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NEW CLASSES OF ANALYTIC FUNCTIONS

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Cluj-Napoca, 2013

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Introduction

Geometric functions theory is the branch of complex analysis which deals with geometric properties of analytic functions. This theory is based on the notion of conformal representation in which univalent functions play an essential role. A remarcable result in this sense is the Riemann mapping theorem. The first important papers in this branch appeared in the early twentieth century, due to P. Koebe [46] in 1907, T. H. Gronwall [28] in 1914, J.W. Alexander [1] in 1915, L. Bieberbach [10] in 1916. The Bieberbach conjecture, solved by Louis de Branges in 1984, determined various approaches and directions of studies in the geometric functions theory, one of these directions was defining new subclasses of univalent functions for which the conjecture could be verified. Therewith appeared and developed new research methods such as Löwner parametric method and integral representation method introduced by Herglotz.

It is worth mentioning that romanian mathematicians brought valuable contribution in developing this area of mathematics. G. Călugăreanu obtained in 1931 necessary and sufficient condition for univalence in the open unit disk. Continuing the work of G. Călugăreanu, P.T. Mocanu obtained importand results in the field: introduced α-convex functions; obtained univalence criteria for non-analytic functions; developed, in collaboration with S.S. Miller, the method of differential subordinations and superordinations.

This thesis consists of five chapters. Chapter 1 is structured in nine sections in which are presented some fundamental definitions and results which constitutes the backgrounds for the remaining chapters. Therefore, are given important results regarding univalent functions, functions with positive real part, starlike functions, convex functions, close-to-starlike and close-to-convex functions, spirallike functions and also results regarding harmonic mappings.

Chapter 2 deals with differential subordinations problems. In first section we give some definitions and results concerning differential subordinations and briefly present the method of admisible functions. The next two sections of the chapter contains original results presented in [87] and [43]. In the second section we provide some examples of subordinations involving the disk and the half-plane by applying the admisibility conditions introduced in previous section. In the last section we study differential subordinations involving harmonic means of the expresions $p(z), p(z)+zp'(z)$ and $p(z)+\dfrac{zp'(z)}{p(z)}$, when p is an analytic function in the unit disk, such that $p(0)=1, p(z)\not\equiv 1$, and present some applications in geometric functions theory.

In Chapter 3 we define four new subclasses of bi-univalent functions for which we obtain estimates of coefficients a_2 and a_3. A function f is said to be bi-univalent in U if both f and its inverse are univalent in U. The results in this chapter are original and are presented in [86] and [88].

In Chapter 4 we define and study new classes of functions defined by means of certain operators. The results in this chapter are original and are contained in the papers: [85], [89], [90], [19] and [91]. In first section we define a new class of analytic function by means of Carlson-Shaffer and Cho-Srivastava operators. We provide sufficient condition for a function to be in this function class and we find some angular estimates. In the next section we define and study a subclass of harmonic univalent and sense preserving functions also connected with a generalized operator. We determine neccesary and sufficient condition for a function to be in this class, extreme points, distorsion bounds and also an inclusion result related to convolution. In the third section we define and study a new class of analytic functions connected with Sălăgean integral operator. In particular, we derive an inclusion property, a subordination result, extreme points and coefficient bounds for this function class. In Section 4.4, by means of Sălăgean integral operator, we introduce two new subclasses of analytic functions involving λ-spirallikeness of order α. For this function classes we establish some inclusion results. In the last section of this chapter we introduce a new class of generalized close-to-starlike functions connected with Srivastava-Attiya operator. For this class we provide inclusion results, coefficient bounds and we give an integral representation. Also, we will show that this function class is closed under the convolution operation by convex functions.

In Chapter 5 we present Löwner chains and their utility in obtaining new univalence criteria. In first section we present the general Löwner chain theory and in the second and the third chapter we present some original results, [92], where we obtain new condition for univalence by applying this subordination chains method.

The bibliography contains 91 titles, 9 signed by the author, two of them in collaboration.

I would like to express my warm and sincere thanks to my thesis supervisors Professor G. Şt. Sălăgean and Professor S. Kanas for guidings, encouragements, suport and for pertinent observations that I received during the research.

I also would like to thank Professor D. Răducanu for continuous help and support.

Many thanks for the financial support provided from the Sectoral Operational Programme for Human Resources Development 2007-2013, co-financed by the European Social Fund, under the project number POSDRU/107/1.5/S/76841 with the title Modern Doctoral Studies: Internationalization and Interdisciplinarity.

Keywords: analytic function, univalent function, bi-univalent function, harmonic function, starlikeness, convexity, spirallikeness, differential subordination, differential operator, integral operator univalence criteria, Löwner chain.

Chapter 1

Preliminary Results

1.1 Definitions and notations

In this section are presented basic notions about complex functions, analytic continuation and extreme points.

We use the following notations:

- Unit disk: $U = \{z \in \mathbb{C} : |z| < 1\}$,
- $U_r = \{z \in \mathbb{C} : |z| < r\}$,
- $\bar{U} = \{z \in \mathbb{C} : |z| \leq r\}$,
- $\partial U \{z \in \mathbb{C} : |z| = r\}$.

Let $\mathcal{H}(U)$ be the set of holomorphic functions in U. For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ denote

- $\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$,
- $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \cdots\}$, $(\mathcal{A}_1 = \mathcal{A})$.

A function $f \in \mathcal{A}$ has the following Taylor series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

1.2 Univalent functions

Definition 1.2.1. *[20] A single-valued function f is said to be univalent (or schlicht) in a domain $D \subset \mathbb{C}$ if it never takes the same value twice; that is, if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in D with $z_1 \neq z_2$.*

We denote by $\mathcal{S}$ the class of univalent function in the unit disk U normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each function in the class $\mathcal{S}$ has a Taylor series expansion of the form

$$f(z) = z + a_2 z^2 + \cdots, \quad z \in U.$$

Theorem 1.2.1 (**Koebe One-Quarter Theorem**). *[20] The range of every function of class $\mathcal{S}$ contains the disk $\left\{w : |w| < \frac{1}{4}\right\}$.*

In 1916, Bieberbach [10] formulated the following conjecture:

Theorem 1.2.2 (**Bieberbach Conjecture**). *The coefficients of each function $f \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n = 2, 3, \cdots$. Strict inquality holds for all n unless f is the Koebe function or one of its rotations.*

1.3 Functions with positive real part

The class of Caratheodory functions is denoted by

$$\mathcal{P} = \{p \in \mathcal{H}(U) : p(0) = 1, \ \Re p(z) > 0, \ z \in U\}.$$

Theorem 1.3.1. *[65] Let $p \in \mathcal{P}$. Then*

$$\begin{aligned} &|p_n| \leq 2, \ n \geq 1, \\ &\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}. \end{aligned} \tag{1.2}$$

1.4 Starlike functions

Definition 1.4.1. *[57] A function $f \in \mathcal{H}(U)$ is said to be starlike if it is univalent and $f(U)$ is a starlike domain.*

The following theorem gives an analytic characterisation of starlike functions:

Theorem 1.4.1. *[57] A function $f \in \mathcal{H}(U)$ is starlike if and only if $f(0) = 0,\ f'(0) \neq 0$ and*

$$\Re \frac{zf'(z)}{f(z)} > 0, \quad z \in U.$$

The class of starlike functions is denoted by $\mathcal{S}^*$ and consists of the sets of all f in $\mathcal{S}$ for which $f(U)$ is starlike.

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > 0 \right\}. \tag{1.3}$$

Theorem 1.4.2. *[26] If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is in $\mathcal{S}^*$ then*

$$|a_n| \leq n,\ n = 2, 3, \cdots .$$

Equality takes place if and only if f is Koebe function.

1.5 Close-to-starlike functions

Definition 1.5.1. *[67] A function $f \in \mathcal{A}$ is close-to-starlike if and only if there exists a function $g \in \mathcal{S}^*$ such that*

$$\Re \left(\frac{f(z)}{g(z)} \right) \geq 0, \quad z \in U.$$

We denote by $\mathcal{CS}^*$ the class of close-to-starlike functions.

Theorem 1.5.1. *[67] If $f \in \mathcal{A}$ is close-to-starlike then the coefficients satisfy the inequality*

$$|a_n| \leq n^2,\ n = 2, 3, \cdots ,$$

with the equality for the Robertson functions starlike in one direction [70].

1.6 Convex functions

Definition 1.6.1. *[57] A function $f \in \mathcal{H}(U)$ is said to be convex if it is univalent and $f(U)$ is a convex domain.*

Theorem 1.6.1. *[57] A function $f \in \mathcal{H}(U)$ is convex if and only if $f'(0) \neq 0$ and*

$$\Re\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0, \quad z \in U.$$

The class of convex functions is denoted by $\mathcal{K}$ and consists of the sets of all f in $\mathcal{S}$ for which $f(U)$ is convex.

An analytic description of $\mathcal{K}$ is given by:

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \Re\left[\frac{zf''(z)}{f'(z)} + 1\right] > 0 \right\}. \tag{1.4}$$

Theorem 1.6.2. *[26] If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is in $\mathcal{K}$ then*

$$|a_n| \leq 1, \;\; n = 2, 3, \cdots .$$

Equality takes place if and only if f has the following form:

$$f(z) = \frac{z}{1 + e^{i\sigma}z}, \;\; \sigma \in \mathbb{R}.$$

1.7 Close-to-convex functions

Definition 1.7.1. *[67] A function $f \in \mathcal{A}$ is close-to-convex if and only if there exists a function $g \in \mathcal{K}$ such that*

$$\Re\left(\frac{f'(z)}{g'(z)}\right) \geq 0, \quad z \in U.$$

We denote by $\mathcal{CK}$ the class of close-to-convex functions.

Theorem 1.7.1. *[67] If $f \in \mathcal{A}$ is close-to-convex then the coefficients satisfy the inequality*

$$|a_n| \leq n, \;\; n = 2, 3, \cdots .$$

1.8 Spirallike functions

Theorem 1.8.1. *[65] A function $f \in \mathcal{A}$ is said to be λ- spirallike, $\left(-\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right)$ if and only if*

$$\Re\left[e^{i\lambda}\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in U.$$

In 1967, R. Libera [52] extended the definition to functions λ-spirallike of order α.

Definition 1.8.1. *For $0 \leq \alpha < 1$ and $|\lambda| < \pi/2$, a function $f \in \mathcal{A}$ is said to be λ-spirallike of order α in U if*

$$\Re\left[e^{i\lambda}\frac{zf'(z)}{f(z)}\right] > \alpha\cos\lambda, \quad z \in U. \tag{1.5}$$

1.9 Harmonic mappings

In this section we present some basic notions and properties related to harmonic functions. These functions are closely connected to holomorphic functions since the real and the imaginary parts of any holomorphic function are harmonic functions and every harmonic function on a simply connected domain D in $\mathbb{C}$ is the real (imaginary) part of a holomorphic function in D.

Definition 1.9.1. *[21] Let $D \subseteq \mathbb{C}$ be a region. A real valued function $u(x, y)$ is harmonic in D if it satisfies Laplace's equation*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Proposition 1.9.1. *[21] Let $f = u + iv$ be a holomorphic function in D. Then u and v are harmonic functions in D.*

Remark 1.9.1. *The converse of Proposition* 1.9.1 *is also true but only if D is a simply connected domain, i.e. when every path between two points in D can be continuously transformed, staying within D, into any other path while presetving the two endpoints in question.*

Theorem 1.9.1. *[21] Suppose u is harmonic on the simply connected domain D. Then there exists a harmonic function v such that $f = u + iv$ is holomorphic in D.*

Remark 1.9.2. *The function v is called a harmonic conjugate of u.*

Theorem 1.9.2 (the mean value property). *[21] Let u be a harmonic real-valued function on a open set D. If D contains a closed disk of radius r centered at z_0 then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \mathrm{d}\theta.$$

Theorem 1.9.3 (the maximum principle). *[21] Let u be a harmonic real-valued function on a open connected set D. If u attains its maximum value at some point $z_0 \in D$ then u is constant.*

The next theorem allows to recover values of a harmonic function in a disk from its values on the disk boundary. The result is an analogue of the Cauchy integral formula for holomorphic functions.

Theorem 1.9.4 (Poisson integral formula). *[21] Let $r > 0$ and $u : \bar{U}(0,r) \to \mathbb{R}$ a harmonic function in $U(0,r)$ and continuous in barU(0,r). Then*

$$u(\rho e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - \rho^2}{r^2 - 2\rho\cos(\theta - \varphi) + \rho^2} u(re^{i\theta}) \mathrm{d}\theta,$$

for all $\rho \in [0,r)$ and $\varphi \in \mathbb{R}$.

Chapter 2

Differential subordinations

In [56] and [58], S.S. Miller and P.T. Mocanu extended the study of differential inequalities for real-valued functions to complex-valued functions defined in the unit disk. They developed a new method in geometric theory of analytic functions known as the method of differential subordinations or the method of admisible functions. This method proved to be very effective in obtaining new results in geometric functions theory or proving, in a simple manner, several results already known.

2.1 Basic definitions and results

Definition 2.1.1. *[65] If f and g are two functions analytic in U, we say that f is subordinate to g, written as*

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function ω (i.e. analytic in U, with $\omega(0) = 0$ and $|\omega(z)| < 1$, for all $z \in U$) such that

$$f(z) = g(\omega(z)), \; z \in U.$$

Theorem 2.1.1. *[65] Let $f, g \in \mathcal{H}(U)$ and let g be univalent in U. Then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

Let $\Omega, \Delta \subset \mathbb{C}$, $p \in \mathcal{H}(U)$ with $p(0) = a$, $a \in \mathbb{C}$ and let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. The method of differential subordinations (or the method of admisible functions) deals with generalizations of the following implication:

$$\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\} \subset \Omega \Rightarrow p(U) \subset \Delta. \tag{2.1}$$

Definition 2.1.2. *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{2.2}$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (2.2).

A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (2.2) is said to be the best dominant of (2.2). (Note that the best dominant is unique up to a rotation of U).

Definition 2.1.3. *[57] Denote by Q the set of functions q that are analytic and injective on $\bar{U} \setminus E(q)$, where*

$$E(q) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\right\},$$

such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

If $q \in Q$ then $\Delta = q(U)$ is a simply connected domain.

Lemma 2.1.1. *[57] Let $q \in Q$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \cdots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,*

i) $p(z_0) = q(\zeta_0)$,

ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$,

iii) $\Re \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\Re\left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right]$.

Next, we define the class of admisible functions:

Definition 2.1.4. *[57] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admisible fuctions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admisibility condition*

$$\begin{cases} \psi(r, s, t; z) \notin \Omega, \ \textit{whenever} \\ r = q(\zeta), \ s = m\zeta q'(\zeta), \ \Re\left(1 + \dfrac{t}{s}\right) \geq m\Re\left(1 + \dfrac{\zeta q''(\zeta)}{q'(\zeta)}\right), \\ \zeta \in \partial U \setminus E(q), \ z \in U, \ m \geq n. \end{cases}$$

Theorem 2.1.2. *[57] Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$.*
If $p \in \mathcal{H}[a, n]$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ analytic in U then

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \Rightarrow p(z) \prec q(z).$$

Definition 2.1.5. *[57] Let h be an univalent function in U with $h(0) = a$ and let $p \in \mathcal{H}[a, n]$ satisfy*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \delta} \prec h(z).$$

The first-order differential subordination is called Briot-Bouquet differential subordination.

Remark 2.1.1. *The name of Briot-Bouquet subordination derives from the fact that a differential equation of the form*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \delta} = h(z)$$

is called a differential equation of Briot-Bouquet type [32].

Lemma 2.1.2. *[57] Let h be a convex function in U with $\Re[\beta h(z) + \delta] > 0$, $z \in U$. If q is an analytic function in U such that $q(0) = h(0)$ and*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \delta} \prec h(z),$$

then $q(z) \prec h(z)$.

2.2 Examples

In this section we present some examples of subordinations involving the disk and the half-plane by applying the admisibility conditions introduced in previous section.

In our first three examples we consider different conditions for functions A, B, C such that

$$\Re[A(z)p(z) + B(z)zp'(z) + C(z)z^2p''(z)] > 0 \Rightarrow Rep(z) > 0,\ z \in U.$$

Example 2.2.1. *[87] Let $A, B : U \to \mathbb{C}$, $C : U \to \mathbb{R}$ such that $\Im A(z) \leq 1$ and $\Re B(z) \leq 1 + C(z)$, $z \in U$. If $p(z) = 1 + a_1 z + \cdots$ then*

$$\Re[A(z)p(z) + B(z)zp'(z) + C(z)z^2p''(z)] > 0 \Rightarrow Rep(z) > 0,\ z \in U.$$

Example 2.2.2. *[87] Let $A, B : U \to \mathbb{C}$, $C : U \to \mathbb{R}$ such that $\Re A(z) > 0$, $\Re B(z) \leq \Re A(z) + C(z)$, $C(z) > 0$ and $\Im^2 A(z) \leq \Re^2 A(z)$, $z \in U$. If $p(z) = 1 + a_1 z + \cdots$ then*

$$\Re[A(z)p(z) + B(z)zp'(z) + C(z)z^2p''(z)] > 0 \Rightarrow Rep(z) > 0,\ z \in U.$$

Example 2.2.3. *[87] Let $A, B, C : U \to \mathbb{C}$ such that $\Im C(z) < 0$, $\Re B(z) = \Re C(z)$, $\Re C(z) > 0$ and $\Im^2 A(z) \leq \Im^2 C(z)$, $z \in U$. If $p(z) = 1 + a_1 z + \cdots$ then*

$$\Re[A(z)p(z) + B(z)zp'(z) + C(z)z^2p''(z)] > 0 \Rightarrow Rep(z) > 0,\ z \in U.$$

In the next three examples we consider different conditions for functions A, B, C such that

$$\left|A(z)p(z) + B(z)zp'(z) + C(z)z^2p''(z)\right| < 1 \Rightarrow |p(z)| < 1,\ z \in U.$$

Example 2.2.4. *[87] Let $A, B : U \to \mathbb{C}$, $C : U \to \mathbb{R}$ such that $C(z) \geq 1$, $\Re A(z) \geq 1 + \Im A(z)$ and $(\Re B(z) - 1)^2 \leq 4\Im A(z)$, $z \in U$. If $p \in \mathcal{H}[0, n]$ then*

$$\left|A(z)p(z) + B(z)zp'(z) + C(z)z^2p''(z)\right| \Rightarrow |p(z)| < 1,\ z \in U.$$

Example 2.2.5. *[87] Let $A, B, C : U \to \mathbb{C}$ such that $\Re C(z) = \Re B(z)$, $\Re A(z) \geq$*

$1 - \Re C(z)$ *and* $\Re C(z) \geq 0$, $z \in U$. *If* $p \in \mathcal{H}[0,n]$ *then*

$$\left|A(z)p(z) + B(z)zp'(z) + C(z)z^2p''(z)\right| \Rightarrow |p(z)| < 1, \ z \in U.$$

Example 2.2.6. *[87] Let* $A, B, C : U \to \mathbb{C}$ *such that* $\Re C(z) = \Re B(z) - 2\Re A(z)$, $\Re B(z) \geq 1 + 2\Re A(z)$ *and* $\Re A(z) \geq 1 + \Re^2 A(z)$, $z \in U$. *If* $p \in \mathcal{H}[0,n]$ *then*

$$\left|A(z)p(z) + B(z)zp'(z) + C(z)z^2p''(z)\right| \Rightarrow |p(z)| < 1, \ z \in U.$$

2.3 Differential subordinations involving harmonic mean

In this section we study the harmonic mean, as a suplementary to the well known arithmetics and geometric Pythagorean means. In addition, a new mean brings along a wide range of new possibilities for exploiting harmonic ideas in connection of several quantities or functionals in the geometric function theory.

Theorem 2.3.1. *[43] Let $p(z) = 1 + a_1 z + \cdots$ be analytic in U with $p(z) \not\equiv 1$. Then*

$$\Re\left\{\frac{2p(z)\left[p(z) + zp'(z)\right]}{2p(z) + zp'(z)}\right\} > 0 \Rightarrow \Re p(z) > 0. \tag{2.3}$$

Remark 2.3.1. *We only note that the expression of the left hand side of* (2.3) *is of the harmonic form of two elements $x_1 = p(z)$ and $x_2 = p(z) + zp'(z)$ $(z \in U)$.*

Setting $p(z) = \dfrac{f(z)}{z}$ in the previous theorem we obtain the following corollary:

Corollary 2.3.1. *[43] Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in U. Then*

$$\Re\frac{2f(z)f'(z)}{f(z) + zf'(z)} > 0 \Rightarrow \Re\frac{f(z)}{z} > 0.$$

Theorem 2.3.2. *[43] Let $p(z) = 1 + a_1 z + \cdots$ be analytic in U with $p(z) \not\equiv 1$. Then*

$$\Re\left[\frac{2p(z) + 2zp'(z)}{1 + p^2(z) + zp(z)p'(z)}\right] > 0 \Rightarrow \Re p(z) > 0.$$

Setting $p(z) = \dfrac{f(z)}{z}$ we obtain:

Corollary 2.3.2. *[43] Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in U. Then*

$$\Re\left[\frac{2\dfrac{z}{f(z)}f'(z)}{\dfrac{z}{f(z)} + f'(z)}\right] > 0 \Rightarrow \Re\frac{f(z)}{z} > 0.$$

Theorem 2.3.3. *[43] Let $p(z) = 1 + a_1 z + \cdots$ be analytic in U with $p(z) \not\equiv 1$. Then*

$$\Re\left\{\frac{2\left[p(z)+\dfrac{zp'(z)}{p(z)}\right]}{2+\dfrac{zp'(z)}{p^2(z)}}\right\} > 0 \Rightarrow \Re p(z) > 0.$$

Setting $p(z) = \dfrac{zf'(z)}{f(z)}$ we obtain:

Corollary 2.3.3. *[43] Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in U. Then*

$$\Re\left\{\frac{2\dfrac{zf'(z)}{f(z)}\left[1+\dfrac{zf''(z)}{f'(z)}\right]}{1+\dfrac{zf'(z)}{f(z)}+\dfrac{zf''(z)}{f'(z)}}\right\} > 0 \Rightarrow \Re\frac{zf'(z)}{f(z)} > 0.$$

Theorem 2.3.4. *[43] Let $p(z) = 1 + a_1 z + \cdots$ be analytic in U with $p(z) \not\equiv 1$. Then*

$$\Re\left\{\frac{2\left[p(z)+\dfrac{zp'(z)}{p(z)}\right]}{1+p^2(z)+zp'(z)}\right\} > 0 \Rightarrow \Re p(z) > 0.$$

Setting $p(z) = \dfrac{zf'(z)}{f(z)}$ we obtain:

Corollary 2.3.4. *[43] Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in U. Then*

$$\Re\left\{\frac{2\dfrac{f(z)}{zf'(z)}\left[1+\dfrac{zf''(z)}{f'(z)}\right]}{1+\dfrac{f(z)}{zf'(z)}+\dfrac{zf''(z)}{f'(z)}}\right\} > 0 \Rightarrow \Re\frac{zf'(z)}{f(z)} > 0.$$

Theorem 2.3.5. *[43] Let $p(z) = 1 + a_1 z + a_2 z^2 + \cdots$ be analytic in U with $p(z) \not\equiv 1$, and let $0 < M < \frac{1}{3}$. Then*

$$\left|\frac{2p(z)\left[p(z)+zp'(z)\right]}{2p(z)+zp'(z)} - 1\right| < M \Rightarrow |p(z) - 1| < M. \tag{2.4}$$

Let $p(z) = \dfrac{f(z)}{z}$. Then the previous theorem reduces o the following corollary:

Corollary 2.3.5. *[43] Let $f(z) = z + a_2z^2 + \cdots$ be analytic in U and let $0 < M < \frac{1}{3}$. Then*

$$\left| \frac{2f'(z)}{2 + \dfrac{zf'(z)}{f(z)}} - 1 \right| < M \Rightarrow \left| \frac{f(z)}{z} - 1 \right| < M.$$

For the case when $p(z) = f'(z)$, Theorem (2.3.5) gives:

Corollary 2.3.6. *[43] Let $f(z) = z + a_2z^2 + \cdots$ be analytic in U and let $0 < M < \frac{1}{3}$. Then*

$$\left| \frac{2(f'(z) + zf''(z))}{2 + \dfrac{zf''(z)}{f'(z)}} - 1 \right| < M \Rightarrow |f'(z) - 1| < M.$$

Also, letting $p(z) = \dfrac{zf'(z)}{f(z)}$ in Theorem (2.3.5), we conclude:

Corollary 2.3.7. *[43] Let $f(z) = z + a_2z^2 + \cdots$ be analytic in U and let $0 < M < \frac{1}{3}$. Then*

$$\left| \frac{2\dfrac{zf'(z)}{f(z)}\left(2 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}\right)}{3 + \dfrac{zf''(z)}{f'(z)} - \dfrac{zf'(z)}{f(z)}} - 1 \right| < M \Rightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| < M.$$

Theorem 2.3.6. *[43] Let $p(z) = 1 + a_1z + \cdots$ be analytic in U with $p(z) \not\equiv 1$ and let $\gamma \in (0, 1]$. Then*

$$\left| \arg \frac{2p(z)\,[p(z) + zp'(z)]}{2p(z) + zp'(z)} \right| < \gamma\frac{\pi}{2} \Rightarrow |\arg p(z)| < \gamma\frac{\pi}{2}. \tag{2.5}$$

Setting $p(z) = \dfrac{f(z)}{z}$ we obtain the following corollary:

Corollary 2.3.8. *[43] Let $f(z) = z + a_2z^2 + \cdots$ be analytic in U. Then*

$$\left| \arg \frac{2f(z)f'(z)}{f(z) + zf'(z)} \right| < \gamma\frac{\pi}{2} \Rightarrow \left| \arg \frac{f(z)}{z} \right| < \gamma\frac{\pi}{2}.$$

Also, letting $p(z) = \dfrac{zf'(z)}{f(z)}$ in Theorem (2.3.6), we conclude:

Corollary 2.3.9. *Let* $f(z) = z + a_2z^2 + \cdots$ *be analytic in* U. *Then*

$$\left|\arg \frac{2\dfrac{zf'(z)}{f(z)}\left(2+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}\right)}{3+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}}\right| < \gamma\frac{\pi}{2} \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \gamma\frac{\pi}{2}.$$

Theorem 2.3.7. *[43] Let* $p(z) = 1 + a_1z + \cdots$ *be analytic in* U *with* $p(z) \not\equiv 1$ *and let* $\gamma \in (0, 1]$. *Then*

$$\left|\arg \frac{2\left[p(z)+\dfrac{zp'(z)}{p(z)}\right]}{2+\dfrac{zp'(z)}{p^2(z)}}\right| < \gamma\frac{\pi}{2} \Rightarrow |\arg p(z)| < \gamma\frac{\pi}{2}.$$

Setting $p(z) = \dfrac{zf'(z)}{f(z)}$ we obtain:

Corollary 2.3.10. *[43] Let* $f(z) = z + a_2z^2 + \cdots$ *be analytic in* U. *Then*

$$\left|\arg \frac{2\dfrac{zf'(z)}{f(z)}\left[1+\dfrac{zf''(z)}{f'(z)}\right]}{1+\dfrac{zf'(z)}{f(z)}+\dfrac{zf''(z)}{f'(z)}}\right| < \gamma\frac{\pi}{2} \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \gamma\frac{\pi}{2}.$$

Chapter 3

Bi-univalent functions

3.1 Coefficient bounds for the class $R^{\tau}_{\gamma,\sigma}(\varphi)$

Definition 3.1.1. *[86] A function f, given by* (1.1) *is said to be in the class $R^{\tau}_{\gamma,\sigma}(\varphi)$* $(0 \leq \gamma \leq 1,\ \ \tau \in \mathbb{C} \setminus \{0\})$ *if satisfies the following conditions:*

$$f \in \sigma \quad \textit{and} \quad 1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) \prec \varphi(z)$$

$$\textit{and} \quad 1 + \frac{1}{\tau}\left(g'(w) + \gamma w g''(w) - 1\right) \prec \varphi(w),$$

where g is the extension of f^{-1} to U.

If we set $\varphi(z) = \dfrac{1+Az}{1+Bz}$, $\ -1 \leq B < A \leq 1,\ \ z \in U$ we will obtain the subclass $R^{\tau}_{\gamma,\sigma}(A,B)$ of functions f which satisfies:

$$f \in \sigma \quad \textit{and} \quad \left|\frac{f'(z) + \gamma z f''(z) - 1}{\tau(A-B) - B\left(f'(z) + \gamma z f''(z) - 1\right)}\right| < 1,$$

$$\textit{and} \quad \left|\frac{g'(w) + \gamma w g''(w) - 1}{\tau(A-B) - B\left(g'(w) + \gamma w g''(w) - 1\right)}\right| < 1.$$

The class $R^{\tau}_{\gamma}(\varphi)$ of analytic functions which satisfies the condition

$$1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) \prec \varphi(z)$$

was introduced by Bansal in [7].

Theorem 3.1.1. *[86] If $f \in R^{\tau}_{\gamma,\sigma}(\varphi)$ is given by (1.1) then*

$$|a_2| \leq \frac{|\tau| B_1 \sqrt{B_1}}{\sqrt{\left|3\tau B_1^2(1+2\gamma) - 4(B_2 - B_1)(1+\gamma)^2\right|}} \tag{3.1}$$

and

$$|a_3| \leq \frac{B_1|\tau|}{3(1+2\gamma)} + \frac{B_1^2|\tau|^2}{4(1+\gamma)^2}. \tag{3.2}$$

If we choose, in Theorem 3.1.1, $\tau = 1$ and $\gamma = 0$ we obtain the result proved by Ali et al. [5, Th. 2.1].

For $A = 1$ and $B = -1$ in corollary 3.1.1 and for $\tau = 1$ and $\gamma = 0$, we obtain the following estimations for a_2 and a_3

$$|a_2| \leq \sqrt{\frac{2}{3}} \quad \text{and} \quad |a_3| \leq \frac{5}{3}.$$

Setting $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$, $0 < \alpha \leq 1$ we will obtain the following corollary:

Corollary 3.1.1. *[86] If $f \in R^{\tau}_{\gamma,\sigma}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$, then*

$$|a_2| \leq \frac{2|\tau|\alpha\sqrt{\alpha}}{\sqrt{|6\tau\alpha^2(1+2\gamma) - 4\alpha(\alpha-1)(1+\gamma)^2|}}$$

and

$$|a_3| \leq \frac{|\tau|}{1+2\gamma}\left(\frac{2\alpha}{3} + \frac{\alpha^2|\tau|^2}{(1+\gamma)^2}\right).$$

If we set $\tau = 1$ and $\gamma = 0$, corollary 3.1.1 reduces to the result in [81, Th. 1].

3.2 Coefficient bounds for the class $M_{\alpha,\lambda,\sigma}(\varphi)$

Motivated by the class $M(\alpha, \lambda, \rho)$ of analytic functions [29, Def. 1.1], we define a new subclass of bi-univalent functions, as follows:

Definition 3.2.1. *[86] A function f, given by* (1.1) *is said to be in the class* $M_{\alpha,\lambda,\sigma}(\varphi)$, $(\alpha \geq 0,\ \lambda \geq 0)$ *if satisfies the following conditions:*

$$\begin{aligned} & f \in \sigma \\ & \text{and} \left\{ \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} + \lambda \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] \right\} \prec \varphi(z) \\ & \text{and} \left\{ \frac{wg'(w)}{g(w)} \left(\frac{g(w)}{w} \right)^{\alpha} + \lambda \left[1 + \frac{wg''(w)}{g'(w)} - \frac{wg'(w)}{g(w)} + \alpha \left(\frac{wg'(w)}{g(w)} - 1 \right) \right] \right\} \prec \varphi(w), \end{aligned}$$

where g is the extension of f^{-1} to U.

Theorem 3.2.1. *[86] If $f \in M_{\alpha,\lambda,\sigma}(\varphi)$ is given by* (1.1) *then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|\frac{1}{2}B_1^2(\alpha+1)(\alpha+2\lambda+2) + (1+\alpha)^2(1+\lambda)^2(B_1 - B_2)\right|}} \tag{3.3}$$

and

$$|a_3| \leq \frac{2\left(B_1 + |B_2 - B_1|\right)}{(\alpha+1)(\alpha+2\lambda+2)}. \tag{3.4}$$

For $\alpha = 0$ and $\lambda = 0$ we obtain coefficient estimates for bi-starlike functions and for $\alpha = 0$ and $\lambda = 1$ we obtain the following coefficient estimates for bi-convex functions.

For $\alpha = 1$ and $\lambda = 0$ we obtain the following coefficient estimates for the class $\mathcal{H}_\sigma(\varphi)$, introduced by Ali et al. in [5]:

Corollary 3.2.1. *Let f be in the class $M_{1,0,\sigma}(\varphi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|3B_1^2 + 4B_1 - 4B_2|}} \quad \text{and}$$

$$|a_3| \leq \frac{1}{3}\left(B_1 + |B_2 - B_1|\right).$$

We observe that the estimate for coefficient a_3 in corollary 3.2.1 is improved (see [5, Th. 2.1]).

3.3 Coefficient bounds for the class $\mathcal{CK}_\sigma(\varphi)$

Definition 3.3.1. *[88] A function f, given by* (1.1), *is said to be in the class $\mathcal{CK}_\sigma(\varphi)$ if satisfies the following conditions:*

$$f \in \sigma \quad \text{and there exist a function } \phi \in \mathcal{K} \text{ such that } \quad \frac{f'(z)}{\phi'(z)} \prec \varphi(z) \quad \text{and}$$

$$\frac{g'(w)}{\phi'(w)} \prec \varphi(w),$$

where g is the extension of f^{-1} to U.

Theorem 3.3.1. *[88] If $f \in \mathcal{CK}_\sigma(\varphi)$ is given by* (1.1) *then*

$$|a_2| \leq \min\left\{\sqrt{\frac{B_1^2(B_1+1)+4\,|B_2-B_1|}{|3B_1^2-4(B_2-B_1)|}}\ ,\ \frac{B_1}{2}\right\} \tag{3.5}$$

and

$$|a_3| \leq B_1 + \frac{B_1^2}{4} + \frac{4}{3}\frac{|B_2-B_1|}{B_1}. \tag{3.6}$$

Setting $\varphi(z) = \left(\dfrac{1+z}{1-z}\right)^\alpha$, $0 < \alpha \leq 1$ we will obtain the following corollary:

Corollary 3.3.1. *If $f \in \mathcal{CK}_\sigma\left(\left(\dfrac{1+z}{1-z}\right)^\alpha\right)$ is given by* (1.1) *then*

$$|a_2| \leq \sqrt{\frac{2\alpha^2-\alpha+2}{\alpha+2}}$$

and

$$|a_3| \leq \alpha^2 + \frac{2}{3}\alpha + \frac{4}{3}.$$

3.4 Coefficient bounds for the class $\mathcal{CS^*}_\sigma(\varphi)$

Definition 3.4.1. *[88] A function f, given by* (1.1)*, is said to be in the class* $\mathcal{CS^*}_\sigma(\varphi)$ *if satisfies the following conditions:*

$$f \in \sigma \quad \text{and there exist a function } h \in \mathcal{S}^* \text{ such that } \frac{f(z)}{h(z)} \prec \varphi(z) \quad \text{and}$$

$$\frac{g(w)}{h(w)} \prec \varphi(w).$$

where g is the extension of f^{-1} to U.

Theorem 3.4.1. *[88] If $f \in \mathcal{CS^*}_\sigma(\varphi)$ is given by* (1.1) *then*

$$|a_2| \leq \min\left\{\sqrt{\frac{B_1^3 + B_1^2 + 4\,|B_2 - B_1|}{|B_1^2 - B_2 + B_1|}}\ ,\ B_1\right\} \tag{3.7}$$

and

$$|a_3| \leq 3B_1 + B_1^2 + \frac{8|B_2 - B_1|}{B_1}. \tag{3.8}$$

Setting $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $0 < \alpha \leq 1$ we will obtain the following corollary:

Corollary 3.4.1. *If $f \in \mathcal{CS^*}_\sigma\left(\left(\frac{1+z}{1-z}\right)^\alpha\right)$ is given by* (1.1) *then*

$$|a_2| \leq \sqrt{\frac{4\alpha^2 - 2\alpha + 4}{\alpha + 1}}$$

and

$$|a_3| \leq 4\alpha^2 - 2\alpha + 8.$$

Chapter 4

Classes of functions defined by operators

In this chapter we introduce and investigate new classes of analytic functions defined by means of several well-known operators which are presented below:

- **Sălăgean differential operator:**

  For a function $f \in \mathcal{A}$ Sălăgean (see [76]) introduced the operator D^n defined by

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{4.1}$$

$$\vdots$$

$$D^n f(z) = D(D^{n-1} f(z)),\ n = 1, 2, \cdots,\ z \in U.$$

- **Sălăgean integral operator:**

  For a function $f \in \mathcal{A}$ Sălăgean (see [76]) introduced the integral operator I^n defined by

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}\mathrm{d}t, \tag{4.2}$$

$$\vdots$$

$$I^n f(z) = I(I^{n-1} f(z)),\ n = 1, 2, \cdots,\ z \in U.$$

- **Ruscheweyh derivative:**

  For a function $f \in \mathcal{A}$ Ruscheweyh (see [73]) introduced the operator $R^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$R^\lambda f(z) = \frac{1}{(1-z)^{\lambda+1}} * f(z), \quad \lambda > -1, \; z \in U.$$

  In particular, for $\lambda = n$, we have

$$R^n(z) = \frac{z}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{z^{n-1}f(z)\right\}, \quad n \in \mathbb{N}, \; z \in U. \tag{4.3}$$

- **Carlson-Shaffer operator:**

  Let the function $\phi(a, c; z)$ be given by

$$\phi(a, c; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad (c \neq 0, -1, -2, ...; z \in U)$$

  where $(x)_k$ is the *Pochhammer symbol* defined by

$$(x)_k := \begin{cases} 1, & k = 0 \\ x(x+1)(x+2)...(x+k-1), & k \in \mathbb{N}^* \end{cases}$$

  Carlson and Shaffer [15] introduced a linear operator $L(a, c)$, corresponding to the function $\phi(a, c; z)$, defined by the following Hadamard product:

$$L(a, c) := \phi(a, c; z) * f(z) = z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+1} z^{k+1}. \tag{4.4}$$

- **Cho-Srivastava operator:**

  In [17], N.E. Cho and H.M. Srivastava introduced a linear operator of the form:

$$\mathcal{I}(m, l)f(z) = z + \sum_{k=2}^{\infty} \left(\frac{l+k}{l+1}\right)^m a_k z^k, \quad m \in \mathbb{Z}, \; l \geq 0. \tag{4.5}$$

- **Srivastava-Attiya operator:**

Srivastava and Attiya (see [82]) introduced the linear operator

$$J_{s,b} : \mathcal{A} \to \mathcal{A}$$

defined by

$$J_{s,b}(f)(z) := G_{s,b}(z) * f(z), \quad z \in U, \ b \in \mathbb{C} - \mathbb{Z}_0^-, \ s \in \mathbb{C}, \tag{4.6}$$

where

$$G_{s,b}(z) := (1+b)^s \left[\sum_{n=0}^{\infty} \frac{z^n}{(n+b)^s} - b^{-s} \right], \ z \in U. \tag{4.7}$$

4.1 A new class of analytic functions connected with a generalized operator

Let $\mathcal{L}(m,l,a,c,\lambda)$ be the operator defined by:

$$\mathcal{L}(m,l,a,c,\lambda)f(z) = \lambda \mathcal{I}(m,l)f(z) + (1-\lambda)L(a,c)f(z) \tag{4.8}$$

where $\mathcal{I}(m,l)f(z)$ is of the form (4.4) and $L(a,c)f(z)$ is of the form (4.5).
For $\lambda = 0$ we obtain Carlson-Shaffer operator introduced in [15], for $\lambda = 1$ we obtain linear operator in [17] and for $a = m+1, c = 1, l = 0$ we obtain generalized Sălăgean and Ruscheweyh operator studied by A. Alb Lupaş in [3].

For $c = 1$ and $a = n+1$ we have

$$\mathcal{L}(m,l,n,\alpha)f(z) = z + \sum_{k=2}^{\infty} \left[\alpha \left(\frac{l+k}{l+1} \right)^m + (1-\alpha) C_{n+k-1}^n \right] a_k z^k, \tag{4.9}$$

By means of operator $\mathcal{L}(m,l,a,c,\lambda)$ we introduce the following subclass of analytic functions:

Definition 4.1.1. *[85] We say that a function $f \in \mathcal{A}_n$ is in the class $\mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$, $n,m \in \mathbb{N}, l,\mu,\lambda \geq 0, \alpha \in [0,1)$ if*

$$\left| \frac{\mathcal{L}(m+1,l,a+1,c,\lambda)f(z)}{z} \left(\frac{z}{\mathcal{L}(m,l,a,c,\lambda)f(z)} \right)^{\mu} - 1 \right| < 1-\alpha, \ z \in U. \tag{4.10}$$

Remark 4.1.1. *[85] The class $\mathcal{BL}(m, l, a, c, \mu, \alpha, \lambda)$ includes various classes of analytic univalent functions, such as:*

- $\mathcal{BL}(0, 0, a, c, 1, \alpha, 1) \equiv \mathcal{S}^*(\alpha)$
- $\mathcal{BL}(1, 0, a, c, 1, \alpha, 1) \equiv \mathcal{K}(\alpha)$
- $\mathcal{BL}(0, 0, a, c, 0, \alpha, 1) \equiv \mathcal{R}(\alpha)$
- $\mathcal{BL}(0, 0, a, c, 2, \alpha, 1) \equiv \mathcal{B}(\alpha)$ *introduced by Frasin and Darus in [24]*
- $\mathcal{BL}(0, 0, a, c, \mu, \alpha, 1) \equiv \mathcal{B}(\mu, \alpha)$ *introduced by Frasin and Jahangiri in [23]*
- $\mathcal{BL}(m, 0, a, c, \mu, \alpha, 1) \equiv \mathcal{BS}(m, \mu, \alpha)$ *introduced by A. Alb Lupaş and A. Cătaş in [4]*
- $\mathcal{BL}(m, 0, m+1, 1, \mu, \alpha, 0) \equiv \mathcal{BR}(m, \mu, \alpha)$ *introduced by A. Alb Lupaş and A. Cătaş in [3]*
- $\mathcal{BL}(m, 0, m+1, 1, \mu, \alpha, \lambda) \equiv \mathcal{BL}(m, \mu, \alpha, \lambda)$ *introduced by A. Alb Lupaş and A. Cătaş in [2]*

In the first theorem we provide sufficient condition for functions to be in the class $\mathcal{BL}(m, l, a, c, \mu, \alpha, \lambda)$.

Theorem 4.1.1. *[85] Let $f \in \mathcal{A}_n$, $n, m \in \mathbb{N}, l, \mu, \lambda \geq 0, \alpha \in [1/2, 1)$. If*

$$\lambda \frac{(l+1)\mathcal{I}(m+2,l)f(z) - l\mathcal{I}(m+1,l)f(z)}{\mathcal{L}(m+1,l,a+1,c,\lambda)}$$
$$+(1-\lambda)\frac{(a+1)L(a+2,c)f(z) - aL(a+1,c)f(z)}{\mathcal{L}(m+1,l,a+1,c,\lambda)}$$
$$-\mu\lambda\frac{(l+1)\mathcal{I}(m+1,l)f(z) - l\mathcal{I}(m,l)f(z)}{\mathcal{L}(m,l,a,c,\lambda)}$$
$$+\mu(1-\lambda)\frac{aL(a+1,c)f(z) - (a-1)L(a,c)f(z)}{\mathcal{L}(m,l,a,c,\lambda)}$$
$$+\mu \prec 1 + \frac{3\alpha-1}{2\alpha}z, \quad z \in U, \tag{4.11}$$

then $f \in \mathcal{BL}(m, l, a, c, \mu, \alpha, \lambda)$.

If we take $a = l$ in Theorem 4.1.1, we obtain the following corollary:

Corollary 4.1.1. *[85] Let $f \in \mathcal{A}_n,\ n, m \in \mathbb{N}, l, \mu, \lambda \geq 0, \alpha \in [1/2, 1)$. If*

$$\frac{(l+1)\mathcal{L}(m+2, l, a+2, c, \lambda)}{\mathcal{L}(m+1, l, a+1, c, \lambda)} - \mu\frac{(l+1)\mathcal{L}(m+1, l, a+1, c, \lambda)}{\mathcal{L}(m, l, a, c, \lambda)}$$
$$-l + \mu(l+1) \prec 1 + \frac{3\alpha - 1}{2\alpha} z, \quad z \in U,$$

then $f \in \mathcal{BL}(m, l, a, c, \mu, \alpha, \lambda)$.

Next, we prove the following theorem:

Theorem 4.1.2. *[85] Let $f(z) \in \mathcal{A}$. If $f(z) \in \mathcal{BL}(m, l, l+1, c, \mu, \alpha, \lambda)$, then*

$$\left| \arg \frac{\mathcal{L}(m, l, l+1, c, \lambda)}{z} \right| < \frac{\pi}{2}\,\alpha,$$

for $0 < \alpha \leq 1$ and $2/\pi\ \tan^{-1}(\alpha/(l+1)) - \alpha(\mu - 1) = 1$.

If we get, in Theorem 4.1.2, $m = l = 0, \mu = 2$ and $\lambda = 1$, we obtain the following corollary, proved by B. A. Frasin and M. Darus in [24]:

Corollary 4.1.2. *[24] Let $f(z) \in \mathcal{A}$. If $f(z) \in \mathcal{B}(\alpha)$, then*

$$\left| \arg\left(\frac{f(z)}{z}\right) \right| < \frac{\pi}{2}\,\alpha,\ \ z \in U,$$

for some $\alpha (0 < \alpha < 1)$ and $(2/\pi)\ \tan^{-1}\alpha - \alpha = 1$.

4.2 A subclass of harmonic functions

Let

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad |b_1| < 1. \tag{4.12}$$

We note that the family $\mathcal{S}_{\mathcal{H}}$ reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero $(g \equiv 0)$. Silverman

[77] introduced the subclass of $\mathcal{S}_{\mathcal{H}}$, denoted by $\mathcal{S}_{\overline{\mathcal{H}}}$, which contains functions of the form $f = h + \overline{g}$ where

$$\begin{aligned} h(z) &= z - \sum_{k=2}^{\infty} |a_k| z^k \quad \text{and} \\ g(z) &= \sum_{k=1}^{\infty} |b_k| z^k, \;\; |b_1| < 1. \end{aligned} \tag{4.13}$$

For $f = h + \overline{g}$ given by (4.12), we define the modified operator $\mathcal{L}(m, l, n, \alpha)$ of harmonic univalent function f as

$$\mathcal{L}(m, l, n, \alpha) f(z) = \mathcal{L}(m, l, n, \alpha) h(z) + \overline{\mathcal{L}(m, l, n, \alpha) g(z)}, \tag{4.14}$$

where

$$\mathcal{L}(m, l, n, \alpha) h(z) = z + \sum_{k=2}^{\infty} \left[\alpha \left(\frac{l+k}{l+1} \right)^m + (1-\alpha) C_{n+k-1}^{n} \right] a_k z^k$$

and

$$\mathcal{L}(m, l, n, \alpha) g(z) = \sum_{k=1}^{\infty} \left[\alpha \left(\frac{l+k}{l+1} \right)^n + (1-\alpha) C_{n+k-1}^{n} \right] b_k z^k, |b_1| < 1.$$

We denote by $\mathcal{HL}(m, l, n, \alpha, \gamma)$ the class of harmonic functions f of the form (4.12), such that

$$\Re \left[\frac{z \left(\mathcal{L}(m, l, n, \alpha) f(z) \right)'}{\mathcal{L}(m, l, n, \alpha) f(z)} \right] \geq \gamma, \quad 0 \leq \gamma < 1.$$

For $n = l$, we obtain the class $\mathcal{HL}(m, l, \alpha, \gamma)$

$$\Re \left[\frac{(l+1) \mathcal{L}(m+1, l, l+1, \alpha) f(z)}{\mathcal{L}(m, l, l, \alpha) f(z)} - l \right] \geq \gamma, \tag{4.15}$$

where $\mathcal{L}(m, l, n, \alpha)$ is defined by (4.14).

Also, we denote by $\overline{\mathcal{HL}}(m, l, \alpha, \gamma)$ the class of functions $f = h + \overline{g}$ in $\mathcal{HL}(m, l, \alpha, \gamma)$, where h and g are of the form (4.13).

First we determine a sufficient coefficient bound for functions in $\mathcal{HL}(m, l, \alpha, \gamma)$.

Theorem 4.2.1. *[89] Let $f = h + \overline{g}$ be given by (4.12). If*

$$\sum_{k=2}^{\infty}(k-\gamma)\left[\alpha\left(\frac{l+k}{l+1}\right)^m + (1-\alpha)C_{l+k-1}^l\right](|a_k|+|b_k|) + |b_1| \leq 1-\gamma, \tag{4.16}$$

where $l, m \geq 0$, $a_1 = 1$, $\alpha, \gamma \in [0,1)$, then $f(z)$ is harmonic univalent, sense preserving in U and $f(z) \in \mathcal{HL}(m, l, \alpha, \gamma)$.

If we take $m, l, \gamma = 0$ and $\alpha = 1$ in the previous theorem, we obtain the following theorem, proved by Jahangiry and Silverman in [37].

Corollary 4.2.1. *Let $f = h + \overline{g}$ given by (4.12). If*

$$\sum_{k=2}^{\infty} k(|a_k| + |b_k|) \leq 1 - |b_1|,$$

then f is sense-preserving, harmonic univalent in U and $f \in \mathcal{S}_{\mathcal{H}}^*$ (the functions in $\mathcal{S}_{\mathcal{H}}$ which are starlike in U).

The harmonic function

$$f(z) = z + \sum_{k=2}^{\infty}\frac{1-\gamma}{(k-\gamma)A_k}x_k z^k + \sum_{k=1}^{\infty}\frac{2(1-\gamma)}{[(1-\gamma+k)+|1+\gamma-k|]A_k}\overline{y_k z^k}, \tag{4.17}$$

where

$$A_k = \left[\alpha\left(\frac{l+k}{l+1}\right)^m + (1-\alpha)C_{l+k-1}^l\right].$$

$$\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$$

shows that the coefficient bound given by (4.16) is sharp.

The functions of the form (4.17) are in $\mathcal{HL}(n, l, \alpha, \gamma)$ because

$$\sum_{k=2}^{\infty}(k-\gamma)A_k(|a_k|+|b_k|)+|b_1|$$

$$=\sum_{k=2}^{\infty}(k-\gamma)A_k|a_k|+\sum_{k=1}^{\infty}\frac{(1-\gamma+k)+|1+\gamma-k|}{2}A_k|b_k|$$

$$=(1-\gamma)\left(\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\right)=1-\gamma.$$

In the next theorem we will prove the necessity of condition (4.16) for functions of the form $f = h + \overline{g}$, where h and g are of the form (4.13).

Theorem 4.2.2. *[89] Let $f = h + \overline{g}$ be given by (4.13). Then $f \in \overline{\mathcal{HL}}(m, l, \alpha, \gamma)$ if and only if*

$$\sum_{k=2}^{\infty}(k-\gamma)\left[\alpha\left(\frac{l+k}{l+1}\right)^m+(1-\alpha)C_{l+k-1}^{l}\right](|a_k|+|b_k|)+|b_1|\leq 1-\gamma. \tag{4.18}$$

Theorem 4.2.3. *[89] Let f be given by 4.13. Then $f \in \overline{\mathcal{HL}}(m, l, \alpha, \gamma)$ if and only if*

$$f(z)=\sum_{k=1}^{\infty}\left(X_k h_k(z)+Y_k g_k(z)\right), \tag{4.19}$$

where

$$h_1(z)=z,\ \ h_k(z)=z-\frac{1-\gamma}{(k-\gamma)A_k}z^k,\ \ k\geq 2,$$

$$g_k(z)=z+\frac{2(1-\gamma)}{((1-\gamma+k)+|1+\gamma-k|)A_k}\overline{z}^k,\ \ k\geq 1,$$

$$\sum_{k=1}^{\infty}(X_k+Y_k)=1,\ \ X_k\geq 0,\ \ Y_k\geq 0.$$

The following theorem gives the distortion bounds for functions in the class

$\overline{\mathcal{HL}}(m, l, \alpha, \gamma)$.

Theorem 4.2.4. *[89] Let $f \in \overline{\mathcal{HL}}(m, l, \alpha, \gamma)$. Then, for $|z| = r < 1$, we have*

$$|f(z)| \leq (1 + |b_1|)r + \frac{1}{\alpha\left(\frac{l+2}{l+1}\right)^m + (1-\alpha)(l+1)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1}{2-\gamma}|b_1|\right) r^2$$

and

$$|f(z)| \geq (1 - |b_1|)r - \frac{1}{\alpha\left(\frac{l+2}{l+1}\right)^m + (1-\alpha)(l+1)}\left(\frac{1-\gamma}{2-\gamma} - \frac{1}{2-\gamma}|b_1|\right) r^2.$$

Theorem 4.2.5. *[89] Let $f(z) \in \overline{\mathcal{HL}}(m, l, \alpha, \gamma)$ and $F(z) \in \overline{\mathcal{HL}}(m, l, \alpha, \delta)$, for $0 \leq \delta \leq \gamma < 1$. Then $f(z) * F(z) \in \overline{\mathcal{HL}}(m, l, \alpha, \gamma) \subset \overline{\mathcal{HL}}(m, l, \alpha, \delta)$.*

4.3 A new class of analytic functions connected with Sălăgean integral operator

In this section we provide an investigation of a new class of analytic functions, $\mathcal{L}^n_{\alpha,\beta}$. In particular, we derive an inclusion property, a subordination result, extreme points and coefficient bounds for this function class.

Definition 4.3.1. *[90] For $\alpha \in (-\pi, \pi]$, $\beta \in (0, 1]$ and $n \in \mathbb{N}$, let*

$$\mathcal{L}^n_{\alpha,\beta} := \left\{ f \in \mathcal{A} : \left| \arg\left((I^n f(z))' + \frac{1+e^{i\alpha}}{2} z (I^n f(z))'' \right) \right| < \beta\frac{\pi}{2},\ z \in U \right\} \tag{4.20}$$

We observe that for $\beta = 1$ and $n = 0$ $\mathcal{L}^0_{\alpha,1} := \mathcal{L}_\alpha$.

In our first theorem we obtain an inclusion result for the function class $\mathcal{L}^n_{\alpha,\beta}$:

Theorem 4.3.1. *[90] Assume that there exists a function $\omega(z)$ such that*

$$(I^{n+1} f(z))' + \frac{1+e^{i\alpha}}{2} z (I^{n+1} f(z))'' = \left(\frac{1+\omega(z)}{1-\omega(z)}\right)^\beta, \tag{4.21}$$

where $\omega(0) = 0$. *Then*

$$\mathcal{L}^n_{\alpha,\beta} \subset \mathcal{L}^{n+1}_{\alpha,\beta}, \text{ for each } n \in \mathbb{N}, \alpha \in (-\pi, \pi], \text{ and } \beta \in (0, 1].$$

Theorem 4.3.2. *[90] Let* $\alpha \in (-\pi, \pi)$, $\beta \in (0, 1]$ *and* $n \in \mathbb{N}$. *If* $f \in \mathcal{L}^n_{\alpha,\beta}$ *then*

$$(I^n f(z))' \prec q(z) = \frac{c}{z^c} \int_0^z t^{c-1} \left(\frac{1+t}{1-t}\right)^\beta \mathrm{d}t, \quad z \in U,$$

where $c = \dfrac{2}{1+e^{i\alpha}}$.

The function q *is the best dominant.*

In the next theorem we will find the extreme points of the class $\mathcal{L}^n_{\alpha,\beta}$.

Theorem 4.3.3. *[90] Let* $\alpha \in (-\pi, \pi)$, $\beta \in (0, 1]$ *and* $n \in \mathbb{N}$. *The extreme points of* $\mathcal{L}^n_{\alpha,\beta}$ *are*

$$f_x(z) = z + 2\sum_{k=2}^{\infty} \frac{\lambda_{k-1} k^{n-1}}{e^{i\alpha}(k-1) + (k+1)} x^{k-1} z^k, \tag{4.22}$$

where $|x| = 1$, $z \in U$,

$$\lambda_k = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \binom{\beta}{j} \binom{-\beta}{k-j} (-1)^{k-j} & 0 < \beta < 1, \\ (-2)^{k-1} & \beta = 1. \end{cases} \tag{4.23}$$

and

$$\binom{\beta}{j} = \begin{cases} \dfrac{\beta(\beta-1)\ldots(\beta-j+1)}{j!} & j = 1, \ldots, k, \\ 1 & j = 0. \end{cases}$$

Finally, we present upper bounds on coefficients in $\mathcal{L}^n_{\alpha,\beta}$. The result is sharp in the case $k = 2$.

Theorem 4.3.4. *[90] Let* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{L}^n_{\alpha,\beta}$, *where* $\alpha \in (-\pi, \pi]$, $\beta \in (0, 1]$ *and* $n \in \mathbb{N}$. *Then*

$$|a_k| \le \frac{2\sqrt{2}\beta k^{n-1}}{\sqrt{k^2 + 1 + (k^2-1)\cos\alpha}}, \quad k = 2, 3, \ldots. \tag{4.24}$$

In the case $k = 2$ the result is sharp, the equality holds for the function f_x given in (4.22)

4.4 A subclass of analytic functions involving λ-spirallikeness of order α

In this section, using Sălăgean integral operator, we introduce two subclasses of analytic functions involving λ-spirallikeness of order α and study some inclusion properties for these classes.

First, we consider the integral operator L_c, introduced by Bernardi in [9].

For $f(z) \in \mathcal{A}$ and $c \in \mathbb{N}$, define the integral operator $L_c f(z)$ of the form

$$L_c f(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} \mathrm{d}t. \tag{4.25}$$

Considering the integral operator I^n defined in [76], of the form (4.2), we introduce the following subclasses of analytic functions:

$$S_n^\lambda(\alpha) = \{f \in \mathcal{A} \,|\, I^n f(z) \in S^\lambda(\alpha)\},$$

$$F_n^\lambda(\alpha) = \{f \in \mathcal{A} \,|\, I^n f(z) \in F^\lambda(\alpha)\}.$$

It is easy to see that $f(z) \in F_n^\lambda(\alpha)$ if and only if $zf'(z) \in S_n^\lambda(\alpha)$. Also, $S_0^\lambda(\alpha) = S^\lambda(\alpha)$ and $F_0^\lambda(\alpha) = F^\lambda(\alpha)$.

Theorem 4.4.1. *[19] For all $n \in \mathbb{N}$,*

$$S_n^\lambda(\alpha) \subset S_{n+1}^\lambda(\alpha),$$

where $\lambda \in (-\pi/2, \pi/2)$ and $\alpha \in [0, 1)$.

Theorem 4.4.2. *[19] For all $n \in \mathbb{N}$,*

$$F_n^\lambda(\alpha) \subset F_{n+1}^\lambda(\alpha),$$

where $\lambda \in (-\pi/2, \pi/2)$ and $\alpha \in [0, 1)$.

Theorem 4.4.3. *[19] Let $c \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$, $\alpha \in [0,1)$. If $f(z) \in S_n^\lambda(\alpha)$ then $L_c f(z) \in S_n^\lambda(\alpha)$, for all $z \in U$.*

Theorem 4.4.4. *[19] Let $c \in \mathbb{N}$, $\lambda \in (-\pi/2, \pi/2)$, $\alpha \in [0,1)$. If $f(z) \in F_n^\lambda(\alpha)$ then $L_c f(z) \in F_n^\lambda(\alpha)$, for all $z \in U$.*

4.5 A new class of generalized close-to-starlike functions

In this section we derive some properties for the class $\mathcal{CS}^*_{s,b}(\alpha)$ such as inclusion results, coefficient inequalities and integral representation.

Definition 4.5.1. *[91] A function $f \in \mathcal{A}$ is in the class $\mathcal{CS}^*_{s,b}(\alpha)$, $\alpha \in [0,1]$ if the function $F(f)(z) = (1-\alpha)J_{s,b}(f)(z) + \alpha z J'_{s,b}(f)(z)$ is close-to-starlike, that is* $\displaystyle\int_{\theta_1}^{\theta_2} \Re\left(\frac{zF'(f)(z)}{F(f)(z)}\right) \mathrm{d}\theta > -\pi,\ z = re^{i\theta},\ 0 < r < 1.$

The above condition is equivalent to

$$\int_{\theta_1}^{\theta_2} \Re\left(\frac{zJ'_{s,b}(f)(z) + \alpha z^2 J''_{s,b}(f)(z)}{(1-\alpha)J_{s,b}(f)(z) + \alpha z J'_{s,b}(f)(z)}\right) \mathrm{d}\theta > -\pi,\ z = re^{i\theta},\ 0 < r < 1. \quad (4.26)$$

In view of Definition 1.5.1, we can give an equivalent form of Definition 4.5.1:

Definition 4.5.2. *[91] A function f is in the class $\mathcal{CS}^*_{s,b}(\alpha)$ if and only if there exist a function $g \in \mathcal{S}^*$ such that*

$$\Re\left[\frac{F(f)(z)}{g(z)}\right] > 0,\ z \in U. \quad (4.27)$$

First, we give an inclusion result for the class $\mathcal{CS}^*_{s,b}(\alpha)$.

Theorem 4.5.1. *[91] For $\alpha \in (0,1]$, $\mathcal{CS}^*_{s,b}(\alpha) \subset \mathcal{CS}^*_{s,b}$.*

Theorem 4.5.2. *[91] If the function f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the class $\mathcal{CS}^*_{s,b}(\alpha)$, then*

$$|a_n| \le \frac{n^2}{1-\alpha+\alpha n}\left|\left(\frac{n+b}{1+b}\right)^s\right|, \quad n \in \mathbb{N} \setminus \{1\}.$$

The result is sharp.

Next, we will show that the class $\mathcal{CS}^*_{s,b}(\alpha)$ is closed under the convolution operation by convex functions.

Theorem 4.5.3. *[91] Let $\phi \in \mathcal{K}$ and $f \in \mathcal{CS}^*_{s,b}(\alpha)$. Then $\phi * f \in \mathcal{CS}^*_{s,b}(\alpha)$.*

Finally, we give an integral representation for functions in the class $\mathcal{CS}^*_{s,b}(\alpha)$

Theorem 4.5.4. *[91] If the function f is in the class $\mathcal{CS}^*_{s,b}(\alpha)$, then*

$$f(z) = h_{s,b}(z) * \frac{1}{\alpha z^{1/\alpha - 1}} \int_0^z t^{1/\alpha - 2} g(t) p(t) \mathrm{d}t,$$

where $g \in \mathcal{S}^*$, $p \in \mathcal{P}$ and

$$h_{s,b}(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+b}{1+b} \right)^s z^n.$$

Chapter 5

Univalence criteria

5.1 Löwner chains. Basic definitions and notations

Definition 5.1.1. *A function $L(z,t) : U \times [0,\infty) \to \mathbb{C}$ is said to be a Löwner chain if it satisfies the following conditions:*

i) $L(z,t)$ is analytic and univalent in U for all $t \in [0,\infty)$,

ii) $L(z,s) \prec L(z,t)$ for all $0 \leq s \leq t < \infty$,

where the symbol $' \prec'$ stands for subordination.

5.2 Univalence criteria connected with Sălăgean operator

Theorem 5.2.1. *[92] Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If the inequalities*

$$\left| \frac{2}{p(z)+1} \cdot \frac{zf'(z)}{D^{n+1}f(z)} - 1 \right| \leq 1 \tag{5.1}$$

and

$$\left| \left(\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{D^{n+1}f(z)} - 1 \right) |z|^2 + (1-|z|^2) \left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 + \frac{zp'(z)}{p(z)+1} \right) \right| \leq 1 \tag{5.2}$$

holds true for $z \in U$, then the function f is univalent in U.

By setting $n = 0$ in Theorem 5.2.1 we obtain the corollary due to Lewandowski [49].

For $p \equiv 1$ the following criterion reduces to a well-known criterion found by Becker [8] and Duren et al. [22].

For $n = 1$, Theorem 5.2.1 yields

Corollary 5.2.1. *[92] Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If the inequalities*

$$\left| \frac{2}{p(z)+1} \cdot \frac{f'(z)}{f'(z)+zf''(z)} - 1 \right| \leq 1$$

and

$$\left| \left(\frac{2}{p(z)+1} \cdot \frac{f'(z)}{f'(z)+zf''(z)} - 1 \right) |z|^2 \right.$$

$$\left. + (1-|z|^2) \left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)} + \frac{zp'(z)}{p(z)+1} \right) \right| \leq 1$$

holds true for $z \in U$ then $f \in \mathcal{S}$.

For the Löwner chain

$$L(z,t) := f(e^{-t}z) + (e^t z - e^{-t}z) \cdot \frac{p(e^{-t}z)+1}{2} \cdot \frac{D^{n+1}f(e^{-t}z)}{D^n f(e^{-t}z)}, \quad z \in U, t \in [0,\infty),$$

following the same steps as in the proof of Theorem 5.2.1, we obtain:

Theorem 5.2.2. *[92] Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If the inequalities*

$$\left| \frac{2}{p(z)+1} \cdot f'(z) \frac{D^n f(z)}{D^{n+1} f(z)} - 1 \right| \leq 1$$

and

$$\left| \left(\frac{2}{p(z)+1} \cdot f'(z) \frac{D^n f(z)}{D^{n+1} f(z)} - 1 \right) |z|^2 \right.$$

$$\left. +(1-|z|^2) \left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^n f(z)} + \frac{zp'(z)}{p(z)+1} \right) \right| \leq 1$$

holds true for $z \in U$, then the function f is univalent in U.

Setting $n = 0$ in previous theorem we obtain the following result:

Corollary 5.2.2. *Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If the inequalities*

$$\left|\frac{2}{p(z)+1}\cdot\frac{f(z)}{z}-1\right| \leq 1$$

and

$$\left|\left(\frac{2}{p(z)+1}\cdot\frac{f(z)}{z}-1\right)|z|^2\right.$$

$$\left.+(1-|z|^2)\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\frac{zp'(z)}{p(z)+1}\right)\right| \leq 1$$

holds true for $z \in U$, then the function f is univalent in U.

For $p \equiv 1$ in previous corollary we obtain Corollary 3.5 due to Kanas and Lecko [42].

Setting $p(z) = \dfrac{f(z)}{z}$ we obtain:

Corollary 5.2.3. *[92] Let $f \in \mathcal{A}$ with $\Re\dfrac{f(z)}{z} > 0$. If the inequality*

$$\left|\left(\frac{f(z)}{z}-1\right)|z|^2+(1-|z|^2)\left[1+\frac{zf''(z)}{f'(z)}\left(\frac{f(z)}{z}+1\right)-\frac{zf'(z)}{f(z)}\right]\right| \leq \left|\frac{f(z)}{z}+1\right|$$

holds true for $z \in U$, then the function f is univalent in U.

Now, setting $p(z) = \dfrac{zf'(z)}{f(z)}$ in Corollary 5.2.2, we obtain the following result:

Corollary 5.2.4. *Let $f \in \mathcal{A}$. If the inequalities*

$$\left|2\frac{f(z)}{z}-\frac{zf'(z)}{f(z)}-1\right| \leq \left|1+\frac{zf'(z)}{f(z)}\right|$$

and

$$\left|\left(2\frac{f(z)}{z}-\frac{zf'(z)}{f(z)}-1\right)|z|^2\right.$$

$$\left.+(1-|z|^2)\left(2\frac{zf'(z)}{f(z)+1}\right)\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| \leq \left|1+\frac{zf'(z)}{f(z)}\right|$$

holds true for $z \in U$, then the function f is univalent in U.

5.3 Univalence criteria connected with Ruscheweyh operator

Theorem 5.3.1. *[92] Let $f \in \mathcal{A}$ and let p be an analytic function with $p(0) = 1$. If the inequalities*

$$\left|\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{(n+1)R^{n+1}f(z) - nR^n f(z)} - 1\right| \leq 1 \tag{5.3}$$

and

$$\left|\left(\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{(n+1)R^{n+1}f(z) - nR^n f(z)} - 1\right)|z|^2\right.$$

$$\left.+(1-|z|^2)\left[(n+1)\left(\frac{(n+2)R^{n+2}f(z) - (n+1)R^{n+1}f(z)}{(n+1)R^{n+1}f(z) - nR^n f(z)} - 1\right) + \frac{zp'(z)}{p(z)+1}\right]\right| \leq 1 \tag{5.4}$$

holds true for $z \in U$, then the function f is univalent in U.

For $n = 0$ in Theorem 5.3.1 we obtain the result due to Lewandowski [49] and for $n = 0$ and $p = 1$ we obtain the result due to Becker [8].

By setting $n = 1$ in Theorem 5.3.1, we have

Corollary 5.3.1. *[92] Let $f \in \mathcal{A}$ and let p be an analytic function with $p(0) = 1$. If the inequalities*

$$\left|\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{zf'(z) + z^2 f''(z)} - 1\right| \leq 1$$

and

$$\left|\left(\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{zf'(z) + z^2 f''(z)} - 1\right)|z|^2\right.$$

$$\left.+(1-|z|^2)\left[\frac{zp'(z)}{p(z)+1} + \frac{2zf'(z) + 4z^2 f''(z) + z^3 f'''(z)}{zf'(z) + z^2 f''(z)} - 2\right]\right| \leq 1$$

holds true for $z \in U$, then the function f is univalent in U.

Theorem 5.3.2. *[92] Let $f \in \mathcal{A}$ and let p be an analytic function with $p(0) = 1$. If the inequalities*

$$\left|\frac{2}{p(z)+1} \cdot f'(z)\frac{R^n f(z)}{R^{n+1}f(z)} - 1\right| \leq 1$$

and

$$\left|\left(\frac{2}{p(z)+1}\cdot f'(z)\frac{R^n f(z)}{R^{n+1}f(z)}-1\right)|z|^2\right.$$

$$\left.+(1-|z|^2)\left((n+2)\frac{R^{n+2}f(z)}{R^{n+1}f(z)}-(n+1)\frac{R^{n+1}f(z)}{R^n f(z)}-1+\frac{zp'(z)}{p(z)+1}\right)\right|\leq 1$$

holds true for $z \in U$, *then the function* f *is univalent in* U.

For $n = 0$ we obtain the result in Corollary 5.2.2.

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