Contributions to the study of the coincidence point problem for singlevalued and multivalued operators

Ph.D. Thesis Summary

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Contents

Introduction

The fixed point theory for singlevalued and multivalued operators is a domain of the nonlinear analysis with a dynamic development in the last decades, proofs being a lot of monographs, proceedings and scientific articles appeared in these years.

In the existing literature on the fixed point theory, metric type conditions on the operators play a vital role in proving the existence and uniqueness of a fixed point. The Banach contraction theorem is a basic tool in functional analysis, nonlinear analysis and differential equations.

Following the Banach contraction principle, in 1969 Nadler introduced the concept of multivalued contraction and established that a multivalued contraction possesses a fixed point in a complete metric space (see S. B. Nadler [83]). Subsequently many authors generalized Nadler's fixed point theorem in different way. Various fixed point results for singlevalued contraction have been extended to multivalued operators, see for instance Y. Feng and S. Liu [44], W. A. Kirk and B. Sims [65], D. Klim and D. Wardowski [66], I. A. Rus [102], and references cited there in.

A generalization of Brouwer's fixed point theorem, from 1912, was obtained by Schauder in 1930. The situation is completely different when certain generalizations are considered, in particular those concerning ϕ-contractive or condensing operators. The degree of noncompactness of a set is measured utilizing functions μ called measures of noncompactness. The first such measure was defined in 1930 by K. Kuratowski [67]. Later, other measures were defined by several authors, for instance I. Gohberg, L. S. Gol'denshtein and A. S. Markus [54] and V. I. Istrăţescu [61].

On the other hand, the stability problem of functional equations originated from a question of Stanislaw Ulam [124], posed in 1940, concerning the stability of group homomorphisms. In the next year, Donald H. Hyers [56] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces, that was the first significant breakthrough and a step toward more solutions in this area. Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem. Therefore this type of stability is called the Ulam-Hyers stability. Concerning Ulam stability there are some results for differential equations, integral equations, see T. P. Petru, A. Petruşel and J.-C. Yao [94], I. A. Rus [110], I. A. Rus [111]. For other results in the case of fixed point problems and coincidence point problems, see M. Bota and A. Petruşel [21], V. L. Lazăr [68], T. P. Petru, A. Petruşel and J.-C. Yao [94], I. A. Rus [112], I. A. Rus [108], I.A. Rus, A.

Petruşel and G. Petruşel [114]. For fixed point theory in metric spaces see Q. H. Ansari [4], M. A. Khamsi and W. A. Kirk [64], W. A. Kirk and B. Sims [65], I. A. Rus [102].

From a mathematical point of view, many problems arising from diverse areas of natural science involve the existence of solutions of nonlinear equations with the form

$$t(u) = s(u), \quad u \in M, \tag{1}$$

where M is a nonempty subset of a Banach space X, and $s, t : M \to Y$ are nonlinear operators taking values on another Banach space Y. The problem of finding a solution for Equation (1) is known as a *coincidence problem.* Coincidence theory is a very powerful technique especially in existence of solutions problems in nonlinear equations. For instance, in R.F. Brown [25], A. Buică [29], T. Chen, W. Liu and Z. Hu [32], K. Goebel [52], Y. Mao and J. Lee [73] several of such results are applied to solve boundary value problems.

The coincidence problem can be considered as a generalization of the fixed point problem since if $t : M \subseteq X \to X$ is an operator, to study the existence of a fixed point for t is the same that to find a solution of the coincidence problem where s is the identity operator on M. In this sense, R. Machuca [72] proved a coincidence theorem which is a generalization of the well known Banach contraction principle. Generalizations of this result can be found, for instance in J. Garcia-Falset and O. Mleşniţe [49], K. Goebel [52], O. Mleşniţe [75]. On the other hand, R.E. Gaines and J.L. Mawhin [45] introduced coincidence degree theory in 1970s in analyzing functional and differential equations. The main goal in the coincidence degree theory is to search the existence of solutions of Equation (1) in some bounded and open set M in some Banach space X for t being a linear operator and s nonlinear operator using Leray-Schauder degree theory (see A. Sirma and S. Sevgin [121] to find a sharpening results).

The following problem

$$S(x) \cap T(x) \neq \emptyset, \quad x \in X \tag{2}$$

where X is a metric space and $S, T : X \to P(Y)$ are two multivalued operators is called a *multivalued coincidence problem.* For existence and Ulam-Hyers stability the solution of this type of problems see V. Berinde [16], M. Bota and A. Petruşel [21], A. Buică [29], O. Mleşniţe and A. Petruşel [76], A. Petruşel, C. Urs and O. Mleşniţe [93], T. P. Petru, A. Petruşel and J.-C. Yao [94], I. A. Rus [102], [108].

This thesis is divided into four chapters, each chapter containing several sections.

Chapter 1: Preliminaries.

The aim of this chapter is to recall some notions and basic results which are necessary in the presentation of the following chapters of this Ph.D. thesis. In writing this chapter, we used the following bibliographical sources: J.-P. Aubin and H. Frankowska [12], J. Dugundji and A. Granas [55], S. Hu and N. S. Papageorgiou [57], W.A. Kirk and B. Sims [65], A. Petruşel [90], A. Petruşel [91], I.A. Rus [107], I. A. Rus [109], I.A. Rus, A. Petruşel and G. Petruşel [114]. This chapter contains the following sections:

§1 *Metric spaces. Generalized metric spaces.* In this section we recall the concept of generalized metric space in the sense of Perov with some of their properties.

§2 *Singlevalued weakly Picard operators.* In this section we present a survey of known results from singlevalued Picard operators theory. The concept of Picard operators and weakly Picard operators were introduced by I. A. Rus in [102]. The theory of weakly Picard operators is very useful to study some properties of the solutions of those equations for which the method of successive approximation works. In terms of a weakly Picard operators, some classical results take a very simple form.

§3 *Multivalued weakly Picard operators.* In this section we describe some basic concepts and results for multivalued Picard operators. Some notions of continuity for multivalued operators are also discussed. The first ideas of continuity for multivalued operators appear already in 1926-1927 in the works of mathematicians like W. A. Wilson, L. S. Hill and W. Hurewicz. The notions about the continuity of multivalued operators can be found in books and papers on multivalued analysis such as J.-P. Aubin and A. Cellina [11], J.-P. Aubin and H. Frankowska [12], S. Hu and N. S. Papageorgiou [57], W. A. Kirk and B. Sims [65], A. Petruşel [91].

Chapter 2: Coincidence theorems for singlevalued operators.

It is well know that a coincidence problem is, under appropriate conditions, equivalent to a fixed point problem for a singlevalued operators. Using this approach, we present, in this chapter, some existence, uniqueness and Ulam-Hyers stability theorems for coincidence problem mentioned above. We present also some extensions of our results in generalized metric spaces. Some examples illustrating the main results of the paper are also given. This chapter contains the following sections:

§1 *Covering operators and Ulam-Hyers stability results for coincidence problems.* In this section we present some existence and Ulam-Hyers stability results for coincidence point problems with singlevalued operators. The basic hypothesis in these results is the property of covering operators. Our contributions in this section are: Theorem 2.1.1 which is an existence and Ulam-Hyers stability result for two singlevalued covering operators; Theorem 2.1.2 which is an coincidence and Ulam-Hyers stability result for two singlevalued covering operators with respect two sets. The scientific paper which contain the original results of this section is: O. Mleşniţe [78].

§2 *Existence and Ulam-Hyers stability results for coincidence problems.* In this section we present some existence and Ulam-Hyers stability results for coincidence problems with singlevalued operators. Our contributions in this section are: Lemma 2.2.1 which shows that a coincidence problem is, under appropriate conditions, equivalent to a fixed point problem; Theorem 2.2.1 is a generalization of Banach's contraction principle; Theorems 2.2.3 and 2.2.4 are some data dependence results for the Ulam-Hyers stability of coincidence problems of two pair of singlevalued operators; Theorem 2.2.5 is an Ulam-Hyers stability result for a coincidence problem with respect two strongly equivalent metrics. The scientific paper which contain the original results of this section are: O. Mlesņiţe [74], [75].

§3 *Coincidence problems for generalized contractions.* In this section we present new existence, uniqueness and Ulam-Hyers stability theorems for coincidence problems using generalized contractions and we generalize Goebel's coincidence theorem given, see K. Goebel [52]. Our contributions in this section are: Theorem 2.3.1 is a result on Ulam-Hyers stability for the case of Goebel's coincidence theorem; Theorem 2.3.2 is a result which extends Goebel's theorem by considering the condition of φ-contraction of an operator with respect to an another operator; Theorems 2.3.3 and 2.3.4 are some generalizations of Theorem 2.2.1 respectively Theorem 2.2.2 using generalized contraction; Theorem 2.3.5 is a generalization of Theorem 2.3.3; Corollary 2.3.2; Theorem 2.3.6 is an existence, uniqueness and generalized Ulam-Hyers stability result using separate contractions. The scientific paper which contain the original results of this section are: O. Mlesŋiţe [74], [77], J. Garcia-Falset and O. Mleşniţe [49].

§4 *Coincidence results by fixed point theorems in generalized metric spaces.* In this section we present some existence, uniqueness and Ulam-Hyers stability results for fixed point and coincidence point problems with singlevalued operators in generalized metric spaces. Our contributions in this section are: Theorem 2.4.1 is an extension of Perov's Theorem and a generalization to vector-valued metric spaces of the main theorem from M. Berinde and V. Berinde [17]; Theorem 2.4.2 is an existence and uniqueness result for coincidence problem with singlevalued operators in generalized metric spaces; Theorem 2.4.3 is an approximation and an error estimate for the solution of the coincidence problem. The scientific paper which contain the original results of this section is: O. Mleşniţe [75].

§5 *A Leray-Schauder condition to the coincidence problems.* In this section we obtain several versions, without invoking degree theory, of the coincidence problems, where the singlevalued operators can both become nonlinear. Our contributions in this section are: Theorem 2.5.4 is an extension of Theorem 2.5.2; Theorem 2.5.5 is a sharpening of Theorem 2.5.3 (see W. V. Petryshyn [95]); Corollary 2.5.2. The scientific paper which contain the original results of this section is: J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50].

Chapter 3: Coincidence theorems for multivalued operators.

The purpose of this chapter is to present some existence and Ulam-Hyers stability results for fixed point and coincidence point problems. This approach is based on the weakly Picard operators technique in the setting of generalized metric spaces in the sense of Perov, i.e., spaces endowed with vector metrics $d : X \times X \to \mathbb{R}^m_+$. Using the cartesian product technique for two multivalued operators, our result improve some recent theorems in the literature, see M. Bota and A. Petruşel [21], T. P. Petru, A. Petruşel and J.-C. Yao [94], I. A. Rus [108]. This chapter contains the following sections:

§1 *Metric regularity and Ulam-Hyers stability results for coincidence problems.* Open covering and metric regularity are properties playing an important role in several topics of modern variational analysis. In this paper, we will present some existence and Ulam-Hyers stability results for coincidence point problems with multivalued operators. The basic hypothesis in these results is the property of metric regularity. Our contributions in this section are: Lemma 3.1.1 which shows that a coincidence problem is, under appropriate conditions, equivalent to a fixed point problem with multivalued operators; Theorems 3.1.1 and 3.1.2 are some generalizations of the main theorems from A. V. Blaga [19]. The scientific paper which contain the original results of this section is: O. Mleşniţe [79].

§2 *Existence and Ulam-Hyers stability results for coincidence problems.* In this section we present some existence and Ulam-Hyers stability results for coincidence problems with multivalued operators using the weakly Picard operators technique. Our contributions in this section are: Theorem 3.2.1 is a generalization of Covitz-Nadler's fixed point theorem; Theorem 3.2.2 is a data dependence result for the Ulam-Hyers stability of the multivalued coincidence problems. The scientific paper which contain the original results of this section is: O. Mleşniţe and A. Petruşel [76].

§3 *Coincidence results by fixed point theorems in generalized metric spaces.* In this section we present some existence and Ulam-Hyers stability results for coincidence point problems with multivalued operators using the weakly Picard operator technique in generalized metric spaces. Our contributions in this section are: Theorem 3.3.1 is a generalization of Perov's fixed point theorem. The scientific papers which contain the original results of this section are: O. Mleşniţe and A. Petruşel [76], A. Petruşel, C. Urs and O. Mleşniţe [93].

§4 *A Leray-Schauder condition to the coincidence problems.* In this section we present some existence results for coincidence point problems with multivalued operators using Leray-Schauder type condition and Theorem 2.5.2. Our contributions in this section are: Theorem 3.4.1 is an existence result for coincidence problem and a generalization of Theorem 2.5.2; Theorem 3.4.2 is an existence result for coincidence problem with operators which are condensing but not necessarily k-set contractive; Corollary 3.4.2 and 3.4.3 are consequences of Theorem 3.4.2. The scientific paper which contain the original results of this section is: J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50].

Chapter 4: Applications.

The purpose of this chapter is to present some applications of the results presented in this thesis. Firstly is given an application regarding Ulam-Hyers stability for differential equations and operatorial inclusions and then we study the existence of classical and strong solution to a diferential equation of first order and second order. Finally we present the existence of solution for a Dirichlet problem. This chapter contains the following sections:

§1 *Ulam-Hyers stability for differential equations.* In this section we establish some new existence, uniqueness and Ulam-Hyers stability results for differential equations. Our contributions in this section are: Application 1 for Theorem 2.3.4, it is an Ulam-Hyers stability result for a differential equation; Application 2 for Theorem 2.3.2, it is an Ulam-Hyers stability result for a differential equation using generalized contractions.

The scientific papers which contain the original results of this section are: J. Garcia-Falset and O. Mleşniţe [49], O. Mleşniţe [77].

§2 *Ulam-Hyers stability for operatorial inclusions.* In this section we prov an Ulam-Hyers stability theorem for a multivalued Cauchy problem corresponding to a first order differential inclusion. Our contribution in this section is: Theorem 4.2.1 is a result with respect to the Ulam-Hyers stability of the Cauchy problem. The scientific paper which contain the original results of this section is: O. Mlesŋiţe [74].

§3 *Existence of solution to a differential equation of first order.* In this section, we want to study the existence of strong solutions to a differential equation of first order. Our contributions in this section are: Lemma 4.3.1, Lemma 4.3.2, Lemma 4.3.3, Lemma 4.3.4, Lemma 4.3.5 and Theorem 4.3.1 representing the main result for existence of strong solution to a differential equation of first order. For obtaining this existence result we apply Corollary 2.5.2. The scientific paper which contain the original results of this section is: J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50].

§4 *Existence of solutions to a differential equation of second order.* In this section, we want to study the existence of classical and strong solutions to a differential equation with a non homogeneous Dirichlet conditions. Our contributions in this section are: Lemma 4.4.1, Lemma 4.4.2, Lemma 4.4.3, Lemma 4.4.4, Theorem 4.4.1 and Theorem 4.4.3 representing the main result for existence of classical solution to a differential equation of second order, for obtaining this existence result we apply Theorem 2.3.3 and Corollary 3.4.2; Lemma 4.4.5, Lemma 4.4.6, Lemma 4.4.7 and Theorem 4.4.4 representing the main result for existence of strong solutions to a differential equation of second order, for obtaining this existence result we apply Corollary 3.4.3. The scientific paper which contain the original results of this section is: J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50].

§5 *A nonlinear Dirichlet problem.* In this section we obtain the existence of solution for a Dirichlet problem using the results of coincidence problems. Our contributions in this section is: Theorem 4.5.1 representing the main result the existence of a solution for a Dirichlet problem. The scientific paper which contain the original results of this section is: J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50].

The author's contributions included in this thesis are also part of the following scientific papers:

- **O. Mlesŋiţe**, *Ulam-Hyers stability for operatorial inclusions*, Creat. Math. Inform., 21 (2012), No. 1, 87-94 (MR2984982).
- **O. Mleşniţe**, *Existence and Ulam-Hyers stability results for coincidence problems*, J. Nonlinear Sci. Appl. 6 (2013), 108-116 (MR3017894).
- **O. Mleşniţe** and A. Petruşel, *Existence and Ulam-Hyers stability results for multivalued coincidence problems*, Filomat, 26, 5 (2012), 965-976 (IF: 0.714).

- **O. Mleşniţe**, *Existence and Ulam-Hyers stability result for a coincidence problems with applications*, Miskolc Mathematical Notes, Vol. 14 (2013), No 1, 183-189 (IF: 0.304).
- J. Garcia-Falset and **O. Mleşniţe**, *Coincidence problems for generalized contractions*, submitted for publication.
- J. Garcia-Falset, C. A. Hernández-Linares and **O. Mleşniţe**, *The Leray-Schauder condition in the coincidence problem for two mappings*, submitted for publication.
- **O. Mleşniţe**, *Covering mappings and Ulam-Hyers stability results for coincidence problems*, Carpathian Journal of Mathematics, accepted for publication, (IF: 0.852).
- **O. Mleşniţe**, *Metric regularity and Ulam-Hyers stability results for coincidence problems with multivalued operators*, submitted for publication.
- A. Petruşel, C. Urs and **O. Mleşniţe**, *Vector-valued Metrics in Fixed Point Theory*, Contemporary Math. Series, Amer. Math. Soc., 2013.
- M.-F. Bota, E. Karapinar and **O. Mleşniţe**, *Ulam-Hyers stability results for fixed point problems via $\alpha - \psi$-contractive mapping in (b)-metric space*, Abstract and Applied Analysis, Volume 2013 (2013), Article ID 825293, 6 pages (IF: 1.102).

A significant part of the original results proved in this thesis were also presented at the following scientific conferences:

- The 7^{th} International Conference on Applied Mathematics (ICAM7), September $1^{st} - 4^{th}$, 2010, North University of Baia Mare, Romania.
- International Conference on Nonlinear Operators, Diferential Equations and Applications (ICNODEA), July $5^{th} - 8^{th}$, 2011, Babeş-Bolyai University of Cluj-Napoca, Romania.
- The Fifth International Workshop-Constructive methods for non-linear boundary value problems, 28 June-1 July, 2012, Tokaj, Hungary.
- The 10^{th} International Conference on Fixed Point Theory and its Applications, July 9 - 15, 2012, Babeş-Bolyai University, Cluj-Napoca, Romania.
- The 5^{th} Workshop on Metric Fixed Point Theory, November 15-17, 2012, Valencia, Spain.
- The Fourteenth International Conference on Applied Mathematics and Computer Science, August, 29-31, 2013, Cluj-Napoca, Romania.
- The 9^{th} International Conference on Applied Mathematics (ICAM9), September, 25-28, 2013, North University of Baia Mare, Romania.

Keywords and phrases: singlevalued operator, multivalued operator, generalized metric space, fixed point, coincidence point, Picard operator, weakly Picard operator, covering operator, metric regularity, Ulam-Hyers stability, Leray-Schauder condition, generalized contraction, measure of noncompactness, k-set contractive operator, condensing operator.

Chapter 1

Preliminaries

The purpose of this chapter is to present the basic notions and results which are further considered in the next chapters of this work, allowing us to present the results of this thesis. Through this thesis we will use the classical notations and notions from Nonlinear Analysis. For the fixed point theory in metric spaces see G. Allaire and S.M. Kaber [2], Q. H. Ansari [4], A. Granas and J. Dugundji [55], M. A. Khamsi and W. A. Kirk [64], W. A. Kirk and B. Sims [65], M. A. Khamsi and W. A. Kirk [64], G. Moţ, A. Petruşel and G. Petruşel [82], A. Petruşel [90], I. A. Rus [101], I. A. Rus [102], I. A. Rus [107], I. A. Rus [109], I.A. Rus, A. Petruşel and G. Petruşel [114], R. S. Varga [125] and others.

1.1 Metric spaces. Generalized metric spaces

In many branches of mathematics, it is convenient to have available a notion of distance between elements of an abstract set. For example, the proofs of some of the theorems in real analysis depends only on a few properties of the distance between points and not on the fact that the points. When these properties of distance are abstracted, they lead to the concept of a metric space. Our objective in this section is to define a metric space and afterwards a generalized metric space with some of their properties.

In 1905 M. Frechet introduced the notion of metric space in order to study the properties of functional spaces.

In the late of XX-th century and the beginning of XXI century appear works which treat results where the vector metric takes values in an infinite dimensional space (see W. A. J. Luxemburg and A. C. Zaanen [70], A.C. Zaanen [131]). Next we define the notion of generalized metric space.

Definition 1.1.1 (A. I. Perov [87]). *Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:*

- *(i) $d(x, y) \geq O$ for all $x, y \in X$; if $d(x, y) = O$, then $x = y$; (where $O := \underbrace{(0, 0, \cdots, 0)}_{m-times}$)*

- *(ii)* $d(x,y) = d(y,x)$ *for all* $x, y \in X$*;*
- *(iii)* $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$*.*

A nonempty set X endowed with a vector-valued metric d is called a generalized metric space in the sense of Perov (in short a generalized metric space) and it will be denote by (X,d).

Notice that the generalized metric space in the sense of Perov is a particular case of Riesz spaces (see W. A. J. Luxemburg and A. C. Zaanen [70], A. C. Zaanen [131]).

1.2 Singlevalued weakly Picard operators

The method of successive approximations is one of the basic tool in the theory of operatorial equations, especially in the fixed point theory. The theory of weakly Picard operators is very useful to study some properties of the solutions of those equations for which the method of successive approximation works. In term a weakly Picard operators, some classical results take a very simple form. Throughout this section we follow the terminologies and the notations in I. A. Rus [102] and I.A. Rus [104], A. Petruşel and G. Petruşel [114].

Let X be a nonempty set and $f : X \to X$ an operator. We will use the notation:

$$Fix(f) := \{x \in X \mid f(x) = x\} - \text{ for the fixed point set of the operator } f.$$

Let (X,d), (Y,ρ) be two metric spaces and let $f : X \to Y$ be an operator.

(a) f is called a Lipschitz if there exists a constant $k \geq 0$ such that

$$\rho(f(x), f(y)) \leq k \cdot d(x,y), \text{ for each } x, y \in X.$$

If $k \in [0,1)$ then f is called contraction.

If $k = 1$, then f is called nonexpansive.

(b) f is a dilatation if there exists a constant $h > 1$ such that

$$\rho(f(x), f(y)) \geq h \cdot d(x,y), \text{ for each } x, y \in X.$$

If $h = 1$, then f is said to be expansive.

(c) f is contractive if

$$\rho(f(x), f(y)) < d(x,y), \text{ for each } x, y \in X \text{ with } x \neq y.$$

Main result for self contractions on generalized metric spaces is Perovs fixed point theorem, see A. I. Perov [87].

Theorem 1.2.1 (A. I. Perov [87]). *Let (X, d) be a complete generalized metric space and the operator $f : X \to X$ with the property that there exists a matrix $A \in M_{m,m}(\mathbb{R})$ such that*

$$d(f(x), f(y)) \leq Ad(x, y) \text{ for all } x, y \in X.$$

If A is a matrix convergent towards zero, then:

1) $Fix(f) = \{x^*\}$;

2) *the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x_0)$ is convergent and it has the limit x^*, for all $x_0 \in X$;*

3) *one has the following estimation*

$$d(x_n, x^*) \leq A^n(I - A)^{-1}d(x_0, x_1);$$

4) *If $g : X \to X$ is an operator such that there exists $y^* \in Fix(g)$ and there exists $\eta := (\eta_1, ..., \eta_m) \in \mathbb{R}^m_+$ with $\eta_i > 0$ for each $i \in \{1, 2, ..., m\}$, such that*

$$d(f(x), g(y)) \leq \eta, \text{ for all } x \in X,$$

then

$$d(y^*, x^*) \leq (I - A)^{-1}\eta.$$

5) *If $g : X \to X$ is an operator, $y_n = g^n(x_0)$ and there exists $\eta := (\eta_1, ..., \eta_m) \in \mathbb{R}^m_+$ with $\eta_i > 0$ for each $i \in \{1, 2, ..., m\}$ such that*

$$d(f(x), g(x)) \leq \eta, \text{ for all } x \in X,$$

we have the following estimation

$$d(y_n, x^*) \leq (I - A)^{-1}\eta + A^n(I - A)^{-1}d(x_0, x_1).$$

1.3 Multivalued weakly Picard operators

In this section we describe some basic concepts and results for multivalued operators such as the notions of continuity for multivalued operators (see J.-P. Aubin and A. Cellina [11], J.-P. Aubin and H. Frankowska [12], W. A. Kirk and B. Sims [65], A. Petruşel [91]) as well multivalued weakly Picard operators (A. Petruşel [90], I. A. Rus [108], I. A. Rus [107], I.A. Rus, A. Petruşel and G. Petruşel [114]).

A point $x \in X$ is called fixed point (respectively strict fixed point) for F if

$$x \in F(x) \text{ (respectively } \{x\} = F(x)).$$

Note $Fix(F)$ (or $SFix(F)$) the fixed point set (respectively the strict fixed point set) for the multivalued operator F, i. e.,

$$Fix(F) := \{x \in X \mid x \in F(x)\} - \text{ the fixed point set of } F;$$

$$SFix(F) := \{x \in X \mid \{x\} = F(x)\} - \text{ the strict fixed point set of } F.$$

Definition 1.3.1. *Let (X, d) and (Y, ρ) be two metric spaces and let $F : X \to P_{cl}(X)$ be a multivalued operator. Then*

(a) F is said to be k-Lipschitz if and only if $k > 0$ and

$$H_\rho(F(x), F(y)) \leq k \cdot d(x, y), \text{ for all } x, y \in X.$$

If F is k-Lipschitz with constant $k < 1$, then F is said to be a multivalued k-contraction.

(b) F is said to be φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function and

$$H_\rho(F(x), F(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in X.$$

The following result is known in the literature as Covitz-Nadler fixed point principle (see H. Covitz and S. B. Nadler [35] and S. B. Nadler [83]).

Theorem 1.3.1 (H. Covitz and S. B. Nadler [35], S. B. Nadler [83])**.** *Let (X, d) be a complete metric space and $x_0 \in X$ be arbitrary. If $F : X \to P_{cl}(X)$ is a multivalued k-contraction, then F has at least fixed point and there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.*

The next result, a generalization of Covitz-Nadler fixed point principle is known in the literature as Węgrzyk's fixed point theorem (see R. Węgrzyk [129]).

Theorem 1.3.2 (R. Węgrzyk [129])**.** *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multivalued φ-contraction. Then F has at least fixed point and for any $x_0 \in X$ there exists a sequence of successive approximations of F starting from x_0 which converges to a fixed point of F.*

Chapter 2

Coincidence theorems for singlevalued operators

It is well know that a coincidence problem is, under appropriate conditions, equivalent to a fixed point problem for a singlevalued operators generated by s and t. Using this approach, we will present, in this chapter, some existence, uniqueness, covering operators and Ulam-Hyers stability theorems for coincidence problems. We present also some extensions of our results in generalized metric spaces. Some examples illustrating the main results of the chapter are also given.

The references which were used to develop this chapter are: J. M. Ayerbe Toledano, T. Domínguez-Benavides and G. Lopez Acedo [13], A. V. Arutyunov [7], A. Arutyunov, E. Avakov, B. Gel'man, A. Dmitruk and V. Obukhovskii [10], A. V. Dmitruk [38], K. Goebel [52], L. A. Lyusternik [71], O. Mleşniţe [78], [75], [74], [77], T. P. Petru, A. Petruşel and J.-C. Yao [94], W. V. Petryshyn [95], I. A. Rus [108], [112].

Let X, Y be two nonempty sets and $s, t : X \to Y$ be two singlevalued operators. Let us consider the following coincidence problem

$$\text{find } \ (x, y) \in X \times Y \ \text{ such that } \ s(x) = t(x) = y. \tag{2.1}$$

We denote by $C(s, t)$ the set of all coincidence points for s and t.

A solution of the coincidence problem (2.1) for s and t is a pair $(x^*, y^*) \in X \times Y$ such that

$$s(x^*) = t(x^*) = y^*.$$

Denote by $CP(s, t) \subset X \times Y$ the set of all solution for the coincidence problem (2.1).

Ulam-Hyers stability for the coincidence problem (2.1):

Let (X, d), (Y, ρ) be two metric spaces and $s, t : X \to Y$ be two operators. The coincidence problem (2.1) is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each solution $w^* \in X$ of the approximative coincidence problem

$$\rho(s(w^*), t(w^*)) \leq \varepsilon \tag{2.2}$$

there exists a solution z^* of (2.1) such that

$$d(w^*, z^*) \leq \psi(\varepsilon). \tag{2.3}$$

If there exists $c > 0$ such that $\psi(t) = ct$ for each $t \in \mathbb{R}_+$ then the coincidence problem (2.1) is said to be Ulam-Hyers stable.

For Ulam-Hyers stability results in the case of fixed point problems and coincidence point problems for singlevalued operators see M. Bota and A. Petruşel [21], V. L. Lazăr [68], T. P. Petru, A. Petruşel and J.-C. Yao [94], I. A. Rus [102], [108], [112], I.A. Rus, A. Petruşel and G. Petruşel [114].

2.1 Covering operators and Ulam-Hyers stability results for coincidence problems

In this section, we will present some existence and Ulam-Hyers stability results for coincidence point problems with singlevalued operators. The basic hypothesis in these results is the property of covering operators. These results generalize the fixed point theorems given by A. V. Arutyunov [7], A. Arutyunov, E. Avakov, B. Gel'man, A. Dmitruk and V. Obukhovskii [10].

Definition 2.1.1 (A. Arutyunov [7]). *Let (X, d) and (Y, ρ) two metric spaces. For a given $\alpha > 0$, an operator $f : X \to Y$ is said to be α- covering if for all $x \in X$ and $r > 0$ we have*

$$B_Y(f(x), \alpha r) \subseteq f(B_X(x, r)). \tag{2.4}$$

The supremum over all values α satisfying inclusion (2.4) is called modulus of covering of f and denoted for short by $cov(f)$ (instead of $cov_{X\times Y}(f)$).
Notice that, due to the global validity of inclusion (2.4) one has

$$B_Y(f(x), cov(f)r) \subseteq f(B_X(x, r)), \text{ for all } x \in X, r > 0.$$

The properties of coverings operators were studied in A. Arutyunov [7], A. V. Dmitruk, A. A. Milyutin, N. P. Osmolovskii [39], A. D. Ioffe [59], [60], B. S. Mordukhovich and B. Wang [81].

Following the idea given in A. Uderzo [123] we have the following remark.

Remark 2.1.1. An operator $f : X \to Y$ fulfils Definition 2.1.1 if and only if there exists $k > 0$ such that

$$d(x, f^{-1}(y)) \leq k\rho(f(x), y), \text{ for all } x \in X, y \in Y. \tag{2.5}$$

We say that f is *metrically regular on X*.

The infimum of all values k satisfying inequality (2.5) is called modulus of global metric regularity of f and denoted by $reg(f)$. The following relation between the modulus of global covering and the modulus of global metric regularity is know to hold

$$reg(f) = \frac{1}{cov(f)},$$

where the case $reg(f) = \infty$, coresponding to $cov(f) = 0$, is intended to mean the absence of global open covering/ metric regularity for a given f.

Another characterization of covering/ metric regularity can be obtained in terms of Lipschitz behavior of the inverse multivalued operator. In fact f covers on X if and only if f^{-1} is Lipschitz continuous in Y and it holds

$$lip(f^{-1}) = \frac{1}{cov(f)}.$$

Lemma 2.1.1 (A. Arutyunov, E. Avakov, B. Gel'man, A. Dmitruk, V. Obukhovskii [10]). *Let $f : X \to Y$ be an onto and k-Lypschitz operator with $k > 0$. The inverse multivalued operator $f^{-1} : Y \to \mathcal{P}(X)$, $f^{-1}(x) = \{y \in \mathcal{P}(X) : f(y) = x\}$ is $\frac{1}{k}$-covering.*

Remark 2.1.2. *It should be mentioned that the converse is also true: if f^{-1} is $\frac{1}{k}$-covering, then f is k-Lipschitz.*

Let us consider a relative version of the α-covering property.

Definition 2.1.2 (A. Arutyunov, E. Avakov, B. Gel'man, A. Dmitruk, V. Obukhovskii [10]). *Let $M \subseteq X$ and $N \subseteq Y$ be any sets and $\alpha > 0$. An operator $f : X \to Y$ is said to be α-covering with respect to the sets M and N if for all $x \in M$ and $r > 0$ such that $B_X(x, r) \subseteq M$ we have*

$$B_Y(f(x), \alpha r) \cap N \subseteq f(B_X(x, r)). \tag{2.6}$$

If $N = Y$ we say that f is α-covering on M.

Notice that other definitions of covering maps may be found, for example, in works of A. D. Ioffe [59], [60], B. S. Mordukhovich [80].

Definition 2.1.3. *Let $M \subseteq X$ and $N \subseteq Y$ be closed sets. An operator $f : X \to Y$ is called closed with respect to M and N if the intersection of its graph with $M \times N$ is closed.*

Theorem 2.1.1 (O. Mleşniţe [78]). *Let (X, d) be a complete metric space and (Y, ρ) be a metric space. Suppose that:*

(i) $t : X \to Y$ is open, bijective, and k_t-Lipschitz operator, with constant $k_t > 0$;

(ii) $s : X \to Y$ is a continuous and k_s-covering operator and $k_s > k_t$;

Then the coincidence problem (2.1) has at last one solution (i.e. there exists $x^* \in X$ *such that* $s(x^*) = t(x^*)$*) and we have*

$$\rho(y_0, t(x^*)) \leq \frac{k_t}{k_s - k_t}\rho(y_0, s(t^{-1}(y_0))), \text{ for all } y_0 \in t(X). \tag{2.7}$$

If, additionally, $t : X \to Y$ *is metrically regular on* X *with constant* $\alpha > 0$ *then the coincidence problem (2.1) is Ulam-Hyers stable.*

Theorem 2.1.2 (O. Mleşniţe [78]). *Let* (X, d) *be a complete metric space,* (Y, ρ)*,* $x_0 \in X$ *and* $R_1, R_2 \in (0, \infty]$*. Suppose that:*

(i) $t : X \to Y$ *is open, bijective and* k_t*- Lipschitz operator, with constant* $k_t > 1$*;*

(ii) $s : X \to Y$ *is a continuous and* k_s*- covering operator with respect to the balls* $B_X(x_0, R_1)$ *and* $B_Y(s(x_0), k_s R_2)$ *and* $k_s > k_t$ *such that*

$$\rho(s(x_0), t(x_0)) \leq \left(\frac{k_s}{k_t} - 1\right)\min\{R_1, R_2\}. \tag{2.8}$$

Then the coincidence problem (2.1) has at last one solution (i.e. there exists $x^* \in X$ *such that* $s(x^*) = t(x^*)$*) and we have*

$$\rho(y_0, t(x^*)) \leq \frac{k_t}{k_s - k_t}\rho(y_0, s(t^{-1}(y_0))), \text{ for all } y_0 \in t(X). \tag{2.9}$$

If, additionally, $t : X \to Y$ *is metrically regular on* X *with constant* $\alpha > 0$ *then the coincidence problem (2.1) is Ulam-Hyers stable.*

2.2 Existence and Ulam-Hyers stability results for coincidence problems

The purpose of this section is to present some existence and Ulam- Hyers stability results for coincidence problems with singlevalued operators. Using the cartesian product technique for two singlevalued operators, these results are based on the following works M. Bota and A. Petruşel [21], O. Mleşniţe [74], [75], I. A. Rus [108].

Let (X, d), (Y, ρ) be two metric spaces and $s, t : X \to Y$ be two operators such that t is a injection. Then, t has a left inverse $t_l^{-1} : t(X) \to X$. Suppose also that $s(X) \subseteq t(X)$. Consider $f : X \times t(X) \to X \times t(X)$ defined by

$$f(x_1, x_2) = (t_l^{-1}(x_2), s(x_1)).$$

Lemma 2.2.1. *Under the above mentioned conditions, we have* $CP(s, t) = Fix(f)$*.*

Let (X,d), (Y,ρ) be two metric spaces, let d_Z be the metric (generated by d and ρ) on $Z := X \times Y$ defined by

$$d^*((x_1,x_2),(u_1,u_2)) = d(x_1,u_1) + \rho(x_2,u_2)$$

or

$$d_*((x_1,x_2),(u_1,u_2)) = \max\{d(x_1,u_1),\rho(x_2,u_2)\}$$

for each (x_1,x_2), $(u_1,u_2) \in Z$ and let $s,t : X \to Y$ be two operators. Let us consider the coincidence problem (2.1).

Theorem 2.2.1 (O. Mleşniţe [75]). *Let (X,d) and (Y,ρ) be two complete metric spaces. Suppose that the operator $t : X \to Y$ is a dilatation with constant $k_t > 1$, the operator $s : X \to Y$ is a contraction with constant $k_s < 1$ and $s(X) \subseteq t(X)$. Then the coincidence problem* (2.1) *for s and t has a unique solution.*

Theorem 2.2.2 (O. Mleşniţe [75]). *Let (X,d) and (Y,ρ) be two complete metric spaces. Suppose that all the hypotheses of Theorem 2.2.1 hold and additionally suppose that $t : X \to Y$ is metrically regular on X with constant $\alpha > 0$. Then the coincidence problem* (2.1) *is Ulam-Hyers stable.*

Remark 2.2.1. *Similar proofs for Theorem* 2.2.1 *and Theorem* 2.2.2 *are possible if we consider on $Z := X \times \overline{t(X)}$ the metric $d^* : Z \times Z \to \mathbb{R}_+$ defined by*

$$d^*((x_1,x_2),(u_1,u_2)) = \max\{d(x_1,u_1),\rho(x_2,u_2)\}.$$

Next we prove some data dependence results for the Ulam-Hyers stability of coincidence problems of two pair of singlevalued operators.

Theorem 2.2.3 (O. Mleşniţe [75]). *Let (X,d) and (Y,ρ) be two metric spaces and $f_i, g_i : X \to Y$, $i \in \{1,2\}$ be four operators. Consider the following coincidence equations:*

$$f_1(x) = g_1(x), \quad x \in X, \tag{2.10}$$

$$f_2(x) = g_2(x), \quad x \in X. \tag{2.11}$$

Let us consider the sets:

$$C_{i\varepsilon} := \{x \in X | \rho(f_i(x), g_i(x)) \le \varepsilon\}, i \in \{1,2\}.$$

If the following conditions are satisfied:

(i) $C(f_2,g_2) \subseteq C(f_1,g_1)$;

(ii) the coincidence equation (2.11) *is Ulam-Hyers stable;*

(iii) $C_{1\varepsilon} \subseteq C_{2\varepsilon}$, for each $\varepsilon > 0$;

then the coincidence equation (2.10) *is Ulam-Hyers stable.*

In the particular case $Y := X$ and $g_1 = g_2 := 1_X$, we get the following Ulam-Hyers stability result for a fixed point equation.

Theorem 2.2.4 (O. Mleşniţe [75]). *Let (X, d) be a metric space and $f_1, f_2 : X \to X$ be two operators. Consider the following fixed point equations:*

$$f_1(x) = x, \quad x \in X \tag{2.12}$$

$$f_2(x) = x, \quad x \in X. \tag{2.13}$$

Let us consider the sets:

$$F_{i\varepsilon} := \{x \in X | d(f_i(x), x) \leq \varepsilon\}, i = \{1, 2\}.$$

If the following conditions are satisfied:
(i) $Fix(f_1) = Fix(f_2)$;
(ii) the fixed point equation (2.13) *is Ulam-Hyers stable;*
(iii) $F_{1\varepsilon} \subseteq F_{2\varepsilon}$, for each $\varepsilon > 0$;
then, the fixed point equation (2.12) *is Ulam-Hyers stable.*

If we suppose that $X = Y$ and ρ, d are two strongly equivalent metrics on X, then we can obtain a Ulam-Hyers stability result for the coincidence equation (2.10).

Theorem 2.2.5 (O. Mleşniţe [75]). *Let X be a nonempty set, ρ and d two strongly equivalent metrics. If the coincidence equation* (2.10) *is Ulam-Hyers stable with respect to metric d, then it is Ulam-Hyers stable with respect to metric ρ.*

2.3 Coincidence problems for generalized contractions

In this section, we study the existence, uniqueness and Ulam-Hyers stability for the coincidence problem and thus, we may extend most of the results given in O. Mleşniţe [75], our techniques also allow us to give a generalization of Theorem 2.1 of T. Xiang and R. Yuan [130].

In O. Mleşniţe [74] is presented the following result on Ulam-Hyers stability for the case of Goebel's coincidence theorem.

Theorem 2.3.1 (O. Mleşniţe [74]). *Let $X \neq \emptyset$ be an arbitrary set and let (Y, ρ) be a metric space. Let $s, t : X \to Y$ such that $s(X) \subset t(X)$ and $(t(X), \rho)$ is a complete subspace of Y. Suppose that exists $0 \leq k < 1$ such that $\rho(s(x), s(y)) \leq k\rho(t(x), t(y))$, for all $x, y \in X$. Then:*
a) $C(s, t) \neq \emptyset$ (Goebel's Theorem, see [52]);
b) If additionally:

$$\rho(y, s(t^{-1}(y))) \leq \rho(t(y), s(y)), \text{ for all } y \in t(X), \tag{2.14}$$

then the coincidence point problem (2.1) *is Ulam-Hyers stable.*

In O. Mleşniţe [77] is presented the following result who extends Goebel's theorem (K. Goebel [52]) by considering the condition of φ-contraction of s with respect to t.

Theorem 2.3.2 (O. Mleşniţe [77]). *Let $X \neq \emptyset$ be an arbitrary set and let (Y, ρ) be a metric space. Let $s, t : X \to Y$ such that $s(X) \subseteq t(X)$ and $(t(X), \rho)$ is a complete subspace of Y. Suppose that there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing and $(\varphi^n(t)) \to 0$, $n \to \infty$, for all $t \in \mathbb{R}_+$ such that*

$$\rho(s(x), s(y)) \leq \varphi(\rho(t(x), t(y))), \text{ for all } x, y \in X.$$

Then:

a) $C(s,t) \neq \emptyset$;

b) If additionally, exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that:

$$\rho(y, s(t^{-1}(y))) \leq \psi(\rho(t(y), s(y))), \text{ for all } y \in t(X), \tag{2.15}$$

then the coincidence point problem (2.1) is $(\beta^{-1} \circ \psi)-$ generalized Ulam-Hyers stable, where $\beta(t) := t - \varphi(t)$ increasing and bijective.

Next we present the main results of this section which extend previous ones (for instance see O. Mleşniţe [75, Theorems 1.6, 1.8, 1.11, 1.13]).

Theorem 2.3.3 (J. Garcia-Falset and O. Mleşniţe [49]). *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose that:*

(i) $t : X \to Y$ is an expansive operator,

(ii) the operator $s : X \to Y$ is a ϕ-contraction,

(iii) $s(X) \subseteq t(X)$.

Then the coincidence problem (2.1) has a unique solution.

Regarding the Ulam-Hyers stability problem the ideas given in T. P. Petru, A. Petruşel and J.-C. Yao [94, Theorem 2.3] allow us to obtain the second main result.

Theorem 2.3.4 (J. Garcia-Falset and O. Mleşniţe [49]). *Let (X, d), (Y, ρ) be two complete metric spaces. Suppose that all the hypotheses of Theorem 2.3.3 hold and additionally that the function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, $\beta(r) := r - \phi(r)$ is strictly increasing and onto. Then the coincidence problem (2.1) is generalized Ulam-Hyers stable.*

Since if $t : X \to Y$ is a dilatation, then t is an expansive operator. As a consequence of Theorems 2.3.3 and 2.3.4, we infer.

Corollary 2.3.1 (J. Garcia-Falset and O. Mleşniţe [49]). *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose both that the operator $t : X \to Y$ is a dilatation and $s : X \to Y$ is a ϕ-contraction with $s(X) \subseteq t(X)$. Then*

1. *the coincidence problem (2.1) has a unique solution.*

2. *If in addition, the function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, $\beta(r) := r - \phi(r)$ is strictly increasing and onto, then the coincidence problem (2.1) is generalized Ulam-Hyers stable.*

Theorem 2.3.5 (J. Garcia-Falset and O. Mleşniţe [49]). *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose that:*

(i) $t : X \to Y$ is an operator such that there exists a function $\phi_1 : \mathbb{R}^+ \to \mathbb{R}^+$, continuous, increasing, $\phi_1(r) > r$ and $\phi_1(0) = 0$ and satisfying the following relation:

$$\rho(t(y), t(z)) \geq \phi_1(d(y, z)), \quad \text{for all} \quad y, z \in X,$$

(ii) the operator $s : X \to Y$ is lipschizian with constant $k_s > 0$,
(iii) $s(X) \subseteq t(X)$,
(iv) $r < \phi_1(\frac{r}{k_s})$.
Then the coincidence problem (2.1) has a unique solution.

Next result is a generalization of Theorem 2.1 of T. Xiang and R. Yuan [130].

Corollary 2.3.2 (J. Garcia-Falset and O. Mleşniţe [49]). *Let (X, d) be a complete metric space and let $t : X \to X$ be an onto operator satisfying condition (i) of Theorem 2.3.5. Then it has a unique fixed point.*

Theorem 2.3.6 (J. Garcia-Falset and O. Mleşniţe [49]). *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose that the operator $t : X \to Y$ is expansive and the operator $s : X \to Y$ is a separate contraction and $s(X) \subseteq t(X)$. Then*

(i) If φ is nondecreasing, the coincidence problem (2.1) has a unique solution.

(ii) If ψ is onto, the coincidence problem (2.1) is generalized Ulam-Hyers stable.

Remark 2.3.1. *Other results on Ulam-Hyers stability for fixed point problems using generalized contractions (i.e. $\alpha - \psi$-contractive mappings) in (b)-metric space are presented in M.-F. Bota, E. Karapinar and O. Mleşniţe [20].*

2.4 Coincidence results by fixed point theorems in generalized metric spaces

In this section we will present some existence, uniqueness and Ulam-Hyers stability results for fixed point and coincidence point problems with singlevalued operators in spaces endowed with vector valued metrics. Many other contributions on this topic were given in R. P. Agarwal [1], A. Bucur, L. Guran and A. Petruşel [27], A. D. Filip and A. Petruşel [43], D. O'Regan, N. Shahzad and R. P. Agarwal [85], R. Precup [97], R. Precup and A. Viorel [98], [99].

We present now an extension of Perov's Theorem. In the same time, the result is a generalization to vector-valued metric spaces of the main theorem in M. Berinde and V. Berinde [17].

Theorem 2.4.1 (A. Petruşel, O. Mleşniţe and C. Urs [93]). *Let (X, d) be a generalized complete metric space and let $f : X \to X$ be an (A, B, C, D)-contraction, i.e., $A, B, C, D \in M_{mm}(\mathbb{R}_+)$ are such that the matrices D and $M := (I - D)^{-1}(A + C)$ converge to zero and*

$$d(f(x), f(y)) \le Ad(x, y) + Bd(y, f(x)) + Cd(x, f(x)) + Dd(y, f(y)), \text{ for all } x, y \in X.$$

Then, the following conclusions hold:

(1) f has at least one fixed point and, for each $x_0 \in X$, the sequence $x_n := f^n(x_0)$ of successive approximations of f starting from x_0 converges to $x^*(x_0) \in Fix(f)$ as $n \to \infty$;

(2) For each $x_0 \in X$ we have

$$d(x_n, x^*(x_0)) \le M^n (I - M)^{-1} d(x_0, f(x_0)), \text{ for all } n \in \mathbb{N}$$

and

$$d(x_0, x^*(x_0)) \le (I - M)^{-1} d(x_0, f(x_0));$$

(3) If, additionally, the matrix $A + B$ converges to zero, then f has a unique fixed point in X.

Remark 2.4.1. *In particular, if $B = C = D = O_{mm}$ (where O_{mm} is zero matrix from $M_{mm}(\mathbb{R}_+)$), then we get the classical result of Perov, see Theorem 1.2.1.*

Let us introduce now some vector valued metrics of Perov's type. Let (X, d) and (Y, ρ) be two metric spaces. Let $Z := X \times Y$ and define on $Z \times Z$ the vector metric $d^V : Z \times Z \to \mathbb{R}^2_+$ by

$$d^V(x, u) = d^V((x_1, x_2), (u_1, u_2)) = (d(x_1, u_1), \rho(x_2, u_2)), \tag{2.16}$$

for each $x = (x_1, x_2), u = (u_1, u_2) \in Z$.

Our main result in this section is the following theorem.

Theorem 2.4.2 (O. Mleşniţe [75]). *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose that the operator $t : X \to Y$ is a dilatation with constant $k_t > 0$, the operator $s : X \to Y$ is Lipschitz with the constant $k_s > 0$ and $s(X) \subseteq t(X)$. If $\frac{k_s}{k_t} \in [0, 1)$, then the coincidence problem* (2.1) *has a unique solution.*

Next we have an approximation and an error estimate for the solution of the coincidence problem.

Theorem 2.4.3 (O. Mleşniţe [75]). *Let $(X, d), (Y, \rho)$ be two complete metric spaces. Suppose that all the hypotheses of Theorem 2.4.2 hold and additionally suppose that $t : X \to Y$ is metrically regular on X with constant $\alpha > 0$. Then the coincidence problem* (2.1) *is Ulam-Hyers stable.*

2.5 A Leray-Schauder condition to the coincidence problems

The first step to extend the Schauder theorem to noncompact operators was given by G. Darbo [36] in 1955. The first measure of noncompactness was defined by K. Kuratowski [67] in 1930. Darbo used this measure to generalize Schauder's theorem to a wide class of operators, called k-set-contractive operators, which satisfy the condition $\alpha(T(A)) \leq k\alpha(A)$ for some $k \in [0, 1)$. In 1967, B. N. Sadovskii [117], generalized Darbo's theorem to set-condensing operators.

Measures of noncompactness are very useful tools in the theory of operator equations in Banach spaces. There exists a considerable literature devoted to this subject, see for example R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, [3], J. Banaś and K. Goebel [14], V. I. Istrăţescu [61], [62], J. M. Ayerbe Toledano, T. Domínguez-Benavides and G. Lopez Acedo [13], W. Zhao [134] and references therein.

In this section, we intend to obtain several versions, without invoking degree theory, of the coincidence problem where s and t can both become nonlinear.

Definition 2.5.1. *Let X be a normed space and $\mathcal{B}(X) := \{A \subset X : A \text{ is bounded}\}$. A measure of non-compactness is a function $\mu : \mathcal{B}(X) \to \mathbb{R}^+$ which satisfies:*

1. $\mu(A) = 0 \Leftrightarrow \overline{A}$ *is compact.*
2. $\mu(A) = \mu(\overline{A})$.
3. $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$
4. $\mu(conv(A)) = \mu(A)$.

To avoid confusion when dealing with different spaces, we will in some cases add the name of a subspace as a subscript.

Point 3 in the last definition implies that $\mu(A) \leq \mu(B)$, whenever $A \subset B$.

Some usual measures of noncompactness are the following.

Definition 2.5.2. *Let $(X, \|\cdot\|)$ be a normed space. The Kuratowski measure of non-compactness for a bounded subset A of X is given by*

$$\alpha(A) = \inf \left\{ r > 0 : A \subset \cup_{i=1}^{n} D_i,\ diam(D_i) \leq r \right\}.$$

Definition 2.5.3. *Let $(X, \|\cdot\|)$ be a normed space. The Hausdorff measure of non-compactness for a bounded subset A of X is given by*

$$\chi(A) = \inf \left\{ r > 0 : A \subset \cup_{i=1}^{n} B(x_i, r), x_i \in X \right\}.$$

A detailed account of theory and applications of measures of noncompactness may be found in the monographs R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii [3], J. M. Ayerbe Toledano, T. Domínguez-Benavides and G. Lopez Acedo [13].

Definition 2.5.4. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces endowed with the measures of noncompactness μ_X and μ_Y respectively. If C is a nonempty subset of X and $T : C \to Y$ is an operator,*

(a) Given $k > 0$, the operator T is called (μ_X, μ_Y)-k-set contractive if $\mu_Y(T(A)) \leq k\mu_X(A)$ for all $A \in \mathcal{B}(C)$.

(b) The operator T is called (μ_X, μ_Y)-condensing if $\mu_Y(T(A)) < \mu_X(A)$ for all bounded subset A of C with $\mu_X(A) > 0$.

(c) The operator T is called expansive if the inequality $\|T(x) - T(y)\|_Y \geq \|x - y\|_X$ holds for every $x, y \in C$.

(d) The operator T is called nonexpansive if the inequality $\|T(x) - T(y)\|_Y \leq \|x - y\|_X$ holds for every $x, y \in C$.

(e) The operator T is said to be bounded if there exists $k > 0$ such that $\|T(x)\|_Y \leq k$ for all $x \in C$.

The following well known theorem was proved in 1967 by B. N. Sadovskii [117], it is a generalization of Darbo's fixed point theorem G. Darbo [36]. We refer to J. Appel [5] where the reader will find many applications of these theorems.

Theorem 2.5.1 (B. N. Sadovskii [117])**.** *Suppose that C is a closed convex bounded subset of a Banach space X and $T : C \to C$ a continuous and condensing operator, then T has a fixed point.*

When the domain C, in Sadovskii's theorem, is unbounded the following result is also well known.

Theorem 2.5.2 (J. Garcia-Falset [46])**.** *Suppose that C is a closed convex and unbounded subset of a Banach space X and $T : C \to C$ a continuous and condensing operator. If there exist $R > 0$ and $z \in C$ such that for all $u \in C \cap S_R(z)$*

$$T(u) - z \neq \lambda(u - z), \qquad \forall \lambda > 1,$$

then T has a fixed point.

We recall the following theorem proved by W. V. Petryshyn in [95].

Theorem 2.5.3 (W. V. Petryshyn [95]). *Suppose that U is an open bounded subset of a Banach space X and $T : \overline{U} \to X$ a continuous and condensing operator. If there exists $z \in U$ such that for all $u \in \partial U$*

$$u \neq \lambda T(u) + (1 - \lambda)z, \qquad \forall \lambda \in (0, 1),$$

then T has a fixed point.

Let $T : X \to Y$ be an operator which transforms bounded subsets of X into bounded subsets of Y. For a such operator, we define

$$l(T) := \sup\{r > 0 : r\mu_X(A) \leq \mu_Y(T(A)), A \in \mathcal{B}(X)\}.$$

In the following we are going to use the Kuratowski measure of noncompactness. Assuming that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces and if an operator $T : C \to Y$ is (α_X, α_Y)-condensing we will say simply that T is α-condensing.

Next we propose to establish several coincidence results by using Theorem 2.5.2.

Theorem 2.5.4 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50]). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Consider C a convex closed subset of X such that the operators $t : C \to Y$ and $s : C \to Y$ satisfy:*

1. *$\overline{t(C)}$ is a convex subset of Y and $t^{-1} : t(C) \to C$ is uniformly continuous on bounded subsets of $t(C)$,*
2. *s is a continuous k-set contractive operator,*
3. *$s(C) \subseteq t(C)$,*
4. *$k < l(t)$,*
5. *There are $R > 0$ and $x_0 \in C$ such that for every $x \in C$ with $\|x - x_0\| \geq R$ we have*

$$s(x) - t(x_0) \neq \lambda(t(x) - t(x_0)) \qquad \forall \lambda > 1. \tag{2.17}$$

Then there is $z \in C$ such that $s(z) = t(z)$.

Corollary 2.5.1 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50]). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $t : X \to Y$ be a continuous invertible linear map and C a convex closed subset of X.Let $s : C \to Y$ be a continuous k-set contraction with $k < l(t)$ and satisfying that $s(C) \subset t(C)$. If there are $R > 0$ and $x_0 \in C$ such that*

$$x \in C, \|x - x_0\|_X \geq R \Rightarrow s(x) - t(x_0) \neq \lambda(t(x) - t(x_0)) \qquad \forall \lambda > 1,$$

then there is $z \in C$ such that $s(z) = t(z)$.

Next result may be considered as a sharpening of Theorem 2.5.3.

Theorem 2.5.5 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50]). *Let X be a normed space and let Y be a Banach space. Assume that U is a bounded open subset of X, $t : \overline{U} \to Y$ an expansive operator such that $t(U)$ is an open bounded subset of Y with $\partial(t(U)) \subset t(\partial U)$ and $s : \overline{U} \to Y$ is a continuous condensing operator. If there exists $x_0 \in U$ such that for all $x \in \partial U$*

$$t(x) \neq \lambda s(x) + (1-\lambda)t(x_0) \qquad \forall \lambda \in (0,1) \tag{2.18}$$

then there exists $z_0 \in Y$ such that $t(z_0) = s(z_0)$.

Corollary 2.5.2 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50]). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $t : X \to Y$ be an expansive continuous invertible affine map and U a bounded open subset of X.Let $s : \overline{U} \to Y$ be a continuous condensing operator. If there is $x_0 \in U$ such that for all $x \in \partial U$,*

$$t(x) \neq \lambda s(x) + (1-\lambda)t(x_0) \qquad \forall \lambda \in (0,1),$$

then there is $z \in \overline{U}$ such that $s(z) = t(z)$.

Chapter 3

Coincidence theorems for multivalued operators

The purpose of this chapter is to present some existence, metric regularity and Ulam-Hyers stability results for fixed point and coincidence point problems with multivalued operators. This approach is based on the weakly Picard operator technique in the setting of generalized metric spaces in the sense of Perov, i.e., spaces endowed with vector metrics $d : X \times X \to \mathbb{R}^m_+$. Using the cartesian product technique for two multivalued operators, our result improve some recent theorems in the literature, see M. Bota and A. Petruşel [21], T. P. Petru, A. Petruşel and J.-C. Yao [94], I. A. Rus [108].

Let (X, d) be a metric space, Y be a nonempty set and $S, T : X \to P(Y)$ be two multivalued operators. An element $x^* \in X$ is a coincidence point for S and T if $S(x^*) \cap T(x^*) \neq \emptyset$. We denote by $C(S, T)$ the set of all coincidence points for S and T.

Let (X, d) and (Y, ρ) be two metric spaces and $S, T : X \to P(Y)$ be two multivalued operators. Let d_Z be a traditional scalar metric on $X \times Y$. Let us consider the following multivalued coincidence problem:

$$\text{find } (x, y) \in X \times Y \text{ such that } y \in S(x) \cap T(x). \tag{3.1}$$

By definition, a solution of the coincidence problem (3.1) is a pair $(x^*, y^*) \in X \times Y$ such that

$$y^* \in T(x^*) \cap S(x^*).$$

Denote by $CP(S, T) \subset X \times Y$ the set of all solutions of the coincidence problem for S and T.

It is well know that a coincidence problem is, under appropriate conditions, equivalent to a fixed point problem for a multivalued operator generated by s and t.

Ulam-Hyers stability for the coincidence problem (3.1):

Let (X, d), (Y, ρ) be two metric spaces and $S, T : X \to Y$ be two multivalued operators. The coincidence problem (3.1) is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each solution $w^* \in X$ of the approximative coincidence problem

$$D_\rho(S(w^*), T(w^*)) \leq \varepsilon \tag{3.2}$$

there exists a solution z^* of (3.1) such that

$$d_Z(w^*, z^*) \leq \psi(\varepsilon). \tag{3.3}$$

If there exists $c > 0$ such that $\psi(t) = ct$ for each $t \in \mathbb{R}_+$ then the coincidence problem (3.1) is said to be Ulam-Hyers stable.

For Ulam-Hyers stability of some integral and differential equations see L.P. Castro and A. Ramos [31], S.-M. Jung [63], I.A. Rus [110], I. A. Rus [111], while for Ulam-Hyers stability of the fixed point problems in metric spaces see I.A. Rus [108], M. Bota and A. Petruşel [21], P.T. Petru, A. Petruşel and J.C. Yao [94].

3.1 Metric regularity and Ulam-Hyers stability results for coincidence problems

In general, metric regularity deals with the study of equation of the type $y \in F(x)$, where $y \in X$ is fixed, for a multivalued operator $F : X \to P(Y)$. Many authors have obtained results in the metric regularity field among whom we remind A. L. Dontchev, A. S. Lewis, R. T. Rockafellar [40], A. L. Dontchev, A. S. Lewis [42], A. D. Ioffe [59], [60], L. A. Lyusternik [71] and others. A point x is an approximate solution of a generalized equation $y \in F(x)$ if the distance from the point y to the set $F(x)$ is small.

Let $F : X \to P(Y)$ be a multivalued operator between metric spaces (X, d) and (Y, d) and $U \subseteq X$, $V \subseteq Y$ given subsets. According to A. D. Ioffe [59] and B. S. Mordukhovich [80], F is said to *cover on* (or to be *open at a linear rate*) with respect to $U \times V$ if there exists a positive constant a such that

$$F(B(x, r)) \supseteq B(F(x) \cap V, ar), \text{ for all } x \in U, r > 0 : B(x, r) \subseteq U. \tag{3.4}$$

The supremum of all constants satisfying inclusion (3.4) is called *modulus of open covering* of F with respect to $U \times V$ and is denoted by $cov_{U\times V}F$. In one of its several manifestations, known and widely employed under the name of *metric regularity*, it takes the form of an inequality providing an estimation for how far a point x is from being a solution to the generalized equation $y \in F(x)$. In the most developed theorems of subdifferential calculus, all qualification conditions appear to be regularity/ open covering conditions for certain multivalued operators, see A. D. Ioffe [59] and B. S. Mordukhovich [80].

The notion of open covering to the global case is the case in which $U = X$ and $V = Y$.

Definition 3.1.1 (A. D. Ioffe [59]). *A multivalued operators $F : X \to P(Y)$ between metric spaces (X, d) and (Y, d) is said cover on X (or to be globally open at a linear rate), provided that there exists a constant $a > 0$ such that*

$$F(B(x, r)) \supseteq B(F(x), ar), \text{ for all } x \in X, r > 0. \tag{3.5}$$

The supremum over all values a satisfying inclusion (3.5) *is called modulus of global covering of F and denoted for short by $cov(F)$ (instead of $cov_{X\times Y}F$).*

Remark 3.1.1 (A. Uderzo [123]).

(i) The open covering property of a multivalued operator admits several useful formulation. It is well known that an operator F fulfils Definition 3.1.1 if and only if there exists $l > 0$ such that

$$D(x, F^{-1}(y)) \leq lD(y, F(x)), \text{ for all } x \in X, y \in Y. \tag{3.6}$$

The infimum of all values l satisfying inequality (3.6) is called modulus of global metric regularity of F and denoted by $reg(F)$.

(ii) Another characterization of open covering/ metric regularity can be obtained in terms of Lipschitz behavior of the inverse multivalued operator. In fact F covers on X if and only if F^{-1} is Lipschitz continuous in Y and it holds

$$lip(F^{-1}) = \frac{1}{cov(F)}.$$

Additionally suppose that T and S are onto and let $F : X \times Y \to P(X) \times P(Y)$ be defined by $F(x, y) = T^{-1}(y) \times S(u)$. We can deduce that $F^{-1} : X \times Y \to P(X) \times P(Y)$ is defined by $F^{-1}(u, v) = S^{-1}(v) \times T(u)$.

Lemma 3.1.1 (O. Mleşniţe and A. Petruşel [76]). *Under the above conditions, we have that*

$$CP(S, T) = Fix(F) = Fix(F^{-1}).$$

Let (X, d) and (Y, ρ) be two metric spaces and the following two metrics on $X \times Y$:

$$d^*((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2), \text{ for all } (x_1, y_1), (x_2, y_2) \in X \times Y$$

$$d_*((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), \rho(y_1, y_2)\}, \text{ for all } (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Denote by H_{d^*} and H_{d_*} the Hausdorff-Pompeiu functionals on $P(X \times Y)$ generated by d^* and d_* respectively.

The following main results are some generalizations of the main theorems of A. V. Blaga [19].

Theorem 3.1.1 (O. Mleşniţe [79]). *Let (X, d) and (Y, ρ) be two complete metric spaces. Let $T, S : X \to P(Y)$ be two onto multivalued operators, such that:*

(i) $T : X \to P_{cl}(Y)$ *is a contraction with constant* $k_T < 1$;

(ii) $S : X \to P(Y)$ *is metrically regular on* X *with constant* $k_S \in (0, 1)$ *and* $S^{-1}(y)$ *is closed for each* $y \in Y$.

Then there exists at least one solution of multivalued coincidence problem (3.1).

If, in addition, S^{-1} *and* T *have compact values then the problem* (3.1) *is Ulam-Hyers stable.*

Theorem 3.1.2 (O. Mleşniţe [79]). *Let (X, d) and (Y, ρ) be two complete metric spaces. Let $T, S : X \to P(Y)$ be two onto multivalued operators, such that:*

(i) $T : X \to P_{cl}(Y)$ *satisfy the following relation:*

$$\exists\ k_T > 2 \text{ such that } D_\rho(x, T^{-1}(y)) \geq k_T \cdot D_d(y, T(x)), \text{ for each } (x, y) \in X \times Y;$$

(ii) $S : X \to P(Y)$ *is metrically regular on* X *with constant* $k_S \in (0, \frac{1}{2})$ *and* $S^{-1}(y)$ *is closed, for all* $y \in Y$;

(iii) $(x, y) \in X \times Y$ *if and only if* $(y, x) \in X \times Y$.

Then there exists at least one solution of the multivalued coincidence problem (3.1).

If, in addition, S^{-1} *and* T *have compact values then the problem* (3.1) *is Ulam-Hyers stable.*

3.2 Existence and Ulam-Hyers stability results for coincidence problems

In this section, we will present some existence and Ulam-Hyers stability results for coincidence problems with multivalued operators using the weakly Picard operator technique. These results are based on the following works W. A. Kirk and B. Sims [65], O. Mleşniţe and A. Petruşel [76], A. Petruşel [90], I. A. Rus [108], [102], I.A. Rus, A. Petruşel and A. Sîntămărian [115]

Let (X, d) and (Y, ρ) be two metric spaces and the following two metrics on $X \times Y$:

$$d^*((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \rho(y_1, y_2), \text{ for all } (x_1, y_1), (x_2, y_2) \in X \times Y$$

$$d_*((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), \rho(y_1, y_2)\}, \text{ for all } (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Denote by H_{d^*} and H_{d_*} the Hausdorff-Pompeiu functionals on $P(X \times Y)$ generated by d^* and d_* respectively.

Theorem 3.2.1 (O. Mleşniţe and A. Petruşel [76]). *Let (X, d) and (Y, ρ) be two complete metric spaces. Let $T, S : X \to P(Y)$ be two multivalued operators such that:*

(i) $T : X \to P(Y)$ is an onto strong k_T-dilatation with constant $k_T > 1$ and $T^{-1}(y)$ is closed for each $y \in Y$;

(ii) $S : X \to P_{cl}(Y)$ is a k_S-contraction.

Then there exists at least one solution of the multivalued coincidence problem (3.1). If, in addition the multivalued operators S and T^{-1} have compact values and T is metrically regular on X with constant $l > 0$ then the multivalued coincidence problem (3.1) is Ulam-Hyers stable.

Remark 3.2.1. *A similar result take place if we replace, in the proof of the above theorem, the metric d^* with d_* and H_{d^*} with H_{d_*}.*

Next we present some data dependence results for the Ulam-Hyers stability of the multivalued coincidence problems of two pair of multivalued operators.

Theorem 3.2.2 (O. Mleşniţe and A. Petruşel [76]). *Let (X, d) and (Y, ρ) be two metric spaces and $T_i, S_i : X \to P(Y)$, $i \in \{1, 2\}$ be four multivalued operators. Consider the following coincidence problems with multivalued operators:*

$$T_1(x) \cap S_1(x) \neq \emptyset \tag{3.7}$$

and

$$T_2(x) \cap S_2(x) \neq \emptyset. \tag{3.8}$$

Let us consider the sets:

$$C_{i\varepsilon} := \{x \in X | D_\rho(T_i(x), S_i(x)) \leq \varepsilon\}, i \in \{1, 2\}.$$

If the following conditions are satisfied:

(i) $C(T_2, S_2) \subseteq C(T_1, S_1)$;

(ii) the multivalued coincidence point problem (3.8) *is Ulam-Hyers stable;*

(iii) $C_{1\varepsilon} \subseteq C_{2\varepsilon}$, for each $\varepsilon > 0$;

then, the multivalued coincidence point problem (3.7) *is Ulam-Hyers stable.*

3.3 Coincidence results by fixed point theorems in generalized metric spaces

The purpose of this section is to present some existence and Ulam-Hyers stability results for fixed point problems coincidence point problems with multivalued operators. The approach is based on the weakly Picard operator technique in the setting of generalized metric spaces in the sense of Perov, i.e., spaces endowed with vector valued metrics $d : X \times X \to \mathbb{R}^m_+$. Using the cartesian product technique for two multivalued operators, our results improve some recent theorems in the literature, see M. Bota and A. Petruşel [21], T. P. Petru, A. Petruşel and J.-C. Yao [94], I. A. Rus [108], [110], [111].

Let (X, d) and (Y, ρ) be two metric spaces. Let $Z := X \times Y$ and define on $Z \times Z$ the vector metric $d^V(u, v) := \begin{pmatrix} d(u_1, v_1) \\ \rho(u_2, v_2) \end{pmatrix}$, for each $u = (u_1, u_2), v = (v_1, v_2) \in Z$.

In the same framework as Chapter 1, let us consider a Hausdorff-Pompeiu type vector functional given by $H^* : (P(X) \times P(Y)) \times (P(X) \times P(Y)) \to \mathbb{R}^2_+$ given by

$$H^*(A \times B, U \times V) := \begin{pmatrix} H_d(A, U) \\ H_\rho(B, V) \end{pmatrix}.$$

From the definition, it follows that H^* is a vector metric on $P_{cl}(X) \times P_{cl}(Y)$.

We present now an existence and Ulam-Hyers stability result for the multivalued coincidence problem.

Theorem 3.3.1 (O. Mleşniţe and A. Petruşel [76]). *Let (X, d) and (Y, ρ) be two complete metric spaces. Let $T, S : X \to P(Y)$ be two multivalued operators such that:*

(i) $T : X \to P(Y)$ is an onto strong k_T-dilatation and $T^{-1}(y)$ is closed for each $y \in Y$;

(ii) $S : X \to P_{cl}(Y)$ is k_S-Lipschitz;

(iii) $\frac{k_S}{k_T} < 1$.

Then there exists at least one solution of the multivalued coincidence problem (3.1). If, in addition the multivalued operators S, T^{-1} have compact values and T is metrically regular on X with constant $l > 0$, then the multivalued coincidence problem (3.1) is Ulam-Hyers v-stable.

3.4 A Leray-Schauder condition to the coincidence problems

In this section we establish several coincidence results for multivalued operators by using Leray-Schauder type condition and Theorem 2.5.2. We use the concepts presented in the section 2.5. These results are based on the following works: V. Barbu [15], J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50].

Definition 3.4.1. *An operator $F \subset X \times X$ is said to be accretive if and only if $\|x - y\| \leq \|x - y + \lambda(u - v)\|$ for all $x, y \in Dom(F)$, for each $u \in F(x)$ and $v \in F(y)$, and for all $\lambda > 0$. If moreover, $R(F + I) = X$, we say that F is m-accretive.*

A detailed account of theory and applications of accretive operators may be found, for instance, in the monograph V. Barbu [15].

Theorem 3.4.1 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50]). *Let X be a normed space and let Y be Banach space. Consider a nonempty subset D of X. Suppose that $t : D \to P(Y)$ is a multivalued operator and $s : D \to Y$ is an operator which satisfy:*

1. *$R(t) = Y$ and $t^{-1} : Y \to D$ is a singlevalued continuous and compact operator,*

2. *s is continuous and it maps bounded subsets into bounded subsets,*

3. *There exists $R > 0$ such that*

$$\|x\|_X \geq R,\ x \in D \quad \Rightarrow \quad \lambda s(x) \notin t(x) \quad \forall \lambda \in (0,1). \tag{3.9}$$

Then there exists $x_0 \in D$ with $s(x_0) \in t(x_0)$.

Corollary 3.4.1 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50])**.** *Let X be a Banach space and $F : Dom(F) \to P(X)$ an m-accretive operator such that $0 \in F(0)$ and $s : Dom(F) \to X$ a continuous operator. Suppose that the following conditions are fulfilled:*

1. *J_λ^F is compact,*

2. *there exists $R > 0$ such that $\|s(x)\| \leq a + b\|x\|$ whenever $x \in Dom(F)$ with $\|x\| \geq R$.*

Then given $\rho > b$ there exists $x_0 \in Dom(F)$ such that $s(x_0) \in \rho x_0 + F(x_0)$.

Next result works with operators which are condensing but not necessarily k-set contractive, examples of such operators can be found for instance in J. Appel [5], J. M. Ayerbe Toledano, T. Domínguez-Benavides and G. Lopez Acedo [13].

Theorem 3.4.2 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50])**.** *Let $(X, \|\cdot\|_X)$ be a normed space and let $(Y, \|\cdot\|_Y)$ be a Banach space. Assume that $t : X \to P(Y)$ is a multivalued operator with $R(t) = Y$ such that $t^{-1} : Y \to X$ is singlevalued nonexpansive and $s : Dom(t) \to Y$ a continuous α-condensing operator satisfying that there exists $R > 0$ and $y_0 \in Y$ such that*

$$\|x - t^{-1}y_0\|_X \geq R \Rightarrow \mu s(x) + (1-\mu)y_0 \notin t(x) \qquad \forall \mu \in (0,1). \tag{3.10}$$

Then there exists $x_0 \in X$ such that $s(x_0) \in t(x_0)$.

Corollary 3.4.2 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50])**.** *Let $(X, \|\cdot\|_X)$ be a normed space and let $(Y, \|\cdot\|_Y)$ be a Banach space. Assume that $t : X \to Y$ is an expansive surjection and $s : X \to Y$ a continuous, bounded and α-condensing operator. Then there exists $x_0 \in X$ such that $s(x_0) = t(x_0)$.*

Corollary 3.4.3 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50])**.** *Let $(X, \|\cdot\|_X)$ be a normed space and let $(Y, \|\cdot\|_Y)$ be a Banach space. Assume that $t : X \to Y$ is an expansive surjection and $s : X \to Y$ a continuous, α-condensing operator satisfying that there exists $R > 0$ and $y_0 \in Y$ such that*

$$\|x - t^{-1}y_0\|_X \geq R \Rightarrow \|s(x) - y_0\|_Y \leq \|x - t^{-1}(y_0)\|_X. \tag{3.11}$$

Then there exists $x_0 \in X$ such that $s(x_0) = t(x_0)$.

Chapter 4

Applications

The purpose of this chapter is to present some applications of the results presented in this thesis. Firstly is given an application regarding Ulam-Hyers stability for differential equations and operatorial inclusions and then we study the existence of classical and of strong solution to a diferential equation of first order and second order.

The references which were used to develop this chapter are: J. Appell and P.P. Zabrejko [6], H. Brezis and W. Strauss [23], J. Garcia-Falset [47], J. Garcia-Falset and O. Mleşniţe [49], J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50], K. Goebel [52], O. Mleşniţe [74], [77]

4.1 Ulam-Hyers stability for differential equations

In this section, we establish some new existence, uniqueness and Ulam-Hyers stability results for differential equations. In this case, the following theorem is an application of the Theorems 2.3.3 and 2.3.4.

Application 1.(J. Garcia-Falset and O. Mleşniţe [49]) Let us consider the differential equation

$$\begin{cases} u'(t) = f(t, u(t)), \\ u(0) = \xi \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

- (i) $f(t, \cdot)$ is a continuous function for almost all $t \geq 0$;
- (ii) $f(\cdot, u)$ is a measurable function for all $u \in \mathbb{R}$;
- (iii) Lipschitz inequality, i.e., $|f(t, x) - f(t, y)| \leq L(t)|x - y|$, where $L : \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable function on the interval $(0, \infty)$;
- (iv) $\int_0^t f(\tau, 0)d\tau = O(e^{\int_0^t L(\tau)d\tau}) := \{u \in C([0, \infty)) \ : \ |u(t)| \leq Me^{\int_0^t L(\tau)d\tau} + N\}$.

Then equation (4.1) has a unique solution u_ξ for every $\xi \in \mathbb{R}$,

$$u_\xi \in C([0,+\infty)) := \{u : [0,+\infty) \to \mathbb{R} \text{ continuous }\},$$

and moreover it is Ulam-Hyers stable.

Let us consider the sets

$$X = \{u \in C([0,+\infty)) \mid u(t) = O(e^{\int_0^t L(\tau)d\tau})\},$$

and the metric $d_p : X \times X \to \mathbb{R}^+$ defined by $d_p(u,v) = \sup\limits_{t\in[0,+\infty)} \{|u(t)-v(t)| \cdot e^{-p\int_0^t L(\tau)d\tau}\}$ where $p > 1$, $Y = BC([0,+\infty))$ is the set of bounded continuous functions on $[0,+\infty)$ and we endow this set with the metric $\rho : Y \times Y \to \mathbb{R}^+$ defined by $\rho(u,v) = \|u-v\|_\infty = \sup\limits_{t\in[0,+\infty)} |u(t)-v(t)|$, then (Y,ρ) is a complete metric space.

We have that (X, d_p) and (Y,ρ) are complete metric spaces.

We define the operators $T, S : X \to Y$ by

$$Tu(t) = u(t) \cdot e^{-p\int_0^t L(\tau)d\tau} \quad \text{and} \quad Su(t) = \left\{\int_0^t f(\tau, u(\tau))d\tau + \xi\right\} e^{-p\int_0^t L(\tau)d\tau}.$$

Equation (4.1) can be written as a coincidence problem in the following form:

$$\text{find } u \in X \text{ such that } Tu = Su. \tag{4.2}$$

T and S fulfill the hypotheses of Theorem 2.3.3, so the coincidence problem (4.2) has a unique solution in X, this means that there exists $\bar{x} \in X$ such that $S(\bar{x}) = T(\bar{x})$.

For the second conclusion, if we define $\beta(r) := r - \dfrac{r}{p}$ $(p > 1)$, since β is a continuous strictly increasing function, $\lim\limits_{r\to 0^+} \beta(r) = 0$ and $\lim\limits_{r\to+\infty} \beta(r) = +\infty$, then β is strictly increasing and onto.

All the hypotheses of Theorem 2.3.4 hold, so the coincidence problem (4.2) is Ulam-Hyers stable. So, equation (4.1) is Ulam-Hyers stable.

Next, we present an application of Theorem 2.3.2.

Application 2.(O. Mleşniţe [77]) Let us consider the same differential equation (4.1) where $f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies all the conditions of Application 1. If in addition f satisfies the following condition

- $f(t,\gamma u) \geq \gamma f(t,u)$ for all $\gamma \geq 1, t > 0, u \in \mathbb{R}$,

then the differential equation (4.1) has a unique solution for every $\xi \in \mathbb{R}$ and moreover it is Ulam-Hyers stable.

The operators S and T satisfy the conditions of Theorem 2.3.2, so there exists $\bar{x} \in X$ such that $S(\bar{x}) = T(\bar{x})$.

Next we prove that the equation (4.1) is Ulam-Hyers stable.

We have $(T^{-1}y)(t) = y(t) \cdot e^{p\int_0^t L(\tau)d\tau}$. We prove that $d(y, S(T^{-1}(y))) \leq \alpha d(Ty, Sy)$, for all $y \in T(A)$. We obtain that

$$S(T^{-1}(y))(t) = e^{-p\int_0^t L(\tau)d\tau}\left\{\int_0^t f(\tau, y(\tau)e^{p\int_0^t L(\tau)d\tau})d\tau + \xi\right\}.$$

By calculations we get

$$|y(t) - S(T^{-1}(y))(t)| = \left|y(t) - e^{-p\int_0^t L(\tau)d\tau}\left\{\int_0^t f(\tau, y(\tau)e^{p\int_0^t L(\tau)d\tau})d\tau + \xi\right\}\right| \leq$$

$$\leq e^{p\int_0^t L(\tau)d\tau}|(Sy)(t) - (Ty)(t)|.$$

Since, all the condition of Theorem 2.3.2 hold, then the differential equation (4.1) is Ulam-Hyers stable.

4.2 Ulam-Hyers stability for operatorial inclusions

The aim of this section is to prove an Ulam-Hyers stability theorem for a multivalued Cauchy problem corresponding to a first order differential inclusion.

Let us consider the following multivalued Cauchy problem:

$$\begin{cases} x'(t) \in F(t, x(t)), \text{ a.e. } t \in [a, b]; \\ x(a) = \alpha, \end{cases} \tag{4.3}$$

where $\alpha \in \mathbb{R}^n$ and $F : [a, b] \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ is a multivalued operator. We will denote by $\int_a^b F(s, x(s))ds$ (where $x : [a, b] \to \mathbb{R}^n$ is a given function) the multivalued integral in Aumann' sense, see J.-P. Aubin and H. Frankowska [12].

Definition 4.2.1. *Let $F : [a, b] \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ a multivalued operator, $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}^n$. A function $\varphi : [a, T] \to \mathbb{R}^n$ is called solution to problem* (4.3) *if and only if $T \leq b$, φ is absolutely continuous on $[a, T]$ and satisfy the relations:*

$$\begin{cases} \varphi'(t) \in F(t, \varphi(t)), \quad a.e. \ on \ [a, T]; \\ \varphi(a) = \alpha. \end{cases}$$

The equivalence between the above differential inclusion and an integral inclusion is given by the following lemma:

Lemma 4.2.1. *Let $I \subset \mathbb{R}$ an interval and $F : I \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ be an upper semi-continuous multivalued operator. Then $x : I \to \mathbb{R}$ is a solution for differential inclusion*

$$x'(t) \in F(t, x(t)) \tag{4.4}$$

if and only if

$$x(t_2) \in x(t_1) + \int_{t_1}^{t_2} F(t, x(t))dt, \ \textit{for each } t_1, t_2 \in I. \tag{4.5}$$

Taking into account of Lemma 4.2.1 we deduce that the problem (4.3) is equivalent to an integral inclusion of Volterra type:

$$x(t) \in \alpha + \int_a^t F(s, x(s))ds, \quad t \in [a, b]. \tag{4.6}$$

The result with respect to the Ulam-Hyers stability of the Cauchy problem (4.3) is the following theorem.

Theorem 4.2.1 (O. Mleşniţe [74]). *Let $F : [a, b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ such that:*

(a) there exists an integrable function $M : [a, b] \to \mathbb{R}_+$ such that for each $u \in \mathbb{R}^n$ we have $F(s, u) \subset M(s)B(0, 1)$, a.e. $s \in [a, b]$;

(b) for each $u \in \mathbb{R}^n$, $F(\cdot, u) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is measurable;

(c) for each $u \in \mathbb{R}^n$, $F(\cdot, u) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is lower semi-continuous;

(d) there exists a continuous function $p : [a, b] \to \mathbb{R}_+$ such that for each $s \in [a, b]$ and each $u, v \in \mathbb{R}^n$ we have that:

$$H(F(s, u), F(s, v)) \leq p(s) \cdot |u - v|. \tag{4.7}$$

Then the following conclusions hold:

(i) there exists at least one solution for the Cauchy problem (4.3)*;*

(ii) the Cauchy problem (4.3) *is Ulam-Hyers stable.*

4.3 Existence of solution to a differential equation of first order

In this section, we intend to obtain several version, without invoking degree theory, of the coincidence problems where the operators s and t can both nonlinear. For these types of applications we apply Corollary 2.5.2. These results are based on the following work: J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50].

In this section we are concerned to find an absolutely continuous function $u : [0, 1] \to \mathbb{R}^n$ such that its derivative $u' \in L^1(0, 1; \mathbb{R}^n)$ satisfies almost for every point in $(0, 1)$ the following differential equation

$$\begin{cases} u'(t) - g(t, u(t), u'(t)) = f(t), & t \in (0, 1) \text{ a.e.} \\ u(0) = \xi \in \mathbb{R}^n, \end{cases} \tag{4.8}$$

where $f \in L^1(0, 1; \mathbb{R}^n)$ is a fixed function and $g : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function. A such function u is called strong solution of equation (4.8).

Consider the Banach space $(\mathbb{R}^n, \|\cdot\|_n)$ and let $L^1(0, 1; \mathbb{R}^n)$ be the Banach space of Bochner integrable functions $x : [0, 1] \to \mathbb{R}^n$ endowed with the norm

$$\|x\|_1 = \int_0^1 \|x(t)\|_n dt.$$

It is well known that if $x : [0,1] \to \mathbb{R}^n$ is absolutely continuous, then it is almost everywhere differentiable on $[0,1]$, its derivative $x' \in L^1(0,1;\mathbb{R}^n)$ and

$$x(t) = x(0) + \int_0^t x'(s)ds.$$

First, let us notice that equation (4.8) is equivalent to the differential equation

$$\begin{cases} u'(t) - g(t, u(t) + \xi, u'(t)) = f(t), & t \in (0,1) \text{ a.e.} \\ u(0) = 0, \end{cases} \tag{4.9}$$

Thus, our goal will be to study the existence of a strong solution of equation (4.9).

Let us introduce the Sobolev space $W^{1,1}(0,1;\mathbb{R}^n)$ as the space of all absolutely continuous functions. Then we can write this space as:

$$W^{1,1}(0,1;\mathbb{R}^n) := \left\{u \in L^1(0,1;\mathbb{R}^n) : u' \in L^1(0,1;\mathbb{R}^n)\right\},$$

The space $W^{1,1}(0,1;\mathbb{R}^n)$ can be endowed with the norm

$$\|u\|_{1,1} := \max\{\|u\|_1, \|u'\|_1\},$$

where $\|\cdot\|_1$ is the usual norm in $L^1(0,1;\mathbb{R}^n)$. $(W^{1,1}(0,1;\mathbb{R}^n), \|\cdot\|_{1,1})$ is a Banach space.

Now we can consider the following subspace $X := \{u \in W^{1,1}(0,1;\mathbb{R}^n) : \; u(0) = 0\}$. This is a closed subspace of $(W^{1,1}(0,1;\mathbb{R}^n), \|\cdot\|_{1,1})$ and thus it is also a Banach space.

Lemma 4.3.1. *Let u be an element in X. Then $\|u\|_{1,1} = \|u'\|_1$.*

Lemma 4.3.2. *Let f be a fixed element of $L^1(0,1;\mathbb{R}^n)$. The operator $T : X \to L^1(0,1;\mathbb{R}^n)$ defined by $T(u)(t) = u'(t) - f(t)$ is an expansive bijection.*

Let $\mathcal{M}(0,1;\mathbb{R}^n)$ be the set of all measurable functions $\varphi : [0,1] \to \mathbb{R}^n$. If $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, then f defines an operator $N_f : \mathcal{M}(0,1;\mathbb{R}^n) \to \mathcal{M}(0,1;\mathbb{R}^n)$ by $N_f(\varphi)(t) := f(t, \varphi(t))$. This operator is called the superposition (or Nemytskii) operator generated by f. The next three lemmas are of foremost importance for our subsequent analysis.

Lemma 4.3.3. *Let $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function, if there exist a constant $b \geq 0$ and a function $a(\cdot) \in L^1_+(0,1;\mathbb{R})$ such that*

$$\|f(t,x)\|_n \leq a(t) + b\|x\|_n,$$

then N_f maps continuously $L^1(0,1;\mathbb{R}^n)$ into itself.

If we argue as J. Appell and P.P. Zabrejko in [6, Lemma 9.5] we obtain:

Lemma 4.3.4. *Let $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function, if there exist a constant $b \geq 0$ and a function $a(\cdot) \in L^1_+(0,1;\mathbb{R})$ such that*

$$\|g(t,x,y)\|_n \leq a(t) + b(\|x\|_n + \|y\|_n),$$

then the map $N_g : W^{1,1}(0,1;\mathbb{R}^n) \to L^1(0,1;\mathbb{R}^n)$ defined by

$$N_g(\varphi)(t) = g(t,\varphi(t),\varphi'(t))$$

is continuous.

Lemma 4.3.5. *Let $g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function such that there exist $a \in L^1_+(0,1,\mathbb{R})$, $b, k > 0$ satisfying that*

1. $\|g(t,x,0)\|_n \leq a(t) + b\|x\|_n$,
2. $\|g(t,x,y_1) - g(t,x,y_2)\|_n \leq k\|y_1 - y_2\|_n$.

Then, the operator $N_g : X \to L^1(0,1;\mathbb{R}^n)$ is $2k$-set contractive.

Now, for studying the existence of a strong solution to equation (4.9), we define

$$T : X \to L^1(0,1;\mathbb{R}^n) \;\text{ by }\; T(u) = u' - f$$

and

$$S : X \to L^1(0,1;\mathbb{R}^n) \;\text{ by }\; S(u) = N_{\tilde{g}}(u),$$

where $\tilde{g}(t,x,y) = g(t,x+\xi,y)$.

Thus, to show that equation (4.9) has a solution is to see that the coincidence problem, $T(u) = S(u)$ admits a solution.

Theorem 4.3.1 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50])**.** *If $\max\{b+k, 2k\} < 1$, equation (4.9) has at least a solution in the Sobolev space $W^{1,1}(0,1;\mathbb{R}^n)$.*

Example 4.3.1.

$$\begin{cases} u'(t) - \frac{\cos(u(t))}{\sqrt{t}} - \frac{u(t)+\sin(u'(t))}{2\sqrt{t+2}} = f(t), \quad t \in (0,1) \\ u(0) = \xi, \end{cases} \tag{4.10}$$

has a strong solution since in this example we have that $g(t,x,y) = \frac{\cos(x)}{\sqrt{t}} + \frac{x+\sin(y)}{2\sqrt{t+2}}$ and therefore $|g(t,x,0)| \leq \frac{1}{\sqrt{t}} + \frac{1}{2\sqrt{2}}|x|$ and $|g(t,x,y_1) - g(t,x,y_2)| \leq \frac{1}{2\sqrt{2}}|y_1 - y_2|$, which implies that g fulfills the conditions of Theorem 4.3.1.

4.4 Existence of solutions to a differential equation of second order

In this section, we want to study the existence of classical and strong solutions to a differential equation with a non homogeneous Dirichlet conditions (we refer to equation (4.11)). For these types of applications we apply Theorem 2.3.3 and Corollary 3.4.3. These results are based on the following work: J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50].

First we will study the existence of classical solutions to a differential equation of second order.

Let $Y := (C([0,1]), \|\cdot\|_0)$ be the Banach space of the continuous functions $u : [0,1] \to \mathbb{R}$, where $\|u\|_0 := \sup\{|u(t)| : t \in [0,1]\}$.

Denote $C^2([0,1]) := \{u : [0,1] \to \mathbb{R} : u'' \in C([0,1])\}$. This allows us to introduce the following linear space

$$X := \{u \in C^2([0,1]) : u(0) = u(1) = 0\},$$

if on this linear space we define the norm $\|u\|_2 := \max\{\|u\|_0, \|u'\|_0, \|u''\|_0\}$, then $(X, \|\cdot\|_2)$ is a Banach space.

On the other hand, if $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, then g defines an operator $N_g : Y \to X$ by $N_g(u)(t) = g(t, u(t), u''(t))$. This operator is called superposition (or Nemytskii) operator generated by g. The following lemmas are of foremost importance for our subsequent analysis.

Lemma 4.4.1. *Let u be an element in X. Then $\|u\|_2 = \|u''\|_0$.*

Lemma 4.4.2. *Let f a fixed element of Y. The operator $T : X \to Y$ defined by $T(u)(t) = u''(t) - f(t)$ is an expansive surjection.*

Lemma 4.4.3. *Let $g : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function and there exists $k \in [0,1)$ such that*

$$|g(t,x,y,z) - g(t,u,v,w)| \le k \max\{|x-u|, |y-v|, |z-w|\},$$

for all $(t,x,y,z), (t,u,v,w) \in [0,1] \times \mathbb{R}^3$. Then the superposition operator $N_g : X \to Y$ is k-contractive.

Lemma 4.4.4. *Let $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the superposition operator $N_g : X \to Y$ is a continuous and compact operator.*

We want to study the existence of classical solutions for the differential equation with a non homogeneous Dirichlet conditions

$$\begin{cases} u''(t) - g(t, u(t), u'(t)) = f(t), \ t \in [0,1], \\ u(0) = \xi, \quad u(1) = \nu, \end{cases} \tag{4.11}$$

where $f \in Y$ is a fixed function.

First, let us notice that equation (4.11) is equivalent to the differential equation with the Dirichlet condition

$$\begin{cases} u''(t) - g(t, u(t) + (\nu - \xi)t + \xi, u'(t) - (\nu - \xi)) = f(t), \ t \in [0, 1], \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{4.12}$$

Thus, our goal will be to study the existence of classical solutions to equation (4.12). For this purpose we define

$$T : X \to Y \text{ by } T(u)(t) = u''(t) - f(t)$$

and

$$S : X \to Y \text{ by } S(u)(t) = N_{\tilde{g}}(u)(t),$$

where $\tilde{g}(t, x, y) = g(t, x + (\nu - \xi)t + \xi, y - (\nu - \xi))$.

To show that equation (4.12) has a classical solution is to find an element $u_0 \in X$ such that $T(u_0) = S(u_0)$. That is, to see that the coincidence problem has a solution.

As a consequence of the Lemmas 4.4.1, 4.4.2, 4.4.3 and Theorem 2.3.3 is the following result.

Theorem 4.4.1 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50])**.** *If $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is under the hypotheses of Lemma 4.4.3, then Problem* (4.12) *has a unique solution.*

Another consequence of Lemmas 4.4.1, 4.4.2, 4.4.4 and Corollary 3.4.2 is the following theorem.

Theorem 4.4.2. *If $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then Problem* (4.12) *has a solution.*

Theorem 4.4.3 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50])**.** *If $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with the properties:*

(a) there exists $M > 0$ such that if $|x| > M$, then $xg(t, x + t(\nu - \xi) + \xi, 0) > \max\{0, -xf(t)\}$,

(b) there exists $A, B > 0$ such that if $|x| \leq M$, then $|g(t, x + t(\nu - \xi) + \xi, y)| < A(y + (\nu - \xi))^2 + B$ for all $t \in [0, 1]$ and for all $y \in \mathbb{R}$,

then Problem (4.12) *has a solution.*

Example 4.4.1.

$$\begin{cases} u''(t) - u^3(t) = f(t), t \in [0, 1], \\ u(0) = \xi, \quad u(1) = \nu, \end{cases} \tag{4.13}$$

has a solution since in this example we have that $g(t, x, p) = x^3$ and therefore g fulfills the conditions of Theorem 4.4.3.

Next, we will study the existence of strong solutions to a differential equation of second order.

Consider the interval $I := (0,1)$ and define the Sobolev space

$$W^{2,1}(I) := \{u \in W^{1,1}(I) : u' \in W^{1,1}(I)\}.$$

The space $W^{2,1}(I)$ can be endowed with the norm

$$\|u\|_{2,1} := \max\{\|u\|_1, \|u'\|_1, \|u''\|_1\},$$

where $\|\cdot\|_1$ is the usual norm in $L^1(I)$. It is well know that $(W^{2,1}(I), \|\cdot\|_{2,1})$ is a Banach space.

Now we can consider the following subspace $X := \{u \in W^{2,1}(I) : u(0) = 0, u'(0) = 0\}$. This is a closed subspace of $(W^{2,1}(I), \|\cdot\|_{2,1})$ and thus it is also a Banach space.

On the other hand, if $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, then g defines an operator $N_g : L^1(I) \to \mathcal{M}(I)$ by $N_g(u)(t) = g(t, u(t), u'(t))$, where $\mathcal{M}(I)$ is the set of all measurable functions $\phi : I \to \mathbb{R}$. This operator is called superposition operator generated by g. The following three lemmas are of foremost importance for our subsequent analysis.

Lemma 4.4.5. *Let u be an element in X. Then $\|u\|_{2,1} = \|u''\|_1$.*

Lemma 4.4.6. *Let f a fixed element of $L^1(I)$. The operator $T : X \to L^1(I)$ defined by $T(u)(t) = u''(t) - f(t)$ is an expansive surjection.*

Lemma 4.4.7. *Let $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that there exist $b, c > 0$ and $a \in L^1_+(I)$ satisfying that $|g(t,x,y)| \leq a(t) + b|x| + c|y|$. Then the superposition operator $N_g : X \to L^1(I)$ is a continuous and compact operator.*

We want to study the existence of at least a function $u \in W^{2,1}(I)$ such that

$$\begin{cases} u''(t) - g(t, u(t), u'(t)) = f(t), \ t \in (0,1) \text{ a.e.} \\ u(0) = \xi, u(1) = \nu, \end{cases} \tag{4.14}$$

where $f \in L^1(I)$ is a fixed function.

First, let us notice that equation (4.14) is equivalent to the differential equation

$$\begin{cases} u''(t) - g(t, u(t) + t(\nu - \xi) + \xi, u'(t) + (\nu - \xi)) = f(t), \ t \in (0,1) \text{ a.e.} \\ u(0) = 0 = u(1). \end{cases} \tag{4.15}$$

Thus, our goal will be to look for a function $u \in X$ satisfying equation (4.15). For this purpose we define

$$T : X \to L^1(I) \text{ by } T(u) = u'' - f$$

and

$$S : X \to L^1(I) \text{ by } S(u) = N_{\tilde{g}}(u),$$

where $\tilde{g}(t,x,y) = g(t, x + t(\nu - \xi) + \xi, y + (\nu - \xi))$.

To show that equation (4.15) has a solution is to find an element $u_0 \in X$ such that $T(u_0) = S(u_0)$. That is, to see that the coincidence problem has a solution.

Theorem 4.4.4 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50])**.** *If $b + c < 1$, equation (4.15) has at least a solution in $W^{2,1}(I)$.*

4.5 A nonlinear Dirichlet problem

In this section we will use the results of coincidence problems to obtain the existence of solution for a Dirichlet problem of the form (4.16). In order to find a solution of this type of problems we will apply Theorem 3.4.1. These applications are based on the work J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50].

Let Ω be a measurable subset on $\mathbb{R}^n$ which for simplicity will be assumed to be bounded.

The Sobolev space $W^{m,p}(\Omega)$ is the Banach space of all functions in $L^p(\Omega)$ all of whose weak derivatives up to order m also belong to $L^p(\Omega)$. The norm in this space is given by

$$\|u\|_{m,p} = \|u\|_p + \sum_{1\leq|\alpha|\leq m} \|D^\alpha u\|_p,$$

where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$, and $D^\alpha u = \frac{\partial^{\alpha_1+...+\alpha_n}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}} u$.

$W_0^{m,p}(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Next we shall study the existence of solutions in $L^1(\Omega)$ for the equation

$$\begin{cases} \Delta\rho(u(x)) = f(x, u(x)) & x \in \Omega \\ \rho(u(x)) = 0 & x \in \partial\Omega \end{cases} \tag{4.16}$$

Let us now specify the conditions assuring the existence of a solution for equation (4.16):

1. Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$.
2. $\rho \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}),\ \rho(0) = 0$.
3. There exists $C > 0$ and $\gamma \in \mathbb{R}^+$ with $\gamma > 1$ such that
$$\rho'(r) \geq C|r|^{\gamma-1} \text{ for each } r \in \mathbb{R} \setminus \{0\}.$$
4. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $|f(s, x)| \leq a(s) + b|x|$, where $a \in L^1(\Omega)$ and $b \geq 0$. This condition guarantees that the superposition operator associated to f,
$$N_f(u)(s) = f(s, u(s)),$$
acts form $L^1(\Omega)$ into $L^1(\Omega)$ and is continuous. We refer to [6] for background material on superposition operators.

H. Brezis and W. Strauss in [23] showed that under the above conditions (1) and (2), the operator

$$\begin{cases} D(P) = \{u \in L^1(\Omega) :\ \rho(u) \in W_0^{1,1}(\Omega),\ \Delta\rho(u) \in L^1(\Omega)\} \\ P(u) = \Delta\rho(u),\ u \in D(P) \end{cases} \tag{4.17}$$

is m-dissipative, which means that $-P$ is m-accretive.

Definition 4.5.1. *We say that $v \in L^1(\Omega)$ is a solution of Problem (4.16) whenever $v \in L^1(\Omega)$, $\rho(v) \in W_0^{1,1}(\Omega)$, $\Delta\rho(v) \in L^1(\Omega)$ and $\Delta\rho(v(x)) = f(x, v(x))$ a.e. $x \in \Omega$. That is, whenever $v \in D(P)$ is a solution of the coincidence problem $P(v) = N_f(v)$, where $D(P)$ and P are defined in (4.17).*

Theorem 4.5.1 (J. Garcia-Falset, C. A. Hernández-Linares and O. Mleşniţe [50])**.** *If Conditions (1-4) are fulfilled, then Problem (4.16) has a solution.*

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