let $\Omega \subset \mathbb{C}$, $q \in Q$ and $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. We denote by $\Psi_n[\Omega, q]$ the class of functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ which satisfy the condition

$$\psi(r, s, t; z) \notin \Omega \quad \text{whenever}$$
$$r = q(\zeta),\ s = m\zeta q'(\zeta),\ \operatorname{Re}\left[\frac{t}{s} + 1\right] \geq m\operatorname{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right],$$
$$\text{where } z \in \mathcal{U},\ \zeta \in \partial\mathcal{U}\backslash E(q) \text{ and } m \geq n.$$

$\Psi_n[\Omega, q]$ is called the *set of admissible functions*, and the above condition is called the *admissibility condition.*

Theorem 1.6.7 ([30])**.** *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}\backslash\{0\}$ such that* $\operatorname{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)\,, \tag{1.7}$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{(\gamma/n)-1}dt\,, \quad z \in \mathcal{U}.$$

The function q is convex and is the best dominant of (1.7).

Definition 1.6.8 ([52])**.** Let h be a univalent function in $\mathcal{U}$ such that $h(0) = a$, and let $p \in \mathcal{H}[a, n]$ and β, $\gamma \in \mathbb{C}$ with $\beta \neq 0$. A differential subordination of the type

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \tag{1.8}$$

is called a *Briot-Bouquet differential subordination.*

Theorem 1.6.9 ([24])**.** *Let β, $\gamma \in \mathbb{C}$ with $\beta \neq 0$ and let h be a convex function which satisfies the inequality*

$$\operatorname{Re}\left[\beta h(z) + \gamma\right] > 0, \quad z \in \mathcal{U}.$$

If p is analytic in $\mathcal{U}$ and $p(0) = h(0)$, then the subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies $p(z) \prec h(z)$.

1.7 Subclasses of meromorphic functions

Let

$$\varphi = \zeta + a_0 + \frac{a_1}{\zeta} + \cdots\,, \quad \zeta \in \mathcal{U}^- \tag{1.9}$$

be a meromorphic function from Σ and let $E(\varphi) = \mathbb{C}\backslash\varphi(\mathcal{U}^-)$.

We will use the notation Σ_0 for the subclass of functions $\varphi \in \Sigma$ which do not vanish in the exterior of the unit disc, i.e.

$$\Sigma_0 = \left\{\varphi \in \Sigma\, :\, \varphi(\zeta) \neq 0,\ \zeta \in \mathcal{U}^-\right\}.$$

Definition 1.7.1 ([62]). A function φ having the form (1.9) is called *starlike in* $\mathcal{U}^-$ if φ is univalent in $\mathcal{U}^-$ and the set $E(\varphi)$ is starlike with respect to the origin.

Definition 1.7.2 ([62]). Let

$$f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots, \quad z \in \mathcal{U}^*$$

be a meromorphic function in $\mathcal{U}^*$. We say that f is *starlike in* $\mathcal{U}^*$ if the function $\varphi = f\left(\frac{1}{\zeta}\right)$, $\zeta \in \mathcal{U}^-$ is starlike in $\mathcal{U}^-$.

Theorem 1.7.3 ([62]). *Let* $f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots$, $z \in \mathcal{U}^*$ *be meromorphic in* $\mathcal{U}^*$ *with* $f(z) \neq 0$, $z \in \mathcal{U}^*$. *Then* f *is starlike in* $\mathcal{U}^*$ *if and only if* f *is univalent in* $\mathcal{U}^*$ *and*

$$\operatorname{Re}\left[-\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in \mathcal{U}^*.$$

We give below the definition of the class of *meromorphic starlike functions of order* α in $\mathcal{U}^*$ $(0 \leq \alpha < 1)$, which we shall denote by $\Sigma^*(\alpha)$:

$$\Sigma^*(\alpha) = \left\{ f \text{ meromorphic in } \mathcal{U}^* \,:\, f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots, \ \operatorname{Re}\left[-\frac{zf'(z)}{f(z)}\right] > \alpha, \ z \in \mathcal{U}^* \right\}. \tag{1.10}$$

Definition 1.7.4 ([62]). A function φ having the form (1.9) is called *convex in* $\mathcal{U}^-$ if φ is univalent in $\mathcal{U}^-$ and the set $E(\varphi)$ is convex.

Definition 1.7.5 ([62]). Let

$$f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots, \quad z \in \mathcal{U}^*$$

be a meromorphic function in $\mathcal{U}^*$. We say that f is *convex in* $\mathcal{U}^*$ if the function $\varphi = f\left(\frac{1}{\zeta}\right)$, $\zeta \in \mathcal{U}^-$ is convex in $\mathcal{U}^-$.

Theorem 1.7.6 ([62]). *Let* $f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots$, $z \in \mathcal{U}^*$ *be meromorphic in* $\mathcal{U}^*$ *with* $f(z) \neq 0$, $z \in \mathcal{U}^*$. *Then* f *is convex in* $\mathcal{U}^*$ *if and only if* f *is univalent in* $\mathcal{U}^*$ *and*

$$\operatorname{Re}\left[-\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0, \quad z \in \mathcal{U}^*.$$

Definition 1.7.7 ([1]). Let $\varphi \in \Sigma_0$. We say that φ is *close-to-convex in* $\mathcal{U}^-$ if there exists a functions $\psi \in \Sigma^*$ such that $\operatorname{Re} \dfrac{\zeta\varphi'(\zeta)}{\psi(\zeta)} > 0$, $\zeta \in \mathcal{U}^-$.

Definition 1.7.8 ([1])**.** Let

$$f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots, \quad z \in \mathcal{U}^*$$

be a meromorphic function in $\mathcal{U}^*$. We say that f is *close-to-convex in* $\mathcal{U}^*$ if there exists a function g, meromorphic and starlike in $\mathcal{U}^*$, such that $\varphi(\zeta) = f\left(\frac{1}{\zeta}\right)$, $\zeta \in \mathcal{U}^-$, is close-to-convex in $\mathcal{U}^-$ with respect to $\psi(\zeta) = g\left(\frac{1}{\zeta}\right)$, $\zeta \in \mathcal{U}^-$ (which is starlike in $\mathcal{U}^-$).

Theorem 1.7.9 ([1])**.** *Let $f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots$, $z \in \mathcal{U}^*$ be meromorphic in $\mathcal{U}^*$ with $f(z) \neq 0$, $z \in \mathcal{U}^*$. Then f is close-to-convex in $\mathcal{U}^*$ if and only if f is univalent in $\mathcal{U}^*$ and there exists a function g, meromorphic and starlike in $\mathcal{U}^*$, such that*

$$\operatorname{Re}\left[-\frac{zf'(z)}{g(z)}\right] > 0, \quad z \in \mathcal{U}^*.$$

1.8 Differential and integral operators

We present in this section some well-known differential and integral operators, which shall be later used in obtaining a series of new results.

Let $f, g \in \mathcal{H}(\mathcal{U})$, $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $g(z) = \sum_{j=0}^{\infty} b_j z^j$. We denote by $f * g$ the *convolution* (or *Hadamard product*) of the functions f and g, given by

$$(f * g)(z) \equiv f(z) * g(z) = \sum_{j=0}^{\infty} a_j b_j z^j.$$

Definition 1.8.1 ([75])**.** Let $n \in \mathbb{N}$. The Ruscheweyh differential operator $R^n : \mathcal{A} \to \mathcal{A}$ is given by

$$R^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z), \quad f \in \mathcal{A}, \ z \in \mathcal{U}.$$

Remark 1.8.2 ([75])**.** For $n \in \mathbb{N}$ and $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $R^n f$ has the power series expansion

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U. \tag{1.11}$$

Definition 1.8.3 ([5])**.** Let $n \in \mathbb{N}$ and $\delta \geq 0$. The Al-Oboudi differential operator $D_\delta^n : \mathcal{A} \to \mathcal{A}$, is defined by

$$D^0 f(z) = f(z),$$

$$D^1 f(z) \equiv D_\delta f(z) = zf'(z),$$

$$D_\delta^n f(z) = D_\delta(D_\delta^{n-1} f(z)), \quad z \in\ U.$$

Remark 1.8.4 ([5])**.** If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then

$$D_\delta^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_j z^j, \quad z \in \mathcal{U}. \tag{1.12}$$

Definition 1.8.5 ([79])**.** For $n \in \mathbb{N}$, the integral Sălăgean operator $I^n : \mathcal{A} \to \mathcal{A}$ is defined by

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = I f(z) = \int_0^z f(t) t^{-1} dt,$$

and

$$I^n f(z) = I(I^{n-1} f(z)), \quad f \in \mathcal{A}. \tag{1.13}$$

Definition 1.8.6 ([7])**.** For $c \in \mathbb{N}$, the Bernardi integral operator $L_c : \mathcal{A} \to \mathcal{A}$ is defined by

$$L_c f(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} \mathrm{d}t, \quad f \in \mathcal{A},\ z \in \mathcal{U}. \tag{1.14}$$

Chapter 2

New classes of analytic functions

This chapter contains a series of new results on analytic functions. In Section 2.1, we introduce the operator $D^n_{\lambda\delta}f$ using the Al-Oboudi and Ruscheweyh differential operators and we investigate several differential subordinations. In Section 2.2, the Sălăgean integral operator I^n is used for defining two new classes of analytic functions for which several inclusion and integral-preserving properties are given. Section 2.3 deals with several classes of bi-univalent functions. Most of the original results in this chapter have been published and are included in the papers [16], [21], [17] and [18].

2.1 Differential subordinations obtained by using Al-Oboudi and Ruscheweyh operators

Definition 2.1.1 ([16])**.** Let $n \in \mathbb{N}$, $\delta \geq 0$ and $\lambda \geq 0$ with $\delta \neq (\lambda - 1)/\lambda$. Let $D^n_{\lambda\delta}$ denote the operator $D^n_{\lambda\delta} : \mathcal{A} \to \mathcal{A}$, defined by

$$D^n_{\lambda\delta}f(z) = \frac{1}{1-\lambda+\lambda\delta}[(1-\lambda)D^n_{\delta}f(z) + \lambda\delta R^n f(z)], \quad z \in \mathcal{U}, \tag{2.1}$$

where the operators $D^n_{\delta}f$ and $R^n f$ are given by Definition 1.8.3 and Definition 1.8.1, respectively.

Remark 2.1.2 ([16])**.** When $\lambda = 0$ in (2.1), $D^n_{\lambda\delta}$ reduces to the Al-Oboudi differential operator, and when $\lambda = 1$ we obtain the Ruscheweyh differential operator.

Also, it is easy to see that for $n = 0$ we have

$$D^0_{\lambda\delta}f(z) = \frac{1}{1-\lambda+\lambda\delta}[(1-\lambda)D^0_{\delta}f(z) + \lambda\delta R^0 f(z)] = f(z), \quad z \in \mathcal{U}.$$

Remark 2.1.3 ([16])**.** We observe that $D^n_{\lambda\delta}$ is a linear operator and for $f \in \mathcal{A}$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

by using equations (1.12) and (1.11), we have

$$D^n_{\lambda\delta} f(z) = z + \frac{1}{1-\lambda+\lambda\delta} \sum_{j=2}^{\infty} \left[(1-\lambda)\left(1+(j-1)\delta\right)^n + \lambda\delta C^n_{n+j-1}\right] a_j z^j, \ z \in \mathcal{U}. \tag{2.2}$$

Theorem 2.1.4 ([16]). *Let $0 \le \alpha < 1$, $\delta > 0$ and $f \in \mathcal{A}_m$. If f satisfies*

$$\mathrm{Re}\left[\left(D^{n+1}_{\lambda\delta} f(z)\right)' + \frac{\lambda\delta z(\delta n+\delta-1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)}\right] > \alpha, \quad z \in \mathcal{U} \tag{2.3}$$

then the following inequality holds

$$\mathrm{Re}\ (D^n_{\lambda\delta} f(z))' > \gamma, \quad z \in \mathcal{U},$$

where

$$\gamma = \gamma(\alpha) = 2\alpha - 1 + \frac{2(1-\alpha)}{\delta m} \beta\left(\frac{1}{\delta m}\right)$$

and

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt\,.$$

Example 2.1.5 ([16]). For the case $f \in \mathcal{A}$, $n = 1$, $\lambda = 1/2$, $\delta = 1$ and $\alpha = 1/2$, we have $\gamma(\alpha) = \ln 2$ and the inequality

$$\mathrm{Re}\ \left[f'(z) + 3zf''(z) + z^2 f'''(z)\right] > \frac{1}{2}, \quad z \in \mathcal{U},$$

implies that

$$\mathrm{Re}\ \left[f'(z) + zf''(z)\right] > \ln 2, \quad z \in \mathcal{U}.$$

Theorem 2.1.6 ([16]). *Let $m \in \mathbb{N}$, $\delta > 0$, r a convex function with $r(0) = 1$ and h a function with the property*

$$h(z) = r(z) + m\delta z r'(z), \quad z \in \mathcal{U}.$$

If $f \in \mathcal{A}_m$, then the following subordination

$$\left(D^{n+1}_{\lambda\delta} f(z)\right)' + \frac{\lambda\delta z(\delta n+\delta-1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \prec h(z) = r(z) + m\delta z r'(z) \tag{2.4}$$

implies that

$$(D^n_{\lambda\delta} f(z))' \prec r(z).$$

The result is sharp.

Theorem 2.1.7 ([16]). *Let $m \in \mathbb{N}$ and let r be a convex function with $r(0) = 1$ and h a function such that*

$$h(z) = r(z) + mzr'(z) \quad (z \in \mathcal{U}).$$

If $f \in \mathcal{A}_m$, then the following subordination

$$(D^n_{\lambda\delta} f(z))' \prec h(z) = r(z) + mzr'(z) \tag{2.5}$$

implies that

$$\frac{D^n_{\lambda\delta} f(z)}{z} \prec r(z).$$

The result is sharp.

2.2 Some subclasses of analytic functions involving λ-spirallikeness of order α

We establish in what follows some results on two new subclasses of analytic functions f, defined by requiring that I^f is in the class $\mathcal{S}^\lambda(\alpha)$ of λ-spirallike functions of order α, or in $\mathcal{F}^\lambda_n(\alpha)$, respectively, where I^n is the Sălăgean integral operator given in Section 1.8.

Definition 2.2.1 ([21])**.** For $\lambda \in (-\pi/2, \pi/2)$ and $\alpha \in [0, 1)$, let $\mathcal{S}^\lambda_n(\alpha)$ be the class of all analytic functions $f \in \mathcal{A}$ with the property

$$I^n f \in \mathcal{S}^\lambda(\alpha),$$

where $\mathcal{S}^\lambda(\alpha)$ is the class of λ-spirallike functions of order α, given by Definition 1.5.8.

Also, let $\mathcal{F}^\lambda_n(\alpha)$ be the class of functions $f \in \mathcal{A}$ which satisfy the relation

$$I^n f \in \mathcal{F}^\lambda(\alpha),$$

where $\mathcal{F}^\lambda(\alpha)$ is the class defined in Definition 1.5.9.

Remark 2.2.2 ([21])**.** It is easy to see that $f(z) \in \mathcal{F}^\lambda_n(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^\lambda_n(\alpha)$. Also, $\mathcal{S}^\lambda_0(\alpha) = \mathcal{S}^\lambda(\alpha)$ and $\mathcal{F}^\lambda_0(\alpha) = \mathcal{F}^\lambda(\alpha)$.

Theorem 2.2.3 ([21])**.** *Let $\lambda \in (-\pi/2, \pi/2)$ and $\alpha \in [0, 1)$. Then for any $n \in \mathbb{N}$, the following inclusion holds:*

$$\mathcal{S}^\lambda_n(\alpha) \subset \mathcal{S}^\lambda_{n+1}(\alpha).$$

Theorem 2.2.4 ([21])**.** *Let $\lambda \in (-\pi/2, \pi/2)$ and $\alpha \in [0, 1)$. Then for any $n \in \mathbb{N}$, we have*

$$\mathcal{F}^\lambda_n(\alpha) \subset \mathcal{F}^\lambda_{n+1}(\alpha).$$

Theorem 2.2.5 ([21])**.** *Let $c \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$ and $\alpha \in [0, 1)$. If $f \in \mathcal{S}^\lambda_n(\alpha)$ then $L_c f \in \mathcal{S}^\lambda_n(\alpha)$.*

Theorem 2.2.6 ([21])**.** *Let $c \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$, $\alpha \in [0, 1)$. If $f \in \mathcal{F}^\lambda_n(\alpha)$ then $L_c f \in \mathcal{F}^\lambda_n(\alpha)$.*

2.3 Coefficient estimates for certain subclasses of bi-univalent functions

In this section, we find coefficient estimates for several subclasses of bi-univalent functions.

A function $f \in \mathcal{A}$ is said to be *bi-univalent in* $\mathcal{U}$ if both f and the analytic extension of f^{-1} to $\mathcal{U}$ are univalent in $\mathcal{U}$. We shall denote by σ the class of bi-univalent functions in $\mathcal{U}$.

Some examples of functions in the class σ are $\dfrac{z}{1-z}$, $-\log(1-z)$ or $\dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$. However, the familiar Koebe function does not belong to σ. Other common examples of functions in $\mathcal{S}$ which are not members of σ are $z - \dfrac{z^2}{2}$ and $\dfrac{z}{1-z^2}$.

Lewin [44] was the first to investigate the class of bi-univalent functions, showing that the second coefficient of the Taylor series expansion of a bi-univalent function satisfies $|a_2| < 1.51$. Recently, several authors studied different subclasses of bi-univalent functions, obtaining (non-sharp) estimates of the first two coefficients, a_2 and a_3 (see, for example [4], [25], [82]).

Throughout this section, ϕ is an analytic function with positive real part in $\mathcal{U}$, with $\phi(0) = 1$, $\phi'(0) > 0$ and such that $\phi(\mathcal{U})$ is starlike with respect to 1 and symmetric with respect to the real axis. Therefore, ϕ has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \quad B_1 > 0. \tag{2.6}$$

Definition 2.3.1 ([17])**.** A function $f \in \sigma$ is said to be in the class $\mathcal{S}^*_{s,\sigma}(\phi)$ if both f and f^{-1} are in $\mathcal{S}^*_s(\phi)$, where $\mathcal{S}^*_s(\phi)$ is the class of functions given in Definition 1.4.12.

Definition 2.3.2 ([17])**.** A function $f \in \sigma$ is said to be in the class $\mathcal{K}_{s,\sigma}(\phi)$ if both f and f^{-1} are functions from $\mathcal{K}_s(\phi)$, where $\mathcal{K}_s(\phi)$ is the class of functions given in Definition 1.4.13.

We next give some coefficient estimates for functions belonging to the two classes defined above.

Theorem 2.3.3 ([17])**.** *If the function f having the form* (3.1) *belongs to the class $\mathcal{S}^*_{s,\sigma}(\phi)$, then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{2|B_1^2 + 2B_1 - 2B_2|}} \quad \text{and} \quad |a_3| \le \frac{1}{2}B_1\left(1 + \frac{1}{2}B_1\right). \tag{2.7}$$

Theorem 2.3.4 ([17])**.** *If the function f given by* (3.1) *belongs to the class $\mathcal{K}_{s,\sigma}(\phi)$, then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{2|3B_1^2 + 8B_1 - 8B_2|}} \quad \text{and} \quad |a_3| \le \frac{1}{2}B_1\left(\frac{1}{3} + \frac{1}{8}B_1\right). \tag{2.8}$$

Two interesting particular cases, stated in the corollaries below, are obtained when the function ϕ is given by

$$\phi(z) = \frac{1 + (1-2\gamma)z}{1-z} = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \cdots,$$

where $0 \le \gamma < 1$.

Corollary 2.3.5 ([17])**.** *Let $0 \le \gamma < 1$ and $f \in \sigma$ be given by (3.1). If the following inequalities are satisfied*

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z)-f(-z)}\right) > \gamma, \quad z \in \mathcal{U}$$

and

$$\operatorname{Re}\left(\frac{2wg'(w)}{g(w)-g(-w)}\right) > \gamma, \quad w \in \mathcal{U},$$

where g is the analytic extension of f^{-1} to $\mathcal{U}$, then

$$|a_2| \le \sqrt{1-\gamma} \quad \text{and} \quad |a_3| \le (1-\gamma)(2-\gamma).$$

Corollary 2.3.6 ([17])**.** *Let $0 \le \gamma < 1$ and $f \in \sigma$ be given by (3.1). If the following inequalities hold*

$$\operatorname{Re}\left(\frac{2(zf'(z))'}{f'(z)+f'(-z)}\right) > \gamma, \quad z \in \mathcal{U}$$

and

$$\operatorname{Re}\left(\frac{2(wg'(w))'}{g'(w)+g'(-w)}\right) > \gamma, \quad w \in \mathcal{U},$$

where g is the analytic extension of f^{-1} to $\mathcal{U}$, then

$$|a_2| \le \sqrt{\frac{1-\gamma}{3}} \quad \text{and} \quad |a_3| \le \frac{(1-\gamma)(7-3\gamma)}{12}.$$

Also, for

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma + 2\gamma^2 + \cdots \quad (0 < \gamma \le 1),$$

we obtain the corollaries below:

Corollary 2.3.7. *Let $0 < \gamma \le 1$ and $f \in \sigma$ be given by (3.1). If the following inequalities are satisfied*

$$\left|\arg\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)\right| < \gamma\frac{\pi}{2}, \quad z \in \mathcal{U}$$

and

$$\left|\arg\left(\frac{2wg'(w)}{g(w)-g(-w)}\right)\right| < \gamma\frac{\pi}{2}, \quad w \in \mathcal{U},$$

where g is the analytic extension of f^{-1} to $\mathcal{U}$, then

$$|a_2| < \gamma \quad \text{and} \quad |a_3| < \gamma(1+\gamma).$$

Corollary 2.3.8. *Let $0 < \gamma \leq 1$ and $f \in \sigma$ be given by (3.1). If the following inequalities hold*

$$\left|\arg\left(\frac{2(zf'(z))'}{f'(z)+f'(-z)}\right)\right| < \gamma\frac{\pi}{2}, \quad z \in \mathcal{U}$$

and

$$\left|\arg\left(\frac{2(wg'(w))'}{g'(w)+g'(-w)}\right)\right| < \gamma\frac{\pi}{2}, \quad w \in \mathcal{U},$$

where g is the analytic extension of f^{-1} to $\mathcal{U}$, then

$$|a_2| \leq \frac{\gamma}{\sqrt{4-\gamma}} \quad \text{and} \quad |a_3| \leq \frac{\gamma(4+3\gamma)}{12}.$$

Definition 2.3.9 ([18])**.** Let $0 \leq \alpha \leq 1$. A function $f \in \sigma$ is said to be in the class $\mathcal{S}^*_{s,\sigma}(\alpha, \phi)$ if the following subordinations hold:

$$\frac{2[(1-\alpha)zf'(z)+\alpha z(zf'(z))']}{(1-\alpha)(f(z)-f(-z))+\alpha z(f'(z)+f'(-z))} \prec \phi(z)$$

and

$$\frac{2[(1-\alpha)wg'(w)+\alpha w(wg'(w))']}{(1-\alpha)(g(w)-g(-w))+\alpha w(g'(w)+g'(-w))} \prec \phi(z),$$

where g is the extension of f^{-1} to $\mathcal{U}$.

Remark 2.3.10 ([18])**.** When $\alpha = 0$, the class $\mathcal{S}^*_{s,\sigma}(0, \phi)$ represents the class of all bi-univalent Ma-Minda starlike functions with respect to symmetric points, whereas when $\alpha = 1$, $\mathcal{S}^*_{s,\sigma}(1, \phi)$ is the class of all bi-univalent Ma-Minda convex functions with respect to symmetric points, introduced in [17].

Theorem 2.3.11 ([18])**.** *Let $0 \leq \alpha \leq 1$. If $f \in \mathcal{S}^*_{s,\sigma}(\alpha, \phi)$ is given by (3.1) then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2|(1+2\alpha)B_1^2 + 2(1+\alpha)^2(B_1 - B_2)|}} \tag{2.9}$$

and

$$|a_3| \leq \frac{1}{2}B_1\left(\frac{1}{1+2\alpha} + \frac{1}{2(1+\alpha)^2}B_1\right). \tag{2.10}$$

When

$$\phi(z) = \frac{1+(1-2\gamma)z}{1-z} = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \cdots, \quad 0 \leq \gamma < 1,$$

we have the following consequence of Theorem 2.3.11:

Corollary 2.3.12. *Let $0 \leq \alpha \leq 1$, $0 \leq \gamma < 1$ and $f \in \sigma$ be given by (3.1). If the following* *inequalities are satisfied*

$$\operatorname{Re}\frac{2[(1-\alpha)zf'(z)+\alpha z(zf'(z))']}{(1-\alpha)(f(z)-f(-z))+\alpha z(f'(z)+f'(-z))}>\gamma, \quad z\in\mathcal{U}$$

and

$$\operatorname{Re}\frac{2[(1-\alpha)wg'(w)+\alpha w(wg'(w))']}{(1-\alpha)(g(w)-g(-w))+\alpha w(g'(w)+g'(-w))}>\alpha, \quad w\in\mathcal{U},$$

where g is the analytic extension of f^{-1} to $\mathcal{U}$, then

$$|a_2|\le\sqrt{\frac{1-\gamma}{1+2\alpha}} \quad \text{and} \quad |a_3|\le(1-\gamma)\left(\frac{1}{1+2\gamma}+\frac{1-\gamma}{1+\alpha}\right).$$

Definition 2.3.13 ([18])**.** Let $0\le\alpha\le 1$. A function $f\in\sigma$ is said to be in the class $\mathcal{L}_{s,\sigma}(\alpha,\phi)$ if the following subordinations hold:

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f'(z)+f'(-z))}\right)^{1-\alpha}\prec\phi(z)$$

and

$$\left(\frac{2wg'(w)}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2(wg'(w))'}{(g'(w)+g'(-w))}\right)^{1-\alpha}\prec\phi(w)$$

where g is the extension of f^{-1} to $\mathcal{U}$.

Theorem 2.3.14 ([18])**.** *Let $0\le\alpha\le 1$. If $f\in\mathcal{L}_{s,\sigma}(\alpha,\phi)$ is given by (3.1) then*

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{2|(\alpha^2-3\alpha+3)B_1^2+2(2-\alpha)^2(B_1-B_2)|}} \tag{2.11}$$

and

$$|a_3|\le\frac{1}{2}B_1\left(\frac{1}{2(2-\alpha)^2}B_1+\frac{1}{3-2\alpha}\right). \tag{2.12}$$

Definition 2.3.15 ([18])**.** Let $0\le\alpha\le 1$. A function $f\in\sigma$ is said to be in the class $\mathcal{Q}_{s,\sigma}(\alpha,\phi)$ if following subordinations hold:

$$\frac{(1-\alpha)zf'(z)+\alpha z(zf'(z))'}{(1-\alpha)h(z)+\alpha zh'(z)}\prec\phi(z) \tag{2.13}$$

and

$$\frac{(1-\alpha)wg'(w)+\alpha w(wg'(w))'}{(1-\alpha)h(w)+\alpha wh'(w)}\prec\phi(w) \tag{2.14}$$

where h satisfies

$$\operatorname{Re}\left[\frac{(1-\alpha)zh'(z)+\alpha z(zh'(z))'}{(1-\alpha)h(z)+\alpha zh'(z)}\right]>0, \quad z\in\mathcal{U} \tag{2.15}$$

and g is the analytic continuation of f^{-1} to $\mathcal{U}$.

Remark 2.3.16 ([18])**.** When $\alpha = 0$, the class $\mathcal{Q}_{s,\sigma}(0, \phi)$ consists of all bi-univalent close-to-convex functions of Ma-Minda type, whereas when $\alpha = 1$, $\mathcal{Q}_{s,\sigma}(1, \phi)$ represents the class of all bi-univalent quasi-convex functions of Ma-Minda type.

Theorem 2.3.17 ([18])**.** *Let* $0 \le \alpha \le 1$. *If* $f \in \mathcal{Q}_{s,\sigma}(\alpha, \phi)$ *is given by* (3.1), *then*

$$|a_2| \le \sqrt{\frac{B_1^2 + B_1^3 + 4|B_1 - B_2|}{|3(1+2\alpha)B_1^2 + 4(1+\alpha)^2(B_1 - B_2)|}} \tag{2.16}$$

and

$$|a_3| \le B_1 \left(\frac{1}{1+2\alpha} + \frac{B_1}{4(1+\alpha)^2} \right) + \frac{4|B_1 - B_2|}{3(1+2\alpha)B_1} \tag{2.17}$$

Chapter 3

New classes of meromorphic functions

We study in this chapter some subclasses of meromorphic functions. In Section 3.1 we define a Janowski type class of functions, $\Sigma(A, B; \alpha)$, and we present, among other results, a sufficient condition foe a function to belong to this aforementioned class, coefficient estimates or a convolution property. In Section 3.2 we give a series of inclusion results, integral-preserving and convolution properties for two new classes of meromorphic functions defined by using the linear operator $L_p^\lambda(a, c)$, while in Section 3.3 we establish criteria for close-to-convexity of meromorphic multivalent functions. The original results presented in this chapter are included in the papers [20], [14] and [15].

For $p \in \mathbb{N}^*$, let Σ_p denote the class of functions having the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1-p}^{\infty} a_n z^n,\ \ z \in \mathcal{U}^*, \tag{3.1}$$

which are meromorphic and p-valent in $\mathcal{U}^*$. Throughout this chapter, we shall denote Σ_1 by Σ.

3.1 On a class of meromorphic functions of Janowski type

Definition 3.1.1 ([20])**.** Let $-1 \le B < A \le 1$ and $0 \le \alpha \le 1$. A meromorphic function f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n,\ \ z \in \mathcal{U}^*, \tag{3.2}$$

is said to be in the class $\Sigma(A, B; \alpha)$ if there exists $g \in \Sigma^*(1/2)$ such that the following subordination is satisfied:

$$\frac{(1-2\alpha)f'(z) - \alpha z f''(z)}{g(z)g(-z)} \prec \frac{1+Az}{1+Bz}, \tag{3.3}$$

where $\Sigma^*(1/2)$ $(0 \le a < 1)$ is the class of meromorphic starlike functions of order $1/2$ in $\mathcal{U}^*$, given in (1.10).

In [34], Janowski introduced the class $P[A,B]$, where $-1 \le B < A \le 1$, as the set of all functions p analytic in $\mathcal{U}$, with $p(0) = 1$, that are subordinate to $\dfrac{1+Az}{1+Bz}$. This is the reason for which classes defined by means of subordintions to the aforementioned function $\dfrac{1+Az}{1+Bz}$ are often called "of Janowski type".

Remark 3.1.2 ([20])**.** The class $\Sigma(A,B;\alpha)$ provides a generalization of the classes studied by Wang et al. [89] (the case $\alpha = 0$, $A = -1$ and $B = 1$) and Sim and Kwon [80] (the case $\alpha = 0$).

Remark 3.1.3 ([20])**.** If $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, then $\Sigma(A_1, B_1; \alpha) \subset \Sigma(A_2, B_2; \alpha)$. To prove this, let $f \in \Sigma(A_1, B_1; \alpha)$. Then

$$\frac{(1-2\alpha)f'(z) - \alpha z f''(z)}{g(z)g(-z)} \prec \frac{1+A_1 z}{1+B_1 z}.$$

But since $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, the following subordination is true:

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z}. \tag{3.4}$$

Indeed, when $-1 < B_2 \le B_1$, the images of $\mathcal{U}$ under these two functions are two circles orthogonal on the real axis and we also have that

$$\begin{aligned}\min_{z\in\partial\mathcal{U}} \mathrm{Re}\frac{1+A_2 z}{1+B_2 z} &= \frac{1-A_2}{1-B_2} \le \min_{z\in\partial\mathcal{U}} \mathrm{Re}\frac{1+A_1 z}{1+B_1 z} = \frac{1-A_1}{1-B_1}\\ \le \max_{z\in\partial\mathcal{U}} \mathrm{Re}\frac{1+A_1 z}{1+B_1 z} &= \frac{1+A_1}{1+B_1} \le \frac{1+A_2}{1+B_2} = \max_{z\in\partial\mathcal{U}} \mathrm{Re}\frac{1+A_2 z}{1+B_2 z},\end{aligned}$$

which shows that the image of $\mathcal{U}$ under $(1+A_1 z)/(1+B_1 z)$ is included in the image of $\mathcal{U}$ under $(1+A_2 z)/(1+B_2 z)$, and so the subordination (3.4) holds. A similar argument shows the subordination is also true when $-1 = B_1 = B_2$ or $-1 = B_1 < B_2$. It therefore follows that $f \in \Sigma(A_2, B_2; \alpha)$.

In our investigation of the class $\Sigma(A,B;\alpha)$ we shall need the following lemmas:

Lemma 3.1.4 ([89])**.** *Let $g \in \Sigma^*(1/2)$. Then*

$$-zg(z)g(-z) \in \Sigma^*.$$

Lemma 3.1.5 ([89])**.** *Let*

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in \Sigma^*(1/2).$$

Then

$$|B_{2n-1}| \le \frac{1}{n}, \quad n \in \mathbb{N}^*,$$

where

$$B_{2n-1} = 2b_{2n-1} + 2b_1 b_{2n-3} - 2b_2 b_{2n-4} + \cdots + (-1)^{n-1} b_{n-2} b_n + (-1)^n b_{n-1}^2. \tag{3.5}$$

Lemma 3.1.6 ([73]). *Let*

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \quad \text{and} \quad k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$$

be two analytic functions in $\mathcal{U}$*. If* k *is convex and* $h \prec k$*, then*

$$|h_n| \le |k_1|, \quad n \in \mathbb{N}^*.$$

The following result gives a sufficient condition for a function to belong to the investigated class $\Sigma(A, B; \alpha)$.

Theorem 3.1.7 ([20]). *Let* $-1 \le B < A \le 1$, $0 \le \alpha \le 1$ *and*

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathcal{U}^*.$$

If f *given by* (3.2) *is a meromorphic functions which satisfies the condition*

$$\sum_{n=1}^{\infty} [(1 + |B|)|1 - \alpha - \alpha n||a_n|n + (1 + |A|)|B_{2n-1}|] < A - B \tag{3.6}$$

where the coefficients B_{2n-1} *are given by* (3.5)*, then* $f \in \Sigma(A, B; \alpha)$*.*

We next determine the coefficient estimates for functions in $\Sigma(A, B; \alpha)$.

Theorem 3.1.8 ([20]). *Let* $-1 \le B < A \le 1$, $0 \le \alpha \le 1$ *and* $f \in \Sigma(A, B; \alpha)$ *be given by* (3.2)*. Then*

$$|a_1| \le 1,$$

$$|a_{2n}| \le \frac{A - B}{2n|1 - (2n+1)\alpha|}\left(1 + \sum_{k=1}^{n-1} \frac{1}{k}\right), \quad n \in \mathbb{N}^* \tag{3.7}$$

and

$$|a_{2n+1}| \le \frac{A - B}{(2n+1)|1 - (2n+2)\alpha|}\left(1 + \sum_{k=1}^{n} \frac{1}{k}\right), \quad n \in \mathbb{N}^*. \tag{3.8}$$

Theorem 3.1.9 ([20]). *If* $-1 \le B < A \le 1$, $0 \le \alpha \le 1$ *and* $f \in \Sigma(A, B; \alpha)$ *then for* $|z| = r$, $0 < r < 1$*, the following inequalities hold:*

$$\frac{(1-r)^2}{r^2} \frac{1 - Ar}{1 - Br} \le |(1 - 2\alpha)f'(z) - \alpha z f'(z)| \le \frac{(1+r)^2}{r^2} \frac{1 + Ar}{1 + Br}. \tag{3.9}$$

We provide next a convolution property of functions from the class $\Sigma(A, B; \alpha)$ considered.

Theorem 3.1.10 ([20]). *Let* $-1 \le B < A \le 1$, $0 \le \alpha \le 1$, $\gamma \le 1$ *and* $f \in \Sigma(A, B; \alpha)$ *such that the* *corresponding function $g \in \Sigma^*(1/2)$ satisfies the condition*

$$-\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) < \frac{3}{2} - \frac{1}{2}\gamma, \quad z \in \mathcal{U}. \tag{3.10}$$

If $\phi \in \Sigma$ with $z^2\phi(z) \in \mathcal{R}(\gamma)$, then $\phi * f \in \Sigma(A, B; \alpha)$.

3.2 Subclasses of meromorphic multivalent functions involving a certain linear operator

Making use of the linear operator $L^{\lambda}_{p,k}(a, c)$ and the principle of subordination between analytic functions, we introduce and investigate some new subclasses of the meromorphic $p-$valent function class Σ_p.

Definition 3.2.1. The function $\varphi_p(a, c; z)$ is defined by

$$\varphi_p(a, c; z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-p}$$

$$(a, c \in \mathbb{R},\ c \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\},\ z \in \mathcal{U}^*),$$

where $(x)_n$ denotes the Pochhammer symbol given by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & \text{if } k = 0, \\ x(x+1)\ldots(x+n-1), & \text{if } k \in \mathbb{N}^*. \end{cases}$$

Corresponding to the function $\varphi_p(a, c; z)$, Liu and Srivastava [47] and Yang [92] independently introduced the linear operator $L_p(a, c)$ on Σ_p by means of the Hadamard product as follows:

$$L_p(a, c)f(z) = \varphi_p(a, c; z) * f(z), \quad f \in \Sigma_p,\ z \in \mathcal{U}^*.$$

Related to the function $\varphi_p(a, c; z)$, in [6] the authors considered the function $\varphi_p^{\lambda}(a, c; z)$ by

$$\varphi_p(a, c; z) * \varphi_p^{\lambda}(a, c; z) = \frac{1}{z^p(1-z)^{p+\lambda}} \tag{3.11}$$

$$(a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-,\ \lambda > -p,\ p \in \mathbb{N}^*,\ z \in \mathcal{U}^*),$$

and the corresponding family of linear operators $L_p^{\lambda}(a, c)$ analogous to $L_p(a, c)$,

$$L_p^{\lambda}(a, c)f(z) = \varphi_p^{\lambda}(a, c; z) * f(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(c)_n(p+n)_n}{(a)_n n!} a_{n-p} z^{n-p},\ z \in \mathcal{U}^*. \tag{3.12}$$

Remark 3.2.2. From (3.11) and (3.12) it can easily be seen that

$$z(L_{p,k}^{\lambda}(a+1,c)f)'(z) = aL_{p,k}^{\lambda}(a,c)f(z) - (a+p)L_{p,k}^{\lambda}(a+1,c)f(z) \tag{3.13}$$

and

$$z(L_{p,k}^{\lambda}(a,c)f)'(z) = (\lambda+p)L_{p,k}^{\lambda+1}(a,c)f(z) - (\lambda+2p)L_{p,k}^{\lambda}(a,c)f(z). \tag{3.14}$$

Making use of the operator $L_{p,k}^{\lambda}(a,c)$ and the principle of subordination between analytic functions, we introduce and investigate some new subclasses of the meromorphic $p-$valent function class Σ_p.

Throughout this section, let $p, k \in \mathbb{N}^*$, $a, c \notin \mathbb{Z}_0^-$, $\epsilon_k = \exp(2\pi i/k)$ and for $f \in \Sigma_p$, define

$$f_{p,k}^{\lambda}(a,c)(z) = \frac{1}{k}\sum_{j=0}^{k-1} \epsilon_k^{jp}(L_{p,k}^{\lambda}(a,c)f)(\epsilon_k^j z) = z^{-p} + \cdots, \; z \in \mathcal{U}^*. \tag{3.15}$$

Definition 3.2.3 ([14])**.** Let $h \in \mathcal{P}$, h convex and let $f \in \Sigma_p$ such that $f_{p,k}^{\lambda}(a,c)(z) \neq 0$, $z \in \mathcal{U}^*$. The function f is said to belong to the class $\Sigma_{p,k}^{\lambda}(a,c;h)$ if it satisfies the subordination

$$-\frac{z(L_{p,k}^{\lambda}(a,c)f)'(z)}{pf_{p,k}^{\lambda}(a,c)(z)} \prec h(z). \tag{3.16}$$

Also, for$-1 < B < A \leq 1$, we set

$$\Sigma_{p,k}^{\lambda}(a,c;A,B) := \Sigma_{p,k}^{\lambda}\left(a,c;\frac{1+Az}{1+Bz}\right).$$

Remark 3.2.4 ([14])**.** For $k = \lambda = 1$, the class $\Sigma_{p,1}^{1}(a,c;A,B)$ was introduced and studied by Liu and Srivastava in [47]. The class $\Sigma_{p,1}^{\lambda}(a,c;h)$ was considered by Srivastava et al. in [84], while for $\lambda = 1$, $\Sigma_{p,k}^{1}(a,c;h)$ was studied by Aouf et al. in [6].

Definition 3.2.5 ([14])**.** Let $h \in \mathcal{P}$, h convex and let $f \in \Sigma_p$ such that $f_{p,k}^{\lambda}(a,c)(z) \neq 0$, $z \in \mathcal{U}^*$. The function f is said to belong to the class $\mathcal{K}_{p,k}^{\lambda}(a,c;h)$ if there exists a function $g \in \Sigma_{p,k}^{\lambda}(a,c;h)$ such that

$$-\frac{z(L_{p,k}^{\lambda}(a,c)f)'(z)}{pg_{p,k}^{\lambda}(a,c)(z)} \prec h(z), \tag{3.17}$$

where $g_{p,k}^{\lambda}(a,c)$ is defined as in (3.15) with $g_{p,k}^{\lambda}(a,c)(z) \neq 0$, $z \in \mathcal{U}^*$. For $-1 < B < A \leq 1$, we also set

$$\mathcal{K}_{p,k}^{\lambda}(a,c;A,B) := \mathcal{K}_{p,k}^{\lambda}\left(a,c;\frac{1+Az}{1+Bz}\right).$$

Lemma 3.2.6 ([14])**.** *Let $h \in \mathcal{P}$ be a convex function. If $f \in \Sigma_{p,k}^{\lambda}(a,c;h)$, then*

$$-\frac{z(f_{p,k}^{\lambda}(a,c))'(z)}{pf_{p,k}^{\lambda}(a,c)(z)} \prec h(z). \tag{3.18}$$

Theorem 3.2.7 ([14])**.** *Let $h \in \mathcal{P}$, convex, with* $\operatorname{Re} h(z) < 1 + \dfrac{a}{p}$*,* $z \in \mathcal{U}$ *and let* $f \in \Sigma_{p,k}^{\lambda}(a, c; h)$ *such that* $f_{p,k}^{\lambda}(a + 1, c)(z) \neq 0,\ z \in \mathcal{U}^*$*. Then* $f \in \Sigma_{p,k}^{\lambda}(a + 1, c; h)$*.*

When $h(z) = \dfrac{1 + Az}{1 + Bz}$, $z \in \mathcal{U}$ and $-1 < B < A \leq 1$, we obtain the following corollary:

Corollary 3.2.8 ([14])**.** *Let* $-1 < B < A \leq 1$ *with* $\dfrac{1 + A}{1 + B} < 1 + \dfrac{a}{p}$ *and let* $f \in \Sigma_{p,k}^{\lambda}(a, c; A, B)$ *such that* $f_{p,k}^{\lambda}(a + 1, c)(z) \neq 0,\ z \in \mathcal{U}^*$*. Then* $f \in \Sigma_{p,k}^{\lambda}(a + 1, c; A, B)$*.*

Theorem 3.2.9 ([14])**.** *Let $h \in \mathcal{P}$, convex, with* $\operatorname{Re} h(z) < 1 + \dfrac{a}{p}$*,* $z \in \mathcal{U}$ *and let* $f \in \mathcal{K}_{p,k}^{\lambda}(a, c; h)$ *with respect to* $g \in \Sigma_{p,k}^{\lambda}(a, c; h)$*. Then* $f \in \mathcal{K}_{p,k}^{\lambda}(a + 1, c; h)$*, provided that* $g_{p,k}^{\lambda}(a, c)(z) \neq 0,\ z \in \mathcal{U}^*$*.*

Corollary 3.2.10 ([14])**.** *Let* $-1 < B < A \leq 1$ *with* $\dfrac{1 + A}{1 + B} < 1 + \dfrac{a}{p}$ *and let* $f \in \mathcal{K}_{p,k}^{\lambda}(a, c; A, B)$ *with respect to* $g \in \Sigma_{p,k}^{\lambda}(a, c; A, B)$*. Then* $f \in \mathcal{K}_{p,k}^{\lambda}(a + 1, c; A, B)$*, provided that* $g_{p,k}^{\lambda}(a, c)(z) \neq 0$*,* $z \in \mathcal{U}^*$*.*

Theorem 3.2.11 ([14])**.** *Let $h \in \mathcal{P}$, convex, with* $\operatorname{Re} h(z) < 2 + \dfrac{\lambda}{p}$*,* $z \in \mathcal{U}$ *and let* $f \in \Sigma_{p,k}^{\lambda+1}(a, c; h)$ *such that* $f_{p,k}^{\lambda+1}(a + 1, c)(z) \neq 0,\ z \in \mathcal{U}^*$*. Then* $f \in \Sigma_{p,k}^{\lambda}(a, c; h)$*.*

Corollary 3.2.12 ([14])**.** *Let* $-1 < B < A \leq 1$ *satisfying the inequality* $\dfrac{1 + A}{1 + B} < 2 + \dfrac{\lambda}{p}$ *and let* $f \in \Sigma_{p,k}^{\lambda+1}(a, c; A, B)$ *such that* $f_{p,k}^{\lambda+1}(a + 1, c)(z) \neq 0,\ z \in \mathcal{U}^*$*. Then* $f \in \Sigma_{p,k}^{\lambda}(a, c; A, B)$*.*

Theorem 3.2.13 ([14])**.** *Let $h \in \mathcal{P}$ be a convex function with* $\operatorname{Re} h(z) < 2 + \dfrac{\lambda}{p}$*,* $z \in \mathcal{U}$ *and let* $f \in \mathcal{K}_{p,k}^{\lambda+1}(a, c; h)$ *with respect to* $g \in \Sigma_{p,k}^{\lambda+1}(a, c; h)$*. We then have* $f \in \mathcal{K}_{p,k}^{\lambda}(a, c; h)$*, provided that* $g_{p,k}^{\lambda}(a, c)(z) \neq 0,\ z \in \mathcal{U}^*$*.*

Corollary 3.2.14. *[14] Let* $-1 < B < A \leq 1$ *satisfying the inequality* $\dfrac{1 + A}{1 + B} < 2 + \dfrac{\lambda}{p}$ *and let* $f \in \mathcal{K}_{p,k}^{\lambda+1}(a, c; A, B)$ *with respect to* $g \in \Sigma_{p,k}^{\lambda+1}(a, c; A, B)$*. We then have* $f \in \mathcal{K}_{p,k}^{\lambda}(a, c; A, B)$*, provided that* $g_{p,k}^{\lambda}(a, c)(z) \neq 0,\ z \in \mathcal{U}^*$*.*

We define next the integral operator $F_{\mu,p} : \Sigma_p \to \Sigma_p$ $(\mu > 0)$, given by

$$F_{\mu,p}(f)(z) = \frac{\mu - p}{z^{\mu}} \int_0^z t^{\mu-1} f(z)dt, \quad f \in \Sigma_p,\ z \in \mathcal{U}^*. \tag{3.19}$$

From (3.19) it follows that

$$\mu L_{p,k}^{\lambda}(a, c) F_{\mu,p}(f)(z) + z \left(L_{p,k}^{\lambda}(a, c) F_{\mu,p}(f) \right)'(z) = (\mu - p) L_{p,k}^{\lambda}(a, c) f(z). \tag{3.20}$$

The operator $F_{\mu,p}$ was investigated by many authors (see for example [38], [87], [92]).

Theorem 3.2.15. *[14] Let $h \in \mathcal{P}$ be a convex function with* $\operatorname{Re} h(z) < \dfrac{\operatorname{Re}\mu}{p}$, $z \in \mathcal{U}$ *and let* $f \in \Sigma_{p,k}^{\lambda}(a,c;h)$. *Then* $F_{\mu,p}(f) \in \Sigma_{p,k}^{\lambda}(a,c;h)$.

Theorem 3.2.16 ([14])**.** *Let $h \in \mathcal{P}$ be a convex function with* $\operatorname{Re} h(z) < \dfrac{\operatorname{Re}\mu}{p}$, $z \in \mathcal{U}$ *and let* $f \in \mathcal{K}_{p,k}^{\lambda}(a,c;h)$ *with respect to* $g \in \Sigma_{p,k}^{\lambda}(a,c;h)$. *Then* $F_{\mu,p}(f) \in \mathcal{K}_{p,k}^{\lambda}(a,c;h)$ *with respect to the function* $G = F_{\mu,p}(g)$, *provided that* $G_{p,k}^{\lambda}(a,c)(z) \neq 0$, $z \in \mathcal{U}^*$.

Theorems 3.2.17 and 3.2.18 stated below involve some convolution properties of the classes $\Sigma_{p,k}^{\lambda}(a,c)$ and $\mathcal{K}_{p,k}^{\lambda}(a,c)$ respectively.

Theorem 3.2.17 ([14])**.** *Let $h \in \mathcal{P}$ be a convex function verifying the condition* $\operatorname{Re} h(z) < 1+\dfrac{1-\alpha}{p}$, $z \in \mathcal{U}$, $\alpha < 1$. *Let also*

$$g \in \Sigma_p \quad \textit{with} \quad z^{p+1}g(z) \in \mathcal{R}(\alpha). \tag{3.21}$$

If $f \in \Sigma_{p,k}^{\lambda}(a,c;h)$, *then* $f * g \in \Sigma_{p,k}^{\lambda}(a,c;h)$.

Theorem 3.2.18 ([14])**.** *Let $h \in \mathcal{P}$ be convex and such that* $\operatorname{Re} h(z) < 1 + \dfrac{1-\alpha}{p}$, $z \in \mathcal{U}$, $\alpha < 1$ *and let* $g \in \Sigma_p$ *with* $z^{p+1}g(z) \in \mathcal{R}(\alpha)$. *If* $f \in \mathcal{K}_{p,k}^{\lambda}(a,c;h)$ *with respect to* $\psi \in \Sigma_{p,k}^{\lambda}(a,c;h)$, *then we also have* $f * g \in \mathcal{K}_{p,k}^{\lambda}(a,c;h)$ *with respect to* $\psi * g$.

Corresponding to the cases $\alpha = 0$ and $\alpha = 1/2$, Theorem 3.2.17 and Theorem 3.2.18 reduce respectively to the two corollaries stated below:

Corollary 3.2.19 ([14])**.** *Let $h \in \mathcal{P}$ be a convex function and $g \in \Sigma_p$ such that one of the following conditions is satisfied:*

(i) $\operatorname{Re} h(z) < 1 + \dfrac{1}{p}$, $z \in \mathcal{U}$ *and* $z^{p+1}g(z)$ *is a convex univalent function in* $\mathcal{U}$

or

(ii) $\operatorname{Re} h(z) < 1 + \dfrac{1}{2p}$, $z \in \mathcal{U}$ *and* $z^{p+1}g(z) \in \mathcal{S}^*(1/2)$.

If $f \in \Sigma_{p,k}^{\lambda}(a,c;h)$, *then* $f * g \in \Sigma_{p,k}^{\lambda}(a,c;h)$.

Corollary 3.2.20 ([14])**.** *Let $h \in \mathcal{P}$ and $g \in \Sigma_p$ such that one of the following conditions is satisfied:*

(i) $\operatorname{Re} h(z) < 1 + \dfrac{1}{p}$, $z \in \mathcal{U}$ *and* $z^{p+1}g(z)$ *is a convex univalent function in* $\mathcal{U}$

or

(ii) $\operatorname{Re} h(z) < 1 + \dfrac{1}{2p}$, $z \in \mathcal{U}$ *and* $z^{p+1}g(z) \in \mathcal{S}^*(1/2)$.

If $f \in \mathcal{K}_{p,k}^{\lambda}(a,c;h)$, *then* $f * g \in \mathcal{K}_{p,k}^{\lambda}(a,c;h)$.

3.3 Some criteria for meromorphic multivalent close-to-convex functions

For $0 \leq \alpha < p$, a functions $f \in \Sigma_p$ is said to be in the class $\mathcal{MC}_p(\alpha)$ of close-to-convex meromorphic p-valent functions if the following inequality is satisfied:

$$\mathrm{Re}[z^{p+1}f'(z)] < -\alpha, \quad z \in \mathcal{U}^*.$$

We next present some sufficient conditions for meromorphic $p-$valent functions to belong to $\mathcal{MC}_p(\alpha)$.

Theorem 3.3.1 ([15])**.** *If $f \in \Sigma_p$ satisfies*

$$\mathrm{Re}\left[z^{p+1}\left(f'(z) + zf''(z)\right)\right] < \alpha p + \frac{p-\alpha}{2}, \quad z \in \mathcal{U}^*, \tag{3.22}$$

then $f \in \mathcal{MC}_p(\alpha)$.

Theorem 3.3.2 ([15])**.** *If $f \in \Sigma_p$ satisfies the inequality*

$$\mathrm{Re}\left(\frac{1}{z^p f'(z)}(z^{p+1}f'(z))'\right) > \begin{cases} \dfrac{\alpha}{2(\alpha-p)}, & 0 \leq \alpha \leq \dfrac{p}{2}, \\ \dfrac{\alpha-p}{2\alpha}, & \dfrac{p}{2} \leq \alpha < p, \end{cases} \quad , \quad z \in \mathcal{U}^*, \tag{3.23}$$

then $f \in \mathcal{MC}_p(\alpha)$.

Theorem 3.3.3 ([15])**.** *If $f \in \Sigma_p$ satisfies the inequality*

$$\left|\frac{p}{z^{p+1}f'(z)}\left(p+1+\frac{zf''(z)}{f'(z)}\right)\right| < \alpha, \quad z \in \mathcal{U}^*, \tag{3.24}$$

then $f \in \mathcal{MC}_p\left(\dfrac{p}{1+p\alpha}\right)$.

Theorem 3.3.4 ([15])**.** *Let $\mu \in [0, 1/2]$ and $f \in \Sigma_p$ satisfying*

$$\left|p+1+\frac{zf''(z)}{f'(z)}\right| < 1-\mu, \quad z \in \mathcal{U}^*. \tag{3.25}$$

Then $f \in \mathcal{MC}_p(p\mu)$.

Chapter 4

Differential subordinations involving the arithmetic and geometric means

The arithmetic and geometric means of some functions and expressions are frequently used in mathematics, especially in geometric function theory. Making use of the arithmetic means, Mocanu [59] introduced the class of α-convex functions (see Definition 1.4.1), which, in some case, proclaims the class of starlike, and in the other, convex functions. In general, the class of α-convex functions determines the *arithmetic bridge* between starlikeness and convexity. In a similar manner, but using the geometric means, Lewandowski et al. [42] defined the class of γ-starlike functions (see Definition 1.4.3) which constitutes the *geometric bridge* between starlikeness and convexity. Many authors have subsequently studied these classes of functions or similar ones that generalize them (the reader is referred, for example, to [26], [78], [35], [70], [83], [39]).

This chapter contains some new results with expressions involving both the arithmetic and the geometric means, proved by using the method of differential subordinations and also some geometric arguments. With the exception of Lemma 4.1.1, the results contained in this chapter are original and were published in the paper [19].

We give below a lemma due to Nunokawa [66], only in a slightly different but equivalent form, which is more convenient for our next considerations.

Lemma 4.1.1 ([66])**.** *Let p be an analytic function in $\mathcal{U}$ such that $p(0) = 1$, $p(z) \not\equiv 1$. If $z_0 \in \mathcal{U}$ satisfies*

$$|\arg p(z_0)| = \max\{\arg p(z)\ : |z| \le |z_0|\} = \gamma\frac{\pi}{2}, \quad \textit{and} \quad p(z_0) = (ix)^\gamma,$$

then

$$\left|\arg[z_0 p'(z_0)]\right| = (\gamma+1)\frac{\pi}{2} \quad \textit{and} \quad |z_0 p'(z_0)| = \left|\frac{\gamma x^\gamma}{2}\left(x + \frac{1}{x}\right)\right|.$$

Definition 4.1.2. Let $\alpha \in [0,1]$, $\delta \in [1,2]$ and $\mu \in [1, 3/2]$. By $\mathcal{H}(\alpha,\delta,\mu)$ we will denote the class of all functions p analytic in $\mathcal{U}$ with $p(0)=1$, $p\not\equiv 1$ for which the function

$$Q(z)=\alpha[p(z)]^{\delta}+(1-\alpha)\left[p(z)+\frac{zp'(z)}{p(z)}\right]^{\mu},\quad z\in\mathcal{U},\ Q(0)=1,$$

is well defined in $\mathcal{U}$ (where all powers are chosen as principal ones).

Theorem 4.1.3 ([19])**.** *Let $\alpha\in[0,1]$, $a\in[0,1)$, $\delta\in[1,2]$ and $\mu\in[1,3/2]$. Also, let $p\in\mathcal{H}(\alpha,\delta,\mu)$. If*

$$\operatorname{Re}\left(\alpha[p(z)]^{\delta}+(1-\alpha)\left[p(z)+\frac{zp'(z)}{p(z)}\right]^{\mu}\right)>a,\quad z\in\mathcal{U},\tag{4.1}$$

then

$$\operatorname{Re}p(z)>a,\quad z\in\mathcal{U}.\tag{4.2}$$

Remark 4.1.4. We observe that when $a=0$, Theorem 4.1.3 is true for $\mu\in[1,2]$. This will also come as a consequence of our next result.

Theorem 4.1.5 ([19])**.** *Let $\alpha\in[0,1]$, $\gamma\in(0,1]$, $\delta\in[1,2]$ and $\mu\in[1,2]$. Also, let $p\in\mathcal{H}(\alpha,\delta,\mu)$. If*

$$\left|\arg\left(\alpha[p(z)]^{\delta}+(1-\alpha)\left[p(z)+\frac{zp'(z)}{p(z)}\right]^{\mu}\right)\right|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U},\tag{4.3}$$

then

$$|\arg p(z)|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U}.\tag{4.4}$$

Definition 4.1.6. Let $\alpha\in[0,1]$, $\delta\in[1,2]$ and $\mu\in[0,1]$. By $\mathcal{F}(\alpha,\delta,\mu)$ we will denote the class of all functions p analytic in $\mathcal{U}$ with $p(0)=1$, $p\not\equiv 1$, for which the function

$$Q(z)=\alpha[p(z)]^{\delta}+(1-\alpha)[p(z)]^{\mu}\left[p(z)+\frac{zp'(z)}{p(z)}\right]^{1-\mu},\quad z\in\mathcal{U},\ Q(0)=1,$$

is well defined in $\mathcal{U}$ (where all powers are chosen as principal ones).

Theorem 4.1.7 ([19])**.** *Let $\alpha\in[0,1]$, $a\in[0,1)$, $\delta\in[1,2]$, $\mu\in[0,1]$. Also, let $p\in\mathcal{F}(\alpha,\delta,\mu)$. If*

$$\operatorname{Re}\left(\alpha[p(z)]^{\delta}+(1-\alpha)[p(z)]^{\mu}\left[p(z)+\frac{zp'(z)}{p(z)}\right]^{1-\mu}\right)>a,\quad z\in\mathcal{U},\tag{4.5}$$

then

$$\operatorname{Re}p(z)>a,\quad z\in\mathcal{U}.\tag{4.6}$$

Theorem 4.1.8 ([19])**.** *Let $\alpha\in[0,1]$, $\gamma\in(0,1]$, $\delta\in[1,2]$, $\mu\in[0,1]$. Also, let $p\in\mathcal{F}(\alpha,\delta,\mu)$. If*

$$\left|\arg\left(\alpha[p(z)]^{\delta}+(1-\alpha)[p(z)]^{\mu}\left[p(z)+\frac{zp'(z)}{p(z)}\right]^{1-\mu}\right)\right|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U},\tag{4.7}$$

then

$$|\arg p(z)| < \gamma\frac{\pi}{2}, \quad z \in \mathcal{U}.$$

We note that, for special selection of the function p, the considered subordinations could generate several different classes of analytic functions that have not been considered yet and which, for particular choice of parameters, reduce to some well known ones. We provide below some of the applications and possibilities to obtain new results for different forms of the function p.

First, by setting $p(z) = zf'(z)/f(z)$ in Theorem 4.1.3 and Theorem 4.1.5, we obtain the following result:

Corollary 4.1.9 ([19])**.** *Let $f \in \mathcal{A}$, $\alpha \in [0,1]$, $a \in [0,1)$, $\delta \in [1,2]$, $\mu \in [1,3/2]$ and $\gamma \in [0,\pi/2]$.*

(i) If

$$\mathrm{Re}\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[1+\frac{zf''(z)}{f'(z)}\right]^{\mu}\right) > a, \quad z \in \mathcal{U},$$

then

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > a, \quad z \in \mathcal{U},$$

and hence f is starlike of order a in $\mathcal{U}$.

(ii) If

$$\left|\arg\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[1+\frac{zf''(z)}{f'(z)}\right]^{\mu}\right)\right| < \gamma\frac{\pi}{2}, \quad z \in \mathcal{U},$$

then

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\right]\right| < \gamma\frac{\pi}{2}, \quad z \in \mathcal{U},$$

or, equivalently, f is strongly starlike of order γ in $\mathcal{U}$.

Remark 4.1.10 ([19])**.** When $\delta = \mu = 1$ and $a = 0$ we obtain the well known result of Mocanu [59] that the class of $\alpha-$convex functions is a subclass of the class of starlike functions in $\mathcal{U}$. Similarly, for $a \in [0,1)$, the class of all $\alpha-$convex functions of order a in $\mathcal{U}$ is included in the class of starlike functions of order a.

The same substitution $p(z) = zf'(z)/f(z)$ in Theorem 4.1.7 and Theorem 4.1.8 leads to the following result:

Corollary 4.1.11 ([19])**.** *Let $f \in \mathcal{A}$. Let also $\alpha \in [0,1]$, $a \in [0,1)$, $\delta \in [1,2]$, $\mu \in [0,1]$ and $\gamma \in [0,\pi/2]$.*

(i) If

$$\mathrm{Re}\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) > a, \quad z \in \mathcal{U},$$

then

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > a, \quad z \in \mathcal{U},$$

so f is starlike of order a in $\mathcal{U}$.

(ii) If

$$\left|\arg\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta}+(1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right)\right|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U},$$

then

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\right]\right|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U},$$

or, equivalently, f is strongly starlike of order γ in $\mathcal{U}$.

Remark 4.1.12 ([19])**.** In the special case $\delta=1$, $\gamma=1$ and $\alpha=0$ we obtain the aforementioned result of Lewandowski et al. [44] that the class of $\mu-$starlike functions is a subclass of the class of starlike functions in $\mathcal{U}$ (Theorem 1.4.4).

The following result is a consequence of Theorem 4.1.3 and Theorem 4.1.5, for the special case $p(z)=f'(z)$, $f\in\mathcal{A}$.

Corollary 4.1.13 ([19])**.** *Let $f\in\mathcal{A}$, $\alpha\in[0,1]$, $a\in[0,1)$, $\delta\in[1,2]$, $\mu\in[1,3/2]$ and $\gamma\in[0,\pi/2]$.*

(i) If

$$\operatorname{Re}\left(\alpha[f'(z)]^{\delta}+(1-\alpha)\left[f'(z)+\frac{zf''(z)}{f'(z)}\right]^{\mu}\right)>a,\quad z\in\mathcal{U},$$

then

$$\operatorname{Re}f'(z)>a\quad ,z\in\mathcal{U},$$

and hence, by Noshiro, Warschawski and Wolff's univalence criterion, f is univalent in $\mathcal{U}$.

(ii) If

$$\left|\arg\left(\alpha[f'(z)]^{\delta}+(1-\alpha)\left[f'(z)+\frac{zf''(z)}{f'(z)}\right]^{\mu}\right)\right|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U},$$

then

$$|\arg f'(z)|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U}.$$

When substituting $p(z)=f'(z)$ in Theorems 4.1.7 and 4.1.8, we obtain the following corollary:

Corollary 4.1.14 ([19])**.** *Let $f\in\mathcal{A}$. Let also $\alpha\in[0,1]$, $a\in[0,1)$, $\delta\in[1,2]$, $\mu\in[0,1]$ and $\gamma\in[0,\pi/2]$.*

(i) If

$$\operatorname{Re}\left(\alpha[f'(z)]^{\delta}+(1-\alpha)[f'(z)]^{\mu}\left[f'(z)+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right)>a,\quad z\in\mathcal{U},$$

then

$$\operatorname{Re}f'(z)>a,\quad z\in\mathcal{U}.$$

Therefore f is univalent in $\mathcal{U}$.

(ii) If

$$\left|\arg\left(\alpha[f'(z)]^{\delta}+(1-\alpha)[f'(z)]^{\mu}\left[f'(z)+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right)\right|<\frac{\pi}{2}\gamma,\quad z\in\mathcal{U},$$

then

$$|\arg f'(z)|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U}.$$

The next corollary results also from Theorem 4.1.3 and Theorem 4.1.5, when substituting $p(z)$ with $f(z)/z$, $f\in\mathcal{A}$.

Corollary 4.1.15 ([19])**.** *Let $f\in\mathcal{A}$, $\alpha\in[0,1]$, $a\in[0,1)$, $\delta\in[1,2]$, $\mu\in[1,3/2]$ and $\gamma\in[0,\pi/2]$.*

(i) If

$$\operatorname{Re}\left(\alpha\left[\frac{f(z)}{z}\right]^{\delta}+(1-\alpha)\left[\frac{f(z)}{z}+\frac{zf'(z)}{f(z)}-1\right]^{\mu}\right)>a,\quad z\in\mathcal{U},$$

then

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)>a,\quad z\in\mathcal{U}.$$

(ii) If

$$\left|\arg\left(\alpha\left[\frac{f(z)}{z}\right]^{\delta}+(1-\alpha)\left[\frac{f(z)}{z}+\frac{zf'(z)}{f(z)}-1\right]^{\mu}\right)\right|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U},$$

then

$$\left|\arg\left(\frac{f(z)}{z}\right)\right|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U}.$$

Finally, if we put $p(z)=f(z)/z$ in Theorems 4.1.7 and 4.1.8, we have our last result:

Corollary 4.1.16 ([19])**.** *Let $f\in\mathcal{A}$. Let also $\alpha\in[0,1]$, $a\in[0,1)$, $\delta\in[1,2]$, $\mu\in[0,1]$ and $\gamma\in[0,\pi/2]$.*

(i) If

$$\operatorname{Re}\left(\alpha\left[\frac{f(z)}{z}\right]^{\delta}+(1-\alpha)\left[\frac{f(z)}{z}\right]^{\mu}\left[\frac{f(z)}{z}+\frac{zf'(z)}{f(z)}-1\right]^{1-\mu}\right)>a,\quad z\in\mathcal{U},$$

then

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)>a,\quad z\in\mathcal{U}.$$

(ii) If

$$\left|\arg\left(\alpha\left[\frac{f(z)}{z}\right]^{\delta}+(1-\alpha)\left[\frac{f(z)}{z}\right]^{\mu}\left[\frac{f(z)}{z}+\frac{zf'(z)}{f(z)}-1\right]^{1-\mu}\right)\right|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U},$$

then

$$\left|\arg\left(\frac{f(z)}{z}\right)\right|<\gamma\frac{\pi}{2},\quad z\in\mathcal{U}.$$

Bibliography

[1] M. Acu, *Some subclasses of meromorphic functions*, Filomat, 21(2)(2007), 1-9.

[2] L.V. Ahlfors, Conformal Invariants. Topics in Geometric Function Theory, McGraw-Hill, New-York, 1973.

[3] J.W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., 17(1915), 12-22.

[4] R.M. Ali, S.K. Lee, V. Ravichandran, S. Supramaniam, *Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions*, Appl. Math. Lett., 25(2012), 344-351.

[5] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci., 27(2004), 1429-1436.

[6] M.K. Aouf, A. Shamandy, A.O. Mostafa, F. El-Emam, *Argument estimates of certain meromorphically p-valent functions associated with a family of linear operators*, Math. Slovaca, 61(2011), 907-920.

[7] S.D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., 135(1969), 429-446.

[8] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. 1916, 940-955.

[9] M. Biernacki, *Sur la représentation conforme des domaines linéairement accessibles*, Prace Mat.-Fiz., 44(1936), 293-314.

[10] D.A. Brannan, W.E. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc., 1(2)(1969), 431-443.

[11] C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annehmen*, Math. Ann., 64(1907), 95-115.

[12] P.N. Chichra, *Regular functions $f(z)$ for which $zf'(z)$ is α-spiral-like*, Proc. Amer. Math. Soc., 49(1)(1975), 151-160.

[13] J.B. Conway, Functions of One Complex Variable, Springer-Verlag, New-York, 1978.

[14] O. Crişan, *Subclasses of meromorphically p-valent functions involving a certain linear operator*, accepted for publishing at Math. Slovaca.

[15] O. Crişan, *Some criteria for meromorphic multivalent close-to-convex functions*, submitted.

[16] O. Crişan, *Differential subordinations obtained by using Al-Oboudi and Ruscheweyh operators*, Stud. Univ. Babeş-Bolyai Math., 56(3)(2011), 45-51.

[17] O. Crişan, *On some subclasses of bi-univalent functions involving starlikeness with respect to symmetric points*, submitted.

[18] O. Crişan, *Coeffcient estimates for certain subclasses of bi-univalent functions*, Gen. Math. Notes, 16(2)(2013), 93-102.

[19] O. Crişan, S. Kanas, *Differential subordinations involving arithmetic and geometric means*, Appl. Math. Comp., 222(2013), 123-131.

[20] O. Crişan, G.Ş. Sălăgean, *On a certain subclass of meromorphic close-to-convex functions*, preprint.

[21] O. Crişan, A.E. Tudor, *Some subclasses of analytic functions involving λ-spirallikeness of order α*, Acta Univ. Apulensis, 31(2012), 47-52.

[22] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math., 154(1985), 137-152.

[23] P.L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.

[24] P.J. Eenigenburg, S.S. Miller, P.T. Mocanu, M.O. Reade, *On a Briot-Bouquet differential subordination*, General Inequalities, 3, International Series of Numerical Mathematics, Vol. 64 Birkhäuser Verlag, Basel 1983, 339-348.

[25] B.A. Frasin, M.K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett., 24(2011), 1569-1573.

[26] S. Fukui, *On $\alpha-$convex functions of order β*, Int. J. Math. Math. Sci., 20(4)(1997), 769-772.

[27] A.W. Goodman, Univalent Functions, I-II, Mariner Publ. Comp., Tampa, Florida, 1983.

[28] I. Graham, G. Kohr, Geometric Function Theory in One and Higher Dimensions, Marcel Dekker Inc., New York, 2003.

[29] T.H. Gronwall, *Some remarks on conformal representation*, Ann. Math., 16(1914/15), 72-76.

[30] D.J. Hallenbeck, S. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc., 52(1975), 191-195.

[31] P. Hamburg, P.T. Mocanu, N. Negoescu, Analiză Matematică (Funcţii Complexe), Ed. Did. şi Ped., Bucureşti, 1982.

[32] D.J. Hallenbeck, T.H. MacGregor, Linear Problems and Convexity Techniques in Geometric Function Theory, Pitman Adv. Publ. Program, Boston-London-Melbourn, 1984.

[33] E. Hille, Ordinary Differential Equations in the Complex Plane, John Wiley, New York, 1976.

[34] W. Janowski, *Some external problems for certain families of analytic functions*, Ann. Polon. Math., 28 (1973), 297-326.

[35] A. Kanas, A. Lecko, J. Stankiewicz, *Differential subordinations and geometric means*, Compl. Var. Theor. Appl., 28(1996), 201-209.

[36] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J., 1(1952), 169-185.

[37] P. Koebe, *Über die Uniformisierung beliebiger analytischer Kurven*, Nachr. Kgl. Ges. Wiss. Göttingen Math. Phys. (1907), 191-210.

[38] V. Kumar, S.L. Shukla, *Certain integrals for classes of p-valent meromorphic functions*, Bull. Austral. Math. Soc., 25(1982), 85-97.

[39] A. Lecko, M. Lecko, *Differential subordinations of arithmetic and geometric means of some functionals related to a sector*, Int. J. Math. Math. Sci., 2011(2011), Article ID 205845, 19 pages.

[40] Z. Lewandowski, *Sur l'identité de certaines classes de fonctions univalentes I*, Ann. Univ. Mariae Curie Skłodowska, 12(1958), 131-146.

[41] Z. Lewandowski, *Sur l'identité de certaines classesde fonctions univalentes II*, Ann. Univ. Mariae Curie-Skłodowska, 14(1960), 19-46.

[42] Z. Lewandowski, S. Miller, E. Złotkiewicz, *Gamma-starlike functions*, Ann. Univ. Mariae Curie-Skłodowska, 28(1974), 32-36.

[43] Z. Lewandowski, S.S. Miller, E. Złotkiewicz, *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc., 56(1976), 111-117.

[44] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Am. Math. Soc., 18(1967), 63-68.

[45] R.J. Libera, *Univalent λ-spiral functions*, Canad. J. Math, 19(1967), 449-456.

[46] E. Lindelöf, *Mémoire sur certaines inégalités dans la théorie des fonctions monogènes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel*, Acta Soc. Sc. Fennicae, 35(7)(1908), 1-35.

[47] J.-L. Liu, H.M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl., 259(2001), 566-581.

[48] K. Löwner, *Untersuchungen über die Verzerrung bei konformen Abbidungen des Einheitskreises* $|z| < 1$, *die durch Funktionen mit nichtverschwindender Ableitung geliefert werden*, S.B. Sächs. Akad. Wiss. Leipzig Berichte, 69(1917), 89-106.

[49] W.C. Ma, D. Minda, *A unified treatment of some special classes of univalent functions*, in: Proceedings of the Conference on Complex Analysis, Tianjin, 1992, pp. 157-169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 1994.

[50] A. Marx, *Untersuchungen über schlichte Abbildungen, Math. Ann.*, 107(1932), 40-67.

[51] S.S. Miller, *Distortion properties of alpha-starlike functions*, Proc. Amer. Math. Soc., 38(1973), 311-318.

[52] S.S. Miller, P.T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York, 2000.

[53] S.S. Miller, P.T. Mocanu, *Differential subordinations and univalent functions*, Michig. Math. J., 28(1981), 157-171.

[54] S.S. Miller, P.T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65(1978), 289-305.

[55] S.S. Miller, P.T. Mocanu, *Briot-Bouquet differential equations and differential subordinations*, Compl. Var., 33(1997), 217-237.

[56] S.S. Miller, P.T. Mocanu, *Differential subordinations and inequalities in the complex plane*, J. Diff. Eqns., 67(1987), 199-211.

[57] S.S. Miller, P.T. Mocanu, M.O. Reade, *All α-convex functions are starlike*, Rev. Roum. Math. Pures Appl., 17(1972), 1395-1397.

[58] S.S. Miller, P.T. Mocanu, M.O. Reade, *Bazilevich functions and generalized convexity*, Rev. Roum. Math. Pures Appl., 19(1974), 213-224.

[59] P.T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica (Cluj), 11(34)(1969), 127-133.

[60] P.T. Mocanu, *On strongly starlike and strongly convex functions*, Stud. Univ. Babeş-Bolyai Math., 31(1986), 16-21.

[61] P.T. Mocanu, *Alpha-convex integral operators and strongly starlike functions*, Stud. Univ. Babeş-Bolyai Math., 34(1989), 18-24.

[62] P.T. Mocanu, T. Bulboacă, G.Ş. Sălăgean, Teoria Geometrică a Funcţiilor Analitice, Casa Cărţii de Ştiinţă, Cluj-Napoca, 1999.

[63] R. Nevanlinna, *Über die konforme Abbildung von Sterngebieten*, Översikt av Finska Vetenskaps-Soc. Förh., 63A(1920-1921), 1-21.

[64] K. Noor, *On quasi-convex functions and related topics*, Internat. J. Math. Math. Sci., 10(2)(1987), 241-258.

[65] K. Noshiro, *On the theory of schlicht functions*, J. Fac. Sci. Hokkaido Univ., 2(1934-35), 129-155.

[66] M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan. Acad., Ser. A, 69(1993), 234-237.

[67] S. Ozaki, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku, A, 40(1935), 167-188.

[68] C. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen 1975.

[69] V. Ravichandran, *Starlike and convex functions with respect to conjugate points*, Acta Math. Acad. Paedagog. Nyiregyhaziensis, 20(2004), 31-37.

[70] D. Răducanu, V.O. Nechita, *On α-convex analytic functions defined by generalized Ruscheweyh operator*, Stud. Univ. Babeş-Bolyai, 53(2008), 109-118.

[71] M.O. Reade, *On close-to-convex univalent functions*, Mich. Math. J., 3(1955-56), 59-62.

[72] M.S. Robertson, *On the theory of univalent functions*, Ann. Math., 37(1936), 374-408.

[73] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc. Series 2, 48(1943), 48-82.

[74] S. Ruschweyh, Convolutions in Geometric Function Theory, Les Presses de l'Université de Montreal, 1982.

[75] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., 49(1975), 109-115.

[76] S. Ruscheweyh, V. Singh, *On a Briot-Bouquet equation related to univalent functions*, Rev. Roumaine Math. Pures Appl., 24(1979), 285-290.

[77] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan, 11(1)(1959), 72-75.

[78] K. Sakaguchi, S. Fukui, *On alpha-starlike functions and related functions*, Bull. Nara Univ. of Education, 28(1979), 5-12.

[79] G.Ş., Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013(1983), 362-372.

[80] Y.J. Sim, O.S. Kwon, *A subclass of meromorphic close-to-convex functions of Janowski's type*, Int. J. Math. Math. Sci., Article ID 682162, 12 pages, 2012. doi:10.1155/2012/682162.

[81] L. Spaček, *Contribution à la théorie des fonctions univalentes*, Časopis Pěst. Mat., 62(1932), 12-19 (in Russian).

[82] H.M. Srivastava, A.K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., 23(2010), 1188-1192.

[83] H.M. Srivastava, A.A. Attyia, *Some applications of differential subordination*, Appl. Math. Lett., 20(2007), 1142-1147.

[84] H.M. Srivastava, D.-G. Yang, N.-E. Xu, *Some subclasses of meromorphically multivalent functions associated with a linear operator*, Appl. Math. Comp., 195(2008), 11-23.

[85] E. Strohhäcker, *Beitrage zur Theorie der schlichten Funktionen*, Math. Z., 37(1933), 356-380.

[86] E. Study, Vorlesungen über ausgewählte Gegenstüde der Geometrie, 2. Heft, Teubner, Leipzig, Berlin, 1913.

[87] B.A. Uralegaddi, C. Somanatha, *Certain classes of meromorphic multivalent functions*, Tamkang J. Math., 23(1992), 223-231.

[88] Z.G. Wang, C.-Y. Gao, S.-M. Yuan, *On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points*, J. Math. Anal. Appl., 322(2006), 97-106.

[89] Z.-W. Wang, Y. Sun, N. Xu, *Some properties of certain meromorphic close-to-convex functions*, Appl. Math. Lett., 25(2012), 454-460.

[90] S.E. Warschawski, *On the higher derivatives at the boundary in conformal mapping*, Trans. Amer. Math. Soc., 38(1935), 310-340.

[91] J. Wolff, *L'intégrate d'une fonction holomorphe et à partie réelle positive dans un demi plan est univalente*, C.R. Acad. Sci. Paris, 198(13)(1934), 1209-1210.

[92] D.-G. Yang, *Certain convolution operators for meromorphic functions*, Southeast Asian Bull. Math., 25(2001), 175-186.