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Fixed point methods for nonlinear differential systems with nonlocal conditions

Ph.D. Thesis Summary

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Contents

Introduction

The purpose of the present thesis is to study nonlinear differential systems with nonlocal conditions. We shall obtain existence and uniqueness results based on an operator approach that uses fixed point theorems.

The vector operator method

We shall apply the fixed point principles of Perov (a vector version of Banach's contraction principle), Schauder and Leray-Schauder. To this aim, the nonlocal problems will be expressed as fixed point equations of the type

$$u = T(u), \tag{1}$$

for some nonlinear operators T. Thus, a key element is represented by the specific expression of the corresponding operator T. In fact, in this thesis, equation (1) has a vector structure, namely

$$\begin{cases} x = T_1(x, y) \\ y = T_2(x, y), \end{cases}$$

where $u = (x, y)$, $T = (T_1, T_2)$, which allows the two terms T_1, T_2 to behave differently one from the other and also with respect to the two variables. This requires the use of matrices instead of constants, when Lipschitz, growth, or "a priori" boundedness conditions are imposed to T_1 and T_2. Correspondingly, the use of vector-valued norms is necessary. We note that we can easily generalize the approach of 2-dimensional systems to the n-dimensional case.

Historically, it was A.I. Perov who in 1964 [63] gave a vector version to Banach's contraction principle and applied it to differential systems, showing the advantage of matrices that are convergent to zero and of vector-valued metrics. More recently, R. Precup [65] showed that this method can be put in connection to other principles of nonlinear analysis, such as Schauder's, Leray-Schauder's and Krasnoselskii's theorems. Here, the author also explained that the use of vector-valued norms and, correspondingly of matrices that are convergent to zero, is more appropriate when treating systems of equations. Systems appear in the mathematical modelling of different processes and phenomena from various domains, such as physics, biology, chemistry, engineering, economy etc., when several quantities vary in time and interact.

The nonlocal problems studied in this thesis

First, in Chapter 2, we consider the problem with discrete nonlocal (polylocal) initial conditions

$$\begin{cases} x'(t) = f(t, x(t), y(t)) \\ y'(t) = g(t, x(t), y(t)) \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0 \qquad\qquad \text{(a.e. on } [0,1]) \\ y(0) + \sum_{k=1}^{m} \widetilde{a}_k y(t_k) = 0, \end{cases}$$

where t_k are given points with $0 \le t_1 \le t_2 \le \ ... \ \le t_m < 1$.

Then, instead of discrete conditions we shall consider nonlocal conditions given by two linear and continuous functionals, $\alpha, \beta : C[0,1] \to \mathbb{R}$, namely

$$x(0) = \alpha[x], \quad y(0) = \beta[y] \quad \text{(uncoupled conditions)}$$

and also

$$x(0) = \alpha[y], \quad y(0) = \beta[x] \quad \text{(coupled conditions).}$$

In Chapter 3, we deal with the n-dimensional case and the general coupled linear condition

$$u(0) = \alpha[u],$$

where $u = (u_1,\ u_2,\ ...,\ u_n)$ and $\alpha : C([0,1], \mathbb{R}^n) \to \mathbb{R}^n$, while in Chapter 4 we discuss second-order differential equations and systems of the form

$$\begin{cases} u''(t) = f(t, u(t), v(t)) \\ v''(t) = g(t, u(t), v(t)) \\ u(0) = 0,\ u(t_0) = \phi(u(\eta), v(\eta)) \\ v(0) = 0,\ v(t_0) = \psi(u(\eta), v(\eta)), \end{cases}$$

$(t \in [0,1])$, under nonlocal conditions on three points $0, \eta, t_0$.

The methods are then addapted in Chapter 5 in order to discuss the problem

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) \\ x(0) = \alpha[x,y] \\ y(0) = \beta[x,y] \end{cases}, \qquad t \in [0,1]$$

with nonlocal conditions given by nonlinear functionals $\alpha, \beta : C[0,1]^2 \to \mathbb{R}$. Here, the nonlinear operators T_1, T_2 will act on the product space $(C[0,1] \times \mathbb{R})^2$ embedding in this way the nonlinear conditions.

The methodology that we use throughout this work is finally applied in Chapter 6 to a special class of problems, namely to impulsive systems of the form

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) \\ \Delta x|_{t=t_0} = I_1(x(t_0)), \quad \Delta y|_{t=t_0} = I_2(y(t_0)), \\ x(0) = \alpha_1[x], \qquad\qquad y(0) = \alpha_2[y], \end{cases} \qquad t \in (0,1),\ t \neq t_0.$$

Here $t_0 \in (0,1)$ and $\Delta v|_{t=t_0}$ denotes the "jump" of the function v in $t = t_0$, that is $\Delta v|_{t=t_0} = v(t_0^+) - v(t_0^-)$, where $v(t_0^-), v(t_0^+)$ are the left and the right limits of v in $t = t_0$.

Nonlocal conditions

The motivation for the study of nonlocal problems is that a nonlocal Cauchy problem has better effect in applications than the classical Cauchy problem since it is usually more precise for physical measurements. Also, the mathematical modeling of real processes, such as heat, fluid, chemical or biological flow, where nonlocal conditions can be seen as feedback controls, have brought into attention the treatment of nonlocal initial value problems.

The study of abstract nonlocal semilinear initial-value problems was initiated by L. Byszewski [19, 20], L. Byszewski and V. Lakshmikantham [21] and afterwards has been continued in many other papers: S. Aizicovici and Y. Gao [3], S. Aizicovici and M. McKibben [5], J. Liang, J.H. Liu and T. Xiao [46], J.H. Liu [47], S.K. Ntouyas and P.Ch. Tsamatos [58] and references therein. As remarked in L. Byszewski [20], nonlocal problems occur naturally when modelling physical problems, for example thermostats G. Infante and J.R.L. Webb [40], beams J.R.L. Webb and G. Infante [76, 77] and suspension bridges G. Infante, F.M. Minhós and P. Pietramala [36]. A unified method for establishing the existence and multiplicity of positive solutions for a large number of nonlinear differential equations of arbitrary order with any allowed number of non-local boundary conditions was given in J.R.L. Webb and G. Infante [76, 77]. Also, other discussions about the importance of nonlocal conditions in different areas of applications, examples of problems with nonlocal conditions and references to other works dealing with nonlocal problems can be found in H.-K. Han and J.-Y. Park [34], D. Jackson [41], H.C. Lee [45] and the references therein.

Many authors have studied different types of nonlocal problems mainly with multi-point boundary conditions (see, for example A. Boucherif and R. Precup [17], A.M.A. El-Sayed, E.O. Bin-Taher [27], O. Nica and R. Precup [52], S.K. Ntouyas [59] and rerefences therein for first order differential equations, S.K. Ntouyas [59], R.P. Agarwal, D. O'Regan and S. Staněk [2], C.P. Gupta, S.K. Ntouyas and P.Ch. Tsamatos [33], G. Infante [35], R. Ma [50], S. Staněk [71] for second-order equations, or M. Eggensperger and N. Kosmatov [26], J.R.L. Webb, G. Infante and D. Franco [80] for higher order equations. Initial value problems involving boundary conditions given by linear and continuous functionals, or equivalently, by Stieltjes integrals were studied, for example, in G. Infante [35], G.L. Karakostats and P.Ch. Tsamatos [42], O. Nica [53, 55], J.R.L Webb and G. Infante [78], J.R.L. Webb, G. Infante and D. Franco [80].

We also mention some other papers on nonlocal problems for several classes of differential equations and systems: S. Aizicovici and H. Lee [4], M. Benchohra and A. Boucherif [7], M. Benchohra, E.P. Gatsori, L. Gorniewicz and S.K. Ntouyas [8], O. Bolojan-Nica, G. Infante and R. Precup [11], O. Bolojan-Nica, G. Infante and R. Precup [12], A. Boucherif [14]-[16], A. Boucherif and R. Precup [18], G. Infante and P. Pietramala [37, 38], G. Infante and J.R.L. Webb [40], G.L. Karakostas and P.Ch. Tsamatos [42], O. Nica [54], S.K. Ntouyas [59], R. Precup [64], R. Precup and D. Trif [66], J.R.L. Webb [74], J.R.L. Webb and G. Infante [75], J.R.L. Webb and K.Q. Lan [79], X. Xue [81, 82] and Z. Yang [83].

Structure of the thesis

The thesis is divided into six chapters, each chapter being organized in several sections.

Chapter 1 is entirely dedicated in presenting some preliminary notions, results and notations that we need throughout this work. Here, in Section 1.1 we introduce the concepts of vector-valued metric and vector-valued norm, while in Section 1.2 we present another essential tool for our investigation, namely the notion of convergent to zero matrix.

Next, in Section 1.3, we recall the classical fixed point principles of Perov, Schauder and Leray-Schauder that we will apply frequently in this thesis. Other auxiliary results will be also presented in the subsequent chapters.

In **Chapter 2** we discuss three types of nonlocal initial value problems for first order differential systems. Section 2.1 contains an overview of the chapter, where we explain the contents of each section and we present the main tools and methods that are used.

Motivated by the paper A. Boucherif and R. Precup [17], in Section 2.2 we deal with the nonlocal initial value problem for the first order differential system

$$\begin{cases} x'(t) = f(t, x(t), y(t)) \\ y'(t) = g(t, x(t), y(t)) \\ x(0) + \sum\limits_{k=1}^{m} a_k x(t_k) = 0 \\ y(0) + \sum\limits_{k=1}^{m} \widetilde{a}_k y(t_k) = 0, \end{cases} \qquad \text{(a.e. on } [0,1])$$

where $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions, t_k are given points with $0 \leq t_1 \leq t_2 \leq \ ... \ \leq t_m < 1$ and a_k, $\widetilde{a}_k$ are real numbers with $1 + \sum\limits_{k=1}^{m} a_k \neq 0$ and $1 + \sum\limits_{k=1}^{m} \widetilde{a}_k \neq 0.$ Section 2.2 contains three subsections, each one being dedicated to the study of the existence of solutions for the problem above. The proofs will rely on the Perov, Schauder and Leray-Schauder fixed point principles which are applied to a nonlinear integral operator splitted into two parts, one of Fredholm type for the subinterval containing the points involved by the nonlocal condition, and another one of Volterra type for the rest of the interval. Vector-valued norms and convergent to zero matrices play a key role in this approach.

In Section 2.3 and Section 2.4, we extend the ideas of Section 2.2 by considering that the nonlocal initial conditions are expressed, more general, by linear continuous functionals, as in the works of J.R.L. Webb [74], J.R.L. Webb and G. Infante [75, 76, 77], J.R.L. Webb and K.Q. Lan [79]. Therefore, in Section 2.3 we discuss the nonlocal initial value problem

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) \\ x(0) = \alpha[x] \\ y(0) = \beta[y], \end{cases} \qquad \text{(a.e. on } [0,1])$$

while in Section 2.4 we study the problem

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) \\ x(0) = \alpha[y] \\ y(0) = \beta[x], \end{cases} \qquad \text{(a.e. on } [0,1])$$

where $f_1, f_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are also Carathéodory functions and $\alpha, \beta : C[0,1] \to \mathbb{R}$ are linear continuous functionals such that $1 - \alpha[1] \neq 0$ and $1 - \beta[1] \neq 0$, in the first case, and $1 - \alpha[1]\beta[1] \neq 0$, in the second case. Each one of the Sections 2.3 and 2.4 is divided into three subsections. Here, the existence and uniqueness results are first obtained under Lipschitz or growth conditions given on the whole interval $[0,1]$, and after that, in the second part of each subsection, under similar conditions given differently on two subintervals $[0, t_0]$ and $[t_0, 1]$. In the second case, the nonlinear integral operator associated to the problem splits into two parts, one of Fredholm type and the other one of Volterra type, which is reflected on the nonlinearities behaviour.

The main results of Section 2.2 are: Theorem 2.2.1, which represents an existence and uniqueness result, as an application of Perov's fixed point theorem; Theorem 2.2.2 and Theorem 2.2.3, two existence theorems which follow from Schauder's and Leray-Schauder's fixed point principle, respectively. The results from this section have been published in the paper O. Nica and R. Precup [52].

The most relevant results in Section 2.3 are: Theorem 2.3.1 and Theorem 2.3.2, two existence and uniqueness theorems that use Perov's fixed point principle; Theorem 2.3.4 and Theorem 2.3.5, which are existence results based on Schauder's fixed point theorem; Theorem 2.3.7 and Theorem 2.3.8, two existence theorems which are applications of Leray-Schauder's principle; Example 2.3.3 and Example 2.3.6 which represent two illustrations of the theory. Most part of these results can be found in the papers O. Nica [53, 55].

Our main contributions in Section 2.4 are: Theorem 2.4.1 and Theorem 2.4.2 of existence and uniqueness; Theorem 2.4.4, Theorem 2.4.5, Theorem 2.4.7, Theorem 2.4.8 of existence; Example 2.4.3 and Example 2.4.6, numerical applications of the theoretical results.

Chapter 3 extends to the general case the results from Chapter 2. More exactly, we consider the n-dimensional system

$$\begin{cases} u_1'(t) = f_1(t, u_1(t), u_2(t), ..., u_n(t)), \\ u_2'(t) = f_2(t, u_1(t), u_2(t), ..., u_n(t)), \\ ... \\ u_n'(t) = f_n(t, u_1(t), u_2(t), ..., u_n(t)), \end{cases}$$

for a.e. t in $[0, 1]$, subject to the coupled nonlocal conditions

$$\begin{cases} u_1(0) = \alpha_{11}[u_1] + \alpha_{12}[u_2] + ... + \alpha_{1n}[u_n], \\ u_2(0) = \alpha_{21}[u_1] + \alpha_{22}[u_2] + ... + \alpha_{2n}[u_n], \\ ... \\ u_n(0) = \alpha_{n1}[u_1] + \alpha_{n2}[u_2] + ... + \alpha_{nn}[u_n]. \end{cases}$$

Here $f_1, f_2, ..., f_n : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ are L^1-Carathéodory functions and $\alpha_{ij} : C[0, 1] \to \mathbb{R}$, $i, j = 1, 2, ..., n$ are linear continuous functionals. The problem can be rewritten in the vector form

$$\begin{cases} u'(t) = f(t, u(t)), \quad \text{a.e. on } [0, 1], \\ u(0) = \alpha[u], \end{cases}$$

where $u = (u_1, u_2, ..., u_n)$, $f = (f_1, f_2, ..., f_n)$, and

$$\alpha[u] = (\alpha_1[u], \alpha_2[u], ..., \alpha_n[u]),$$

$$\alpha_i[u] = \alpha_{i1}[u_1] + \alpha_{i2}[u_2] + ... + \alpha_{in}[u_n] \quad (i = 1, 2, ..., n).$$

Clearly, in this case α is a linear continuous mapping from $C([0, 1], \mathbb{R}^n)$ to $\mathbb{R}^n$. This chapter contains four sections. After an overview given in Section 3.1, we present the existence results based on the fixed point theorems of Perov, Schauder and Leray-Schauder, in Section 3.2, Section 3.3 and Section 3.4, respectively.

The main results in this chapter are: Theorem 3.2.1, Theorem 3.3.1 and Theorem

3.4.1; Example 3.2.2 and Example 3.3.2 that present two numerical applications. These contributions can be found in the paper O. Bolojan-Nica, G. Infante and R. Precup [11].

Chapter 4 is devoted to the study of second order differential equations and systems with nonlinear three-point boundary conditions. The chapter is divided into five sections as follows. An overview on the problems and the contents of the chaper is given in Section 4.1. Then, motivated by the paper of E.V. Castelani and T.F. Ma [23], in Section 4.2, we start our work by studying the three-point boundary value problem for second order differential equations

$$\begin{cases} u'' = f\left(t, u, u'\right) \\ u(0) = 0,\, u(t_0) = g(u(\eta)), \end{cases} \qquad 0 < t < t_0$$

where $0 < \eta < t_0 < 1$ and f, g are continuous functions. In Section 4.3, we give existence results for second-order differential systems

$$\begin{cases} u''(t) = f\left(t, u(t), v(t)\right) \\ v''(t) = g\left(t, u(t), v(t)\right) \\ u(0) = 0,\; u(t_0) = \phi(u(\eta), v(\eta)) \\ v(0) = 0,\; v(t_0) = \psi(u(\eta), v(\eta)), \end{cases}$$

on a given interval $[0, t_0]$. In Section 4.4 we study the problem from Section 4.2 on the larger interval $[0, 1]$, namely

$$\begin{cases} u'' = f\left(t, u, u'\right) \\ u(0) = 0,\; u(t_0) = g(u(\eta)), \end{cases}$$

when $t_0 < 1$. Finally, in Section 4.5, a similar strategy is applied to a system of two second order differential equations. The main results from this chapter are: Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.4 and Theorem 4.3.1, Theorem 4.3.2, existence results on $[0, t_0]$ for equations and systems, respectively; Theorem 4.4.1, Theorem 4.4.2, Theorem 4.4.3, Theorem 4.4.4 and Theorem 4.5.1, Theorem 4.5.2, Theorem 4.5.3, Theorem 4.5.4, existence results for equations and systems on $[0, 1]$. The results from this chapter appear in the paper O. Nica [54].

The purpose of **Chapter 5** is to study the existence of solutions to the nonlocal initial value problem for a first order differential system, with nonlinear nonlocal conditions

$$\begin{cases} x'\left(t\right) = f_1\left(t, x\left(t\right), y(t)\right) \\ y'\left(t\right) = f_2\left(t, x\left(t\right), y(t)\right) \\ x\left(0\right) = \alpha[x, y] \\ y\left(0\right) = \beta[x, y]. \end{cases}, \qquad t \in [0, 1]$$

Here $f_1, f_2 : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, while $\alpha, \beta : C[0, 1]^2 \to \mathbb{R}$ are nonlinear continuous functionals. The existence results are established by means of Perov, Schauder and Leray-Schauder fixed point principles combined with the technique based on vector-valued metrics and convergent to zero matrices. The main contributions here are as follows: Theorem 5.2.1, which represents an existence and uniqueness result based on Perov's fixed point principle; Theorem 5.3.1 and Theorem 5.3.3, two existence results given as direct application of Schauder and Leray-Schauder theorems, respectively; Example 5.2.2 and Example 5.3.2, two numerical applications that illustrate the results given by Theorem 5.2.1 and Theorem 5.3.1. These results are part of the work O. Bolojan-Nica,

G. Infante and R. Precup [13].

The methods that we mainly used in previous chapters are next addapted in **Chapter 6** to the case of impulsive systems with nonlocal initial conditions expressed by means of linear continuous functionals given by Stieltjes integrals

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)), \\ y'(t) = f_2(t, x(t), y(t)), \\ \Delta x|_{t=t_0} = I_1(x(t_0)), \quad \Delta y|_{t=t_0} = I_2(y(t_0)), \\ x(0) = \alpha_1[x], \quad y(0) = \alpha_2[y]. \end{cases} \quad t \in (0,1), \ t \neq t_0.$$

Here $t_0 \in (0,1)$ and $\Delta v|_{t=t_0}$ denotes the "jump" of the function v in $t = t_0$, that is $\Delta v|_{t=t_0} = v(t_0^+) - v(t_0^-)$, where $v(t_0^-), v(t_0^+)$ are the left and the right limits of v in $t = t_0$. The chapter is divided into four sections. After a general overview in Section 6.1, Section 6.2 presents an existence and uniqueness result by applying the fixed point principle of Perov, while in Section 6.3 we provide an existence result as a consequence of Schauder's fixed point theorem. The existence principles that we apply are completed by the technique based on vector-valued norms and matrices that are convergent to zero. Our contributions in this chapter are as follows: Theorem 6.2.1, an existence and uniqueness theorem; Theorem 6.3.1 of existence; Example 6.2.2 and Example 6.3.2. These results are included in the paper O. Bolojan-Nica, G. Infante and P. Pietramala [12].

Some ideas for further work

The methods that we have used throughout the thesis can be applied to other classes of problems, for instance, to systems of evolution equations of the type

$$\begin{cases} x'(t) + A_1 x(t) = f_1(t, x(t), y(t)) \\ y'(t) + A_2 y(t) = f_2(t, x(t), y(t)) \\ x(0) + \sum\limits_{k=1}^{m} a_k x(t_k) = 0 \\ y(0) + \sum\limits_{k=1}^{m} \widetilde{a}_k y(t_k) = 0. \end{cases}$$

Here the linear operator $-A_i : D(A_i) \subseteq X_i \to X_i$ generates a strongly continuous semigroup of contractions $\{S_i(t), t \geq 0\}$ on a Banach space $\left(X_i, |.|_{X_i}\right)$, for $i = 1, 2$. In particular, we can consider systems of parabolic and hyperbolic equations.

Another idea is to use a vector version of Krasnoselskii's fixed point theorem for the sum of two operators (see A. Viorel [73]) together with the method that uses convergent to zero matrices, in order to treat more complicated nonlocal problems, for instance, the problem

$$\begin{cases} x'(t) = g_1(t, x(t), y(t)) + h_1(t, x'(t), y'(t)) \\ y'(t) = g_2(t, x(t), y(t)) + h_2(t, x'(t), y'(t)) \\ x(0) = \alpha[x] \\ y(0) = \beta[y], \end{cases}$$

where $g_i, h_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions and $\alpha, \beta : [0,1] \to \mathbb{R}$ are linear continuous functionals. Of course, all the other problems studied in this work could be generalized this way.

Author's research activity

Most of the results in this thesis are part of the following papers/manuscripts:

- O. Nica and R. Precup, On the nonlocal initial value problem for first order differential systems, Stud. Univ. Babeş-Bolyai Math. 56 (2011), No. 3, 125–137.
- O. Nica, Initial-value problems for first-order differential systems with general nonlocal conditions, Electron. J. Differential Equations 2012 (2012), No. 74, 1–15.
- O. Nica, Existence results for second order three-point boundary value problems, Differ. Equ. Appl. 4 (2012), 547–570.
- O. Nica, Nonlocal initial value problems for first order differential systems, Fixed Point Theory 13 (2012), 603–612.
- O. Nica, Extensions of the Leray-Schauder Principle for integral systems, An. Univ. Oradea Fasc. Mat. 9 (2012), No. 2, 63-81.
- O. Nica, A fixed point approach to first order differential equations and systems, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity 10 (2012), 141-153.
- O. Bolojan-Nica, G. Infante, R. Precup, Existence results for systems with coupled nonlocal initial conditions, Nonlinear Anal. 94 (2014), 231-242.
- O. Bolojan-Nica, G. Infante, P. Pietramala, Existence results for impulsive systems with initial nonlocal conditions, Math. Model. Anal., to appear.
- O. Bolojan-Nica, G. Infante, R. Precup, Existence results for systems with coupled nonlocal nonlinear initial conditions, submitted.

Part of the results was presented at the following scientific conferences, workshops and meetings:

- International Conference on Sciences (ICS), November 11-12, 2011, Oradea, Romania.
- International Conference of Nonlinear Operators, Differential Equations and Applications (ICNODEA), July 5-8, 2011, Cluj-Napoca, Romania.
- The Fifth International Workshop 2012 "Constructive Methods for Non-Linear Boundary Value Problems", June 28-July 1, 2012, Tokaj, Hungary.
- 6^{th} European Congress of Mathematics (6ECM), July 2-6, 2012, Krakow, Poland.
- 10^{th} International Conference on Fixed Point Theory and its Applications (ICFPTAC), July 9-15, 2012, Cluj-Napoca, Romania.
- Research Seminar of the Department of Mathematics, October 22, 2012, University of Calabria, Cosenza, Italy.

- A treia sesiune naţională de comunicări ştiinţifice a doctoranzilor, June 10-16, 2013, Timişoara, Romania.

- The Fourteenth International Conference on Applied Mathematics Computer Science and Mechanics, "Theodor Angheluţă" Seminar, August 29-31, 2013, Cluj-Napoca, Romania.

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Keywords: Nonlinear differential equation; differential system; initial value problem; boundary value problem; impulsive differential equation; nonlocal condition; fixed point; contraction; compact mapping; vector-valued norm; matrix convergent to zero.

Chapter 1

Preliminaries

In this chapter we give some preliminary notions and results that we use throughout the PhD thesis. Vector-valued metrics and norms, convergent to zero matrices and the fixed point principles of Perov, Schauder and Leray-Schauder are the main tools in our work.

1.1 Vector-valued metrics and norms

Definition 1.1.1 *Let X be a nonempty set. By a* vector-valued metric *on X we mean a mapping $d : X \times X \to \mathbb{R}^n$ such that*

(i) $d(x, y) \geq 0$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$
with respect to the natural order relation of $\mathbb{R}^n$. *More exactly, if* $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, *by* $x \leq y$ *we mean* $x_i \leq y_i$ *for* $i = 1, 2, ..., n$.

We call the pair (X, d) a *generalized metric space.*

Definition 1.1.2 *Let X be a linear space. A mapping $\|\cdot\| : X \to \mathbb{R}^n$ is called a* vector-valued norm *if*

(i) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$;

(ii) $\|\lambda x\| = |\lambda| \|x\|$ *for all* $x \in X$ *and* $\lambda \in \mathbb{R}$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

Obviously, every linear space endowed with a vector-valued norm is a generalized metric space with the vector-valued metric

$$d(x, y) = \|x - y\|.$$

1.2 Convergent to zero matrices

Definition 1.2.1 *A square matrix $M \in M_{n\times n}(\mathbb{R}^n)$ with nonnegative elements is said to be* convergent to zero *if*

$$M^k \to 0 \qquad as\ k \to \infty.$$

Lemma 1.2.2 *Let $M \in M_{n\times n}(\mathbb{R}^n)$ be a square matrix of nonnegative numbers. The following statements are equivalent:*

(i) M is a matrix that is convergent to zero;

(ii) $I - M$ is nonsingular and $(I-M)^{-1} = I + M + M^2 + ...$ (where I stands for the unit matrix of the same order as M);

(iii) the eigenvalues of M are located inside the unit disc of the complex plane;

(iv) $I - M$ is nonsingular and $(I-M)^{-1}$ has nonnegative elements.

Lemma 1.2.3 *If A is a square matrix which is convergent to zero and the elements of an other square matrix B are small enough, then $A + B$ is also convergent to zero.*

Definition 1.2.4 *An operator $T : X \to X$ is said to be* a generalized contraction *(with respect to the vector-valued metric d on X) if there exists a convergent to zero matrix M such that*

$$d(T(u), T(v)) \leq Md(u, v) \quad for\ all\ u, v \in X.$$

1.3 Fixed point theorems

Theorem 1.3.1 (Banach Contraction Principle) *Let (K, d) be a complete metric space. Suppose that $T : K \to K$ is a contraction, i.e. there is $\rho \in [0, 1)$ such that*

$$d(T(x), T(y)) \leq \rho d(x, y),$$

for all $x, y \in K$. Then T has a unique fixed point x^* and for any $x \in K$, one has

$$d(T^k(x), x^*) \leq \frac{\rho^k}{1-\rho} d(x, T(x)), \qquad k \in \mathbb{N}.$$

Theorem 1.3.2 (Perov) *Let (X, d) be a complete generalized metric space and $T : X \to X$ a generalized contraction with Lipschitz matrix M. Then T has a unique fixed point x^* and for each $x \in X$ we have*

$$d(T^k(x), x^*) \leq M^k(I-M)^{-1}d(x, T(x)), \qquad k \in \mathbb{N}.$$

Theorem 1.3.3 (Schauder) *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $T : D \to D$ a completely continuous operator (i.e., T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.*

Theorem 1.3.4 (Leray–Schauder) *Let $(X, |\,.\,|_X)$ be a Banach space, $R > 0$ and $T : \overline{B}_X(0; R) \to X$ a completely continuous operator. If $|u|_X < R$ for every solution u of the equation $u = \lambda T(u)$ and any $\lambda \in (0, 1)$, then T has at least one fixed point.*

Chapter 2

Nonlocal initial value problems for first order differential systems

2.1 Overview

This chapter is devoted to existence of solutions to initial value problems for nonlinear first order planar differential systems with nonlocal conditions expressed by means of discrete and continuous linear functionals.

2.2 First order differential systems with polylocal conditions

In this section, we deal with the nonlocal initial value problem for the first order differential system of the type

$$\begin{cases} x'(t) = f(t, x(t), y(t)) \\ y'(t) = g(t, x(t), y(t)) \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0 \\ y(0) + \sum_{k=1}^{m} \widetilde{a}_k y(t_k) = 0. \end{cases} \quad \text{(a.e. on } [0,1]) \tag{2.2.1}$$

Here $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions, t_k are given points with $0 \leq t_1 \leq t_2 \leq \ ... \ \leq t_m < 1$ and a_k, $\widetilde{a}_k$ are real numbers with $1 + \sum_{k=1}^{m} a_k \neq 0$ and $1 + \sum_{k=1}^{m} \widetilde{a}_k \neq 0$. Notice that the nonhomogeneous nonlocal initial conditions

$$\begin{cases} x(0) + \sum_{k=1}^{m} a_k x(t_k) = x_0 \\ y(0) + \sum_{k=1}^{m} \widetilde{a}_k y(t_k) = y_0 \end{cases}$$

can always be reduced to the homogeneous ones (with $x_0 = y_0 = 0$) by the change of variables $x_1(t) := x(t) - a\ x_0$ and $y_2(t) := y(t) - \widetilde{a}\ y_0$, where

$$a=\left(1+\sum_{k=1}^{m}a_k\right)^{-1} \quad \text{and} \quad \widetilde{a}=\left(1+\sum_{k=1}^{m}\widetilde{a}_k\right)^{-1}. \tag{2.2.2}$$

This can be viewed as a fixed point problem in $C[0,1]^2$ for the completely continuous operator $T=(T_1,T_2)$, $T:C[0,1]^2\to C[0,1]^2$, where T_1 and T_2 are given by

$$\begin{aligned}T_1(x,y)(t)&=-a\sum_{k=1}^{m}a_k\int_0^{t_k}f(s,x(s),y(s))\,ds+\int_0^t f(s,x(s),y(s))\,ds,\\ T_2(x,y)(t)&=-\widetilde{a}\sum_{k=1}^{m}\widetilde{a}_k\int_0^{t_k}g(s,x(s),y(s))\,ds+\int_0^t g(s,x(s),y(s))\,ds.\end{aligned}$$

Operators T_1 and T_2 appear as sums of two integral operators, one of Fredholm type, whose values depend only on the restrictions of functions to $[0,t_m]$, and the other one, a Volterra type operator depending on the restrictions to $[t_m,1]$, as this was pointed out in A. Boucherif and R. Precup [17]. Thus, T_1 can be rewritten as $T_1=T_{F_1}+T_{V_1}$, where

$$T_{F_1}(x,y)(t)=\begin{cases}-a\sum_{k=1}^{m}a_k\int_0^{t_k}f(s,x(s),y(s))\,ds+\int_0^t f(s,x(s),y(s))\,ds, & \text{if } t<t_m\\ -a\sum_{k=1}^{m}a_k\int_0^{t_k}f(s,x(s),y(s))\,ds+\int_0^{t_m}f(s,x(s),y(s))\,ds, & \text{if } t\geq t_m\end{cases}$$

and

$$T_{V_1}(x,y)(t)=\begin{cases}0, & \text{if } t<t_m\\ \int_{t_m}^t f(s,x(s),y(s))\,ds, & \text{if } t\geq t_m.\end{cases}$$

Similarly, $T_2=T_{F_2}+T_{V_2}$, where

$$T_{F_2}(x,y)(t)=\begin{cases}-\widetilde{a}\sum_{k=1}^{m}\widetilde{a}_k\int_0^{t_k}g(s,x(s),y(s))\,ds+\int_0^t g(s,x(s),y(s))\,ds, & \text{if } t<t_m\\ -\widetilde{a}\sum_{k=1}^{m}\widetilde{a}_k\int_0^{t_k}g(s,x(s),y(s))\,ds+\int_0^{t_m}g(s,x(s),y(s))\,ds, & \text{if } t\geq t_m\end{cases}$$

and

$$T_{V_2}(x,y)(t)=\begin{cases}0, & \text{if } t<t_m\\ \int_{t_m}^t g(s,x(s),y(s))\,ds, & \text{if } t\geq t_m.\end{cases}$$

2.2.1 Nonlinearities with the Lipschitz property. Application of Perov's fixed point theorem

Here we show that the existence and uniqueness of solutions to problem (2.2.1) follows from Perov's fixed point theorem in case that f,g satisfy Lipschitz conditions in x and y :

$$|f(t,x,y)-f(t,\overline{x},\overline{y})|\leq\begin{cases}b_1|x-\overline{x}|+\widetilde{b}_1|y-\overline{y}|, & \text{if } t\in[0,t_m]\\ c_1|x-\overline{x}|+\widetilde{c}_1|y-\overline{y}|, & \text{if } t\in[t_m,1],\end{cases} \tag{2.2.3}$$

$$|g(t,x,y)-g(t,\overline{x},\overline{y})| \leq \begin{cases} B_1|x-\overline{x}|+\widetilde{B}_1|y-\overline{y}|, & \text{if } \ t\in[0,t_m] \\ C_1|x-\overline{x}|+\widetilde{C}_1|y-\overline{y}|, & \text{if } t\in[t_m,1] \end{cases} \tag{2.2.4}$$

for all $x,y,\overline{x},\overline{y}\in\mathbb{R}$.

In what follows, we denote $A_1 := 1+|a|\sum\limits_{k=1}^{m}|a_k|$, $A_2 = 1+|\widetilde{a}|\sum\limits_{k=1}^{m}|\widetilde{a}_k|$, where a and $\widetilde{a}$ are given by (2.2.2).

Theorem 2.2.1 *If f,g satisfy the Lipschitz conditions* (2.2.3), (2.2.4) *and the matrix*

$$M_0 := \begin{bmatrix} b_1 t_m A_1 & \widetilde{b}_1 t_m A_1 \\ B_1 t_m A_2 & \widetilde{B}_1 t_m A_2 \end{bmatrix} \tag{2.2.5}$$

is convergent to zero, then problem (2.2.1) *has a unique solution.*

2.2.2 Nonlinearities with growth at most linear. Application of Schauder's fixed point theorem

Here we show that the existence of solutions to problem (2.2.1) follows from Schauder's fixed point theorem in case that f,g satisfy instead of the Lipschitz conditions, the more relaxed conditions of growth at most linear:

$$|f(t,x,y)| \leq \begin{cases} b_1|x|+\widetilde{b}_1|y|+d_1, & \text{if } t\in[0,t_m] \\ c_1|x|+\widetilde{c}_1|y|+d_2, & \text{if } t\in[t_m,1], \end{cases} \tag{2.2.6}$$

$$|g(t,x,y)| \leq \begin{cases} B_1|x|+\widetilde{B}_1|y|+D_1, & \text{if } t\in[0,t_m] \\ C_1|x|+\widetilde{C}_1|y|+D_2, & \text{if } t\in[t_m,1], \end{cases} \tag{2.2.7}$$

for all $x,y\in\mathbb{R}$ and some nonnegative coefficients $b_i, c_i, d_i, \widetilde{b}_i, \widetilde{c}_i, \widetilde{d}_i, B_i, C_i, D_i, \widetilde{B}_i, \widetilde{C}_i, \widetilde{D}_i, i = 1,2$.

Theorem 2.2.2 *If f,g satisfy conditions* (2.2.6), (2.2.7) *and matrix* (2.2.5) *is convergent to zero, then problem* (2.2.1) *has at least one solution.*

2.2.3 More general nonlinearities. Application of the Leray-Schauder principle

We now consider that nonlinearities f,g satisfy more general growth conditions, namely:

$$|f(t,u)| \leq \begin{cases} \omega_1(t,|u|_e), & \text{if } t\in[0,t_m] \\ \alpha(t)\beta_1(|u|_e), & \text{if } t\in[t_m,1], \end{cases} \tag{2.2.8}$$

$$|g(t,u)| \leq \begin{cases} \omega_2(t,|u|_e), & \text{if } \ t\in[0,t_m] \\ \alpha(t)\beta_2(|u|_e), & \text{if } \ t\in[t_m,1], \end{cases} \tag{2.2.9}$$

for all $u=(x,y)\in\mathbb{R}^2$, where by $|u|_e$ we mean the euclidean norm in $\mathbb{R}^2$. Here ω_1,ω_2 are Carathéodory functions on $[0,t_m]\times\mathbb{R}_+$, nondecreasing in their second argument, $\alpha\in L^1[t_m,1]$, while $\beta_1,\beta_2:\mathbb{R}_+\to\mathbb{R}_+$ are nondecreasing and $1/(\beta_1+\beta_2)\in L^1_{loc}(0,\infty)$.

Theorem 2.2.3 *Assume that conditions* (2.2.8), (2.2.9) *hold. In addition assume that there exists a positive number* R_0 *such that for* $\rho = (\rho_1, \rho_2) \in (0, \infty)^2$

$$\begin{cases} \frac{1}{\rho_1} \int_0^{t_m} \omega_1(t, |\rho|_e)dt \geq \frac{1}{A_1} \\ \frac{1}{\rho_2} \int_0^{t_m} \omega_2(t, |\rho|_e)dt \geq \frac{1}{A_2} \end{cases} \quad \textit{implies} \quad |\rho|_e \leq R_0 \tag{2.2.10}$$

and

$$\int_{R^*}^{\infty} \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau)} > \int_{t_m}^{1} \alpha(s)ds, \tag{2.2.11}$$

where $R^* = \left[\left(A_1 \int_0^{t_m} \omega_1(t, R_0)dt \right)^2 + \left(A_2 \int_0^{t_m} \omega_2(t, R_0)dt \right)^2 \right]^{1/2}$. *Then problem* (2.2.1) *has at least one solution.*

2.3 The nonlocal conditions $x(0) = \alpha[x], \ y(0) = \beta[y]$

The goal of Section 2.3 is to extend the results established in Section 2.2 to the case where the nonlocal initial conditions are more generally expressed in terms of two linear continuous functionals on $C[0,1]$.

More exactly, we deal with the nonlocal initial value problem for the first order differential system of the type

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) \qquad \text{(a.e. on } [0,1]) \\ x(0) = \alpha[x] \\ y(0) = \beta[y]. \end{cases} \tag{2.3.12}$$

Here $f_1, f_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions, $\alpha, \beta : C[0,1] \to \mathbb{R}$ are linear and continuous functionals such that $1 - \alpha[1] \neq 0$ and $1 - \beta[1] \neq 0$.

This can be viewed as a fixed point problem in $C[0,1]^2$ for the completely continuous operator $T : C[0,1]^2 \to C[0,1]^2$, $T = (T_1, T_2)$, where T_1 and T_2 are given by

$$\begin{aligned} T_1(x,y)(t) &= \tfrac{1}{1-\alpha[1]} \alpha[g_1] + \textstyle\int_0^t f_1(s, x(s), y(s))\, ds, \\ T_2(x,y)(t) &= \tfrac{1}{1-\beta[1]} \beta[g_2] + \textstyle\int_0^t f_2(s, x(s), y(s))\, ds, \end{aligned} \tag{2.3.13}$$

where $g_1(x,y)(t) := \int_0^t f_1(s, x(s), y(s))\, ds, \ g_2(x,y)(t) := \int_0^t f_2(s, x(s), y(s))\, ds$.

We require the following property:

$$x|_{[0,t_0]} = y|_{[0,t_0]} \text{ implies } \alpha[x-y] = 0, \text{ whenever } x, y \in C[0,1]. \tag{2.3.14}$$

Therefore, (2.3.14) reads that the value of functional α on any function x only depends on the restriction of x to the fixed subinterval $[0, t_0]$.
The key property of functional α satisfying (2.3.14) is that

$$|\alpha[x]| \leq \|\alpha\| \cdot |x|_{C[0,t_0]}, \tag{2.3.15}$$

for every $x \in C[0,1]$.

2.3.1 Existence and uniqueness under Lipschitz conditions

First, we show that the existence and uniqueness of solution to problem (2.3.12) follows from Perov's fixed point theorem in case that f_1, f_2 satisfy the Lipschitz conditions in x and y :

$$\begin{cases} |f_1(t,x,y) - f_1(t,\overline{x},\overline{y})| \leq a_1 |x-\overline{x}| + b_1 |y-\overline{y}| \\ |f_2(t,x,y) - f_2(t,\overline{x},\overline{y})| \leq a_2 |x-\overline{x}| + b_2 |y-\overline{y}|, \end{cases} \tag{2.3.16}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$.

Theorem 2.3.1 *If f_1, f_2 satisfy the Lipschitz conditions* (2.3.16) *and matrix*

$$M_{\alpha,\beta} = \begin{bmatrix} a_1 \left(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1 \right) & b_1 \left(\frac{\|\alpha\|}{|1-\alpha[1]|} + 1 \right) \\ a_2 \left(\frac{\|\beta\|}{|1-\beta[1]|} + 1 \right) & b_2 \left(\frac{\|\beta\|}{|1-\beta[1]|} + 1 \right) \end{bmatrix} \tag{2.3.17}$$

is convergent to zero, then problem (2.3.12) *has a unique solution.*

For our second existence and uniqueness result obtained via Perov's fixed point theorem, we shall consider that there is $t_0 \in (0,1)$ such that nonlinearities f_1, f_2 satisfy different Lipschitz conditions on $[0,t_0]$ and $[t_0,1]$, respectively:

$$|f_1(t,x,y) - f_1(t,\overline{x},\overline{y})| \leq \begin{cases} a_1 |x-\overline{x}| + b_1 |y-\overline{y}|, & \text{if } t \in [0,t_0] \\ a_2 |x-\overline{x}| + b_2 |y-\overline{y}|, & \text{if } t \in [t_0,1], \end{cases} \tag{2.3.18}$$

$$|f_2(t,x,y) - f_2(t,\overline{x},\overline{y})| \leq \begin{cases} A_1 |x-\overline{x}| + B_1 |y-\overline{y}|, & \text{if } t \in [0,t_0] \\ A_2 |x-\overline{x}| + B_2 |y-\overline{y}|, & \text{if } t \in [t_0,1], \end{cases} \tag{2.3.19}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$.

In what follows we denote $A_\alpha := \frac{\|\alpha\|}{|1-\alpha[1]|} + 1$, $B_\beta := \frac{\|\beta\|}{|1-\beta[1]|} + 1$.

Theorem 2.3.2 *Assume that α, β satisfy* (2.3.14). *If f_1, f_2 satisfy the Lipschitz conditions* (2.3.18), (2.3.19) *and the matrix*

$$M_0 := \begin{bmatrix} a_1 t_0 A_\alpha & b_1 t_0 A_\alpha \\ A_1 t_0 B_\beta & B_1 t_0 B_\beta \end{bmatrix} \tag{2.3.20}$$

is convergent to zero, then problem (2.3.12) *has a unique solution.*

Example 2.3.3 Consider the following nonlocal initial value problem:

$$\begin{cases} x'(t) = 0.1 + \frac{1}{4} \frac{y^2(t)}{1+y^2(t)} \sin(2x(t)) \equiv f_1(x(t), y(t)) \\ y'(t) = 0.1 + \frac{2}{3} \frac{y^2(t)}{1+y^2(t)} \cos(2x(t)) \equiv f_2(x(t), y(t)) & , \ t \in [0, 40] \\ x(0) = \int_0^{1/2} x(s)ds, \ y(0) = \int_0^{1/2} y(s)ds. \end{cases} \tag{2.3.21}$$

We have that

$$M_0 = \begin{pmatrix} \frac{1}{2} & \frac{3\sqrt{3}}{32} \\ \frac{4}{3} & \frac{\sqrt{3}}{4} \end{pmatrix}$$

whose eigenvalues are $\lambda_1 = 0,\ \lambda_2 = 0.9330....$ Hence M_0 is convergent to zero and Theorem 2.3.2 guarantees that problem (2.3.21) has a unique solution.

2.3.2 Existence under at most linear growth conditions

First, we show that the existence of solutions to problem (2.3.12) follows from Schauder's fixed point theorem in case that f_1, f_2 satisfy instead of the Lipschitz condition, the more relaxed conditions of growth at most linear:

$$\begin{cases} |f_1(t,x,y)| \leq a_1 |x| + b_1 |y| + c_1 \\ |f_2(t,x,y)| \leq a_2 |x| + b_2 |y| + c_2. \end{cases} \tag{2.3.22}$$

for all $x, y \in \mathbb{R}$.

In this first case, we deal with any two linear functionals α, β (i.e., we do not assume (2.3.14)).

Theorem 2.3.4 *If f_1, f_2 satisfy conditions* (2.3.22) *and matrix* (2.3.17) *is convergent to zero, then problem* (2.3.12) *has at least one solution.*

Next, we give another existence result for problem (2.3.12) as an application of Schauder's fixed point theorem in case that f_1, f_2 satisfy the more relaxed condition of growth at most linear differently, on two subintervals $[0, t_0]$ and $[t_0, 1]$,

$$|f_1(t,x,y)| \leq \begin{cases} a_1 |x| + b_1 |y| + c_1, & \text{if } t \in [0, t_0] \\ a_2 |x| + b_2 |y| + c_2, & \text{if } t \in [t_0, 1], \end{cases} \tag{2.3.23}$$

$$|f_2(t,x,y)| \leq \begin{cases} A_1 |x| + B_1 |y| + C_1, & \text{if } t \in [0, t_0] \\ A_2 |x| + B_2 |y| + C_2, & \text{if } t \in [t_0, 1], \end{cases} \tag{2.3.24}$$

and if the functionals α, β satisfy (2.3.14).

Theorem 2.3.5 *If f_1, f_2 satisfy* (2.3.23), (2.3.24) *and the matrix* (2.3.20) *is convergent to zero, then problem* (2.3.12) *has at least one solution.*

Example 2.3.6 Consider the nonlocal initial value problem

$$\begin{cases} x' = -0.9x - 1.8\frac{xy}{2+x^2} + 90 \equiv f_1(x,y) \\ y' = -0.2y - 1.8\frac{xy}{2+x^2} + 750 \equiv f_2(x,y) \\ x(0) = \int_0^{1/2} x(s)ds,\ y(0) = \int_0^{1/2} y(s)ds. \end{cases},\ t \in [0,1] \tag{2.3.25}$$

Since $\left|\frac{x}{2+x^2}\right| \leq \frac{\sqrt{2}}{4}$, we have

$$M_0 = \begin{pmatrix} 0.9 & 0.6364 \\ 0 & 0.8364 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 0.9,\ \lambda_2 = 0.8364$ showing that M_0 is convergent to zero. Then, from Theorem 2.3.5, problem (2.3.25) has at least one solution.

2.3.3 Existence under more general growth conditions

We now consider that nonlinearities f_1, f_2 satisfy more general growth conditions on the entire interval $[0, 1]$, namely:

$$\begin{cases} |f_1(t,u)| \le \omega_1(t, |u|_e) \\ |f_2(t,u)| \le \omega_2(t, |u|_e) \end{cases}, \quad \text{for } t \in [0,1], \tag{2.3.26}$$

for all $u = (x, y) \in \mathbb{R}^2$, where by $|u|_e$ we mean the euclidean norm in $\mathbb{R}^2$. Here, ω_1, ω_2 are Carathéodory functions on $[0, 1] \times \mathbb{R}_+$, nondecreasing in their second argument.

Theorem 2.3.7 *Assume that condition* (2.3.26) *holds. In addition assume that there exists a positive number* R_0 *such that for* $\rho = (\rho_1, \rho_2) \in (0, \infty)^2$

$$\begin{cases} \frac{1}{\rho_1} \int_0^1 \omega_1(t, |\rho|_e) dt \ge \frac{1}{A_\alpha} \\ \frac{1}{\rho_2} \int_0^1 \omega_2(t, |\rho|_e) dt \ge \frac{1}{B_\beta} \end{cases} \quad \textit{implies} \quad |\rho|_e \le R_0. \tag{2.3.27}$$

Then problem (2.3.12) *has at least one solution.*

Then, we suppose that nonlinearities f_1, f_2 satisfy more general growth conditions given differently on $[0, t_0]$ and $[t_0, 1]$, namely:

$$|f_1(t,u)| \le \begin{cases} \omega_1(t, |u|_e), & \text{if } t \in [0, t_0] \\ \gamma(t)\beta_1(|u|_e), & \text{if } t \in [t_0, 1], \end{cases} \tag{2.3.28}$$

$$|f_2(t,u)| \le \begin{cases} \omega_2(t, |u|_e), & \text{if } \ t \in [0, t_0] \\ \gamma(t)\beta_2(|u|_e)\ , & \text{if } \ t \in [t_0, 1], \end{cases} \tag{2.3.29}$$

for all $u = (x, y) \in \mathbb{R}^2$. Here ω_1, ω_2 are Carathéodory functions on $[0, t_0] \times \mathbb{R}_+$, nondecreasing in their second argument, $\gamma \in L^1[t_0, 1]$, while $\beta_1, \beta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing and $1/(\beta_1 + \beta_2) \in L^1_{loc}(\mathbb{R}_+)$.

Theorem 2.3.8 *Assume that the functionals* α, β *satisfy* (2.3.14) *and conditions* (2.3.28), (2.3.29) *hold. In addition assume that there exists a positive number* R_0 *such that for* $\rho = (\rho_1, \rho_2) \in (0, \infty)^2$

$$\begin{cases} \frac{1}{\rho_1} \int_0^{t_0} \omega_1(t, |\rho|_e) dt \ge \frac{1}{A_\alpha} \\ \frac{1}{\rho_2} \int_0^{t_0} \omega_2(t, |\rho|_e) dt \ge \frac{1}{B_\beta} \end{cases} \quad \textit{implies} \quad |\rho|_e \le R_0 \tag{2.3.30}$$

and

$$\int_{R^*}^{\infty} \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau)} > \int_{t_0}^1 \gamma(s) ds, \tag{2.3.31}$$

where

$$R^* = \left[\left(A_\alpha \int_0^{t_0} \omega_1(t, R_0) dt \right)^2 + \left(B_\beta \int_0^{t_0} \omega_2(t, R_0) dt \right)^2 \right]^{1/2}.$$

Then problem (2.3.12) *has at least one solution.*

2.4 The nonlocal conditions $x\,(0)=\alpha[y],\ y\,(0)=\beta[x]$

In this section we introduce in our study coupled nonlocal conditions expressed by linear functionals, namely the conditions

$$x\,(0)=\alpha[y],\qquad y\,(0)=\beta[x].$$

Therefore, we deal with the nonlocal initial value problem for the first order differential system

$$\left\{\begin{array}{l} x'\,(t)=f_1\,(t,x\,(t)\,,y(t))\\ y'\,(t)=f_2\,(t,x\,(t)\,,y(t))\\ x\,(0)=\alpha[y]\\ y\,(0)=\beta[x]. \end{array}\right.\qquad \text{(a.e. on } [0,1]) \tag{2.4.32}$$

Here, $f_1,f_2:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are Carathéodory functions, $\alpha,\beta:C[0,1]\to\mathbb{R}$ are linear and continuous functionals.

This can be viewed as a fixed point problem in $C\,[0,1]^2$ for the completely continuous operator $T:C\,[0,1]^2\to C\,[0,1]^2$, $T=(T_1,T_2)$, where T_1 and T_2 are given by

$$\begin{array}{l} T_1(x,y)(t)=\frac{1}{1-\alpha[1]\beta[1]}\left(\alpha[g_2]+\alpha[1]\beta[g_1]\right)+\int_0^t f_1\,(s,x\,(s)\,,y(s))\,ds,\\ T_2(x,y)(t)=\frac{1}{1-\alpha[1]\beta[1]}\left(\beta[g_1]+\beta[1]\alpha[g_2]\right)+\int_0^t f_2\,(s,x\,(s)\,,y(s))\,ds. \end{array}$$

2.4.1 Problems with Lipschitz conditions

Assume that f_1,f_2 satisfy Lipschitz conditions in x and y :

$$\left\{\begin{array}{l} |f_1(t,x,y)-f_1(t,\overline{x},\overline{y})|\le a_1\,|x-\overline{x}|+b_1\,|y-\overline{y}|\\ |f_2(t,x,y)-f_2(t,\overline{x},\overline{y})|\le A_1\,|x-\overline{x}|+B_1\,|y-\overline{y}|\,, \end{array}\right. \tag{2.4.33}$$

for all $x,y,\overline{x},\overline{y}\in\mathbb{R}$ and denote by

$$\begin{array}{ll} A_\alpha:=\frac{1}{|1-\alpha[1]\beta[1]|}\,\|\alpha\|\,, & A_\beta:=\frac{|\alpha[1]|}{|1-\alpha[1]\beta[1]|}\,\|\beta\|\,,\\ B_\alpha:=\frac{|\beta[1]|}{|1-\alpha[1]\beta[1]|}\,\|\alpha\|\,, & B_\beta:=\frac{1}{|1-\alpha[1]\beta[1]|}\,\|\beta\|\,. \end{array}$$

Theorem 2.4.1 *If f_1,f_2 satisfy the Lipschitz conditions* (2.4.33) *and the matrix*

$$M:=\left[\begin{array}{ll} A_\alpha A_1+(A_\beta+1)\,a_1 & A_\alpha B_1+(A_\beta+1)\,b_1\\ B_\beta a_1+(B_\alpha+1)\,A_1 & B_\beta b_1+(B_\alpha+1)\,B_1 \end{array}\right] \tag{2.4.34}$$

is convergent to zero, then problem (2.4.32) *has a unique solution.*

In case that α,β satisfy condition (2.3.14) for some $t_0\in(0,1)$, we can ask different growth conditions for f_1,f_2, on $[0,t_0]$ and $[t_0,1]$, respectively:

$$|f_1(t,x,y)-f_1(t,\overline{x},\overline{y})|\le\left\{\begin{array}{ll} a_1\,|x-\overline{x}|+b_1\,|y-\overline{y}|\,, & \text{if } t\in[0,t_0]\\ a_2\,|x-\overline{x}|+b_2\,|y-\overline{y}|\,, & \text{if } t\in[t_0,1]\,, \end{array}\right. \tag{2.4.35}$$

$$|f_2(t,x,y)-f_2(t,\overline{x},\overline{y})|\le\left\{\begin{array}{ll} A_1\,|x-\overline{x}|+B_1\,|y-\overline{y}|\,, & \text{if } t\in[0,t_0]\\ A_2\,|x-\overline{x}|+B_2\,|y-\overline{y}|\,, & \text{if } t\in[t_0,1]\,, \end{array}\right. \tag{2.4.36}$$

for all $x,y,\overline{x},\overline{y}\in\mathbb{R}$.

Theorem 2.4.2 *Assume that* α, β *satisfy condition* (2.3.14)*. If* f_1, f_2 *satisfy the Lipschitz conditions* (2.4.35), (2.4.36) *and the matrix*

$$M_0 := \begin{bmatrix} A_\alpha A_1 t_0 + (A_\beta + 1)\, a_1 t_0 & A_\alpha B_1 t_0 + (A_\beta + 1)\, b_1 t_0 \\ B_\beta a_1 t_0 + (B_\alpha + 1)\, A_1 t_0 & B_\beta b_1 t_0 + (B_\alpha + 1)\, B_1 t_0 \end{bmatrix} \tag{2.4.37}$$

is convergent to zero, then problem (2.4.32) *has a unique solution.*

Example 2.4.3 We shall consider the following nonlocal initial value problem:

$$\begin{cases} x'(t) = 0.1 + \frac{1}{4}\frac{y^2(t)}{1+y^2(t)}\sin(2x(t)) \equiv f_1(x(t), y(t)) \\ y'(t) = 0.1 + \frac{1}{3}\frac{y^2(t)}{1+y^2(t)}\cos(2x(t)) \equiv f_2(x(t), y(t)) & ,\ t \in [0,1] \\ x(0) = \int_0^{1/4} x(s)ds, \quad y(0) = \int_0^{1/4} y(s)ds. \end{cases} \tag{2.4.38}$$

We have that $\alpha\,[u] = \beta[u] = \int_0^{1/4} u(s)ds$, $\alpha\,[1] = \beta[1] = \frac{1}{4}$ and $\|\alpha\| = \|\beta\| = \frac{1}{4}$. Also, $t_0 = 1/4$, $A_\alpha = B_\beta = 4/15$, $A_\beta = B_\alpha = 1/15$ and

$$M = \begin{pmatrix} \frac{8}{45} & \frac{\sqrt{3}}{30} \\ \frac{19}{90} & \frac{19\sqrt{3}}{480} \end{pmatrix}$$

whose eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 0.246338122$. Since both λ_1 and λ_2 are less than 1, matrix M is convergent to zero and Theorem 2.4.2 guarantees the existence of a unique solution for problem (2.4.38).

2.4.2 Problems with growth conditions at most linear

For the first existence result, we shall assume that f_1, f_2 satisfy instead of the Lipschitz condition, the more relaxed condition of at most linear growth, uniformly on the entire interval $[0,1]$, and that α, β are general linear continuous functionals on $C[0,1]$. Thus, assume that

$$\begin{cases} |f_1(t,x,y)| \le a_1\,|x| + b_1\,|y| + c_1, \\ |f_2(t,x,y)| \le A_1\,|x| + B_1\,|y| + C_1, \end{cases} \tag{2.4.39}$$

for all $x, y \in \mathbb{R}$.

Theorem 2.4.4 *If* f_1, f_2 *satisfy* (2.4.39) *and matrix* (2.4.34) *is convergent to zero, then problem* (2.4.32) *has at least one solution.*

For our second existence result, another application of Schauder's fixed point theorem, we shall impose for nonlinearities f_1, f_2 conditions of growth at most linear that will differ on $[0, t_0]$ and $[t_0, 1]$, respectively:

$$|f_1(t,x,y)| \le \begin{cases} a_1\,|x| + b_1\,|y| + c_1, & \text{if } t \in [0, t_0] \\ a_2\,|x| + b_2\,|y| + c_2, & \text{if } t \in [t_0, 1]\,, \end{cases} \tag{2.4.40}$$

$$|f_2(t,x,y)| \le \begin{cases} A_1\,|x| + B_1\,|y| + C_1, & \text{if } t \in [0, t_0] \\ A_2\,|x| + B_2\,|y| + C_2, & \text{if } t \in [t_0, 1]\,, \end{cases} \tag{2.4.41}$$

for all $x, y \in \mathbb{R}$ and some nonnegative coefficients $a_i, b_i, c_i, A_i, B_i, C_i, i = 1, 2$.

Theorem 2.4.5 *Assume that* α, β *satisfy* (2.3.14). *If* f_1, f_2 *satisfy* (2.4.40), (2.4.41) *and the matrix* (2.4.37) *is convergent to zero, then problem* (2.4.32) *has at least one solution.*

Example 2.4.6 We study the nonlocal problem given by

$$\left\{ \begin{array}{l} x' = -0.9x - 1.8\frac{xy}{2+x^2} + 1 \equiv f_1(x, y) \\ y' = -0.2y - 1.8\frac{xy}{2+x^2} + 0.7 \equiv f_2(x, y) \\ x(0) = \int_0^{1/4} x(s)ds \\ y(0) = \int_0^{1/4} y(s)ds. \end{array} \right. , \quad t \in [0, 1] \tag{2.4.42}$$

We obtain that

$$M = \left(\begin{array}{cc} 0.24 & 0.08094757079 \\ 0.06 & 0.04094757081 \end{array} \right)$$

has the eigenvalues (rounded to the third decimal place) $\lambda_1 = 0.262 < 1$, $\lambda_2 = 0.019 < 1$.This shows that M is convergent to zero. Then, from Theorem 2.4.5, problem (2.4.42) has at least one solution.

2.4.3 Problems with more general growth conditions

As we have already seen, the Leray-Schauder theorem guarantees the existence of solutions under more general growth conditions on f_1, f_2 :

$$\left\{ \begin{array}{l} |f_1(t, u)| \leq \omega_1(t, |u|_e) \\ |f_2(t, u)| \leq \omega_2(t, |u|_e) \end{array} \right. \tag{2.4.43}$$

for all $u = (x, y) \in \mathbb{R}^2$, where by $|u|_e$ we have ment the euclidean norm in $\mathbb{R}^2$. As until now, ω_1, ω_2 are Carathéodory functions on $[0, 1] \times \mathbb{R}_+$, nondecreasing in their second argument.

Theorem 2.4.7 *Assume that condition* (2.4.43) *holds. In addition assume that there exists a positive number* R_0 *such that for* $\rho = (\rho_1, \rho_2) \in (0, \infty)^2$

$$\left\{ \begin{array}{l} \frac{A_\beta+1}{\rho_1} \int_0^1 \omega_1(t, |\rho|_e)dt + \frac{A_\alpha}{\rho_1} \int_0^1 \omega_2(t, |\rho|_e)dt \geq 1 \\ \frac{B_\beta}{\rho_2} \int_0^1 \omega_1(t, |\rho|_e)dt + \frac{B_\alpha+1}{\rho_2} \int_0^1 \omega_2(t, |\rho|_e)dt \geq 1 \end{array} \right. \quad \textit{implies} \quad |\rho|_e \leq R_0 \tag{2.4.44}$$

Then problem (2.4.32) *has at least one solution.*

If the functionals α, β satisfy (2.3.14) for some number $t_0 \in (0, 1)$, then we may ask for f_1, f_2 to satisfy general growth conditions, differently on each of the intervals $[0, t_0]$ and $[t_0, 1]$, namely:

$$|f_1(t, u)| \leq \left\{ \begin{array}{ll} \omega_1(t, |u|_e), & \text{if } t \in [0, t_0] \\ \gamma(t)\beta_1(|u|_e), & \text{if } t \in [t_0, 1], \end{array} \right. \tag{2.4.45}$$

$$|f_2(t, u)| \leq \left\{ \begin{array}{ll} \omega_2(t, |u|_e), & \text{if } t \in [0, t_0] \\ \gamma(t)\beta_2(|u|_e) , & \text{if } t \in [t_0, 1], \end{array} \right. \tag{2.4.46}$$

for all $u = (x, y) \in \mathbb{R}^2$. Again, ω_1, ω_2 are Carathéodory functions on $[0, t_0] \times \mathbb{R}_+$, nondecreasing in their second argument, $\gamma \in L^1 [t_0, 1]$, while $\beta_1, \beta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing and $1/ (\beta_1 + \beta_2) \in L^1_{loc}(\mathbb{R}_+)$.

Theorem 2.4.8 *Assume that* α, β *satisfy* (2.3.14) *and conditions* (2.4.45), (2.4.46) *hold. In addition assume that there exists a positive number* R_0 *such that for* $\rho = (\rho_1, \rho_2) \in (0,\infty)^2$

$$\left\{ \begin{array}{l} \frac{A_\beta+1}{\rho_1}\int_0^{t_0}\omega_1(t,|\rho|_e)dt + \frac{A_\alpha}{\rho_1}\int_0^{t_0}\omega_2(t,|\rho|_e)dt \geq 1 \\ \frac{B_\beta}{\rho_2}\int_0^{t_0}\omega_1(t,|\rho|_e)dt + \frac{B_\alpha+1}{\rho_2}\int_0^{t_0}\omega_2(t,|\rho|_e)dt \geq 1 \end{array} \right. \quad \textit{implies} \quad |\rho|_e \leq R_0 \tag{2.4.47}$$

and

$$\int_{R^*}^{\infty} \frac{d\tau}{\beta_1(\tau)+\beta_2(\tau)} > \int_{t_0}^{1} \gamma(s)ds, \tag{2.4.48}$$

where

$$\begin{aligned} R^* \quad = \quad & \left\{ \left[(A_\beta+1)\int_0^{t_0}\omega_1(t,R_0)dt + A_\alpha\int_0^{t_0}\omega_2(t,R_0)dt \right]^2 \right. \\ & \left. + \left[B_\beta\int_0^{t_0}\omega_1(t,R_0)dt + (B_\alpha+1)\int_0^{t_0}\omega_2(t,R_0)dt \right]^2 \right\}^{1/2}. \end{aligned}$$

Then problem (2.4.32) *has at least one solution.*

Chapter 3

Systems with general coupled nonlocal initial conditions

3.1 Overview

Having in mind the problems and techniques that have been considered in Chapter 2, in this chapter an existence theory is developed for first-order n-dimensional systems with coupled nonlocal conditions given by general linear functionals.

In this chapter, we discuss the first order differential system

$$\begin{cases} u_1'(t) = f_1(t, u_1(t), u_2(t), ..., u_n(t)), \\ u_2'(t) = f_2(t, u_1(t), u_2(t), ..., u_n(t)), \\ ... \\ u_n'(t) = f_n(t, u_1(t), u_2(t), ..., u_n(t)), \end{cases} \tag{3.1.1}$$

for a.e. t in $[0,1]$, subject to the coupled nonlocal conditions

$$\begin{cases} u_1(0) = \alpha_{11}[u_1] + \alpha_{12}[u_2] + ... + \alpha_{1n}[u_n], \\ u_2(0) = \alpha_{21}[u_1] + \alpha_{22}[u_2] + ... + \alpha_{2n}[u_n], \\ ... \\ u_n(0) = \alpha_{n1}[u_1] + \alpha_{n2}[u_2] + ... + \alpha_{nn}[u_n]. \end{cases} \tag{3.1.2}$$

Here $f_1, f_2, ..., f_n : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are L^1-Carathéodory functions and $\alpha_{ij} : C[0,1] \to \mathbb{R}$, $i, j = 1, 2, ..., n$ are linear and continuous functionals. As was shown in R. Precup and D. Trif [66], the problem (3.1.1)-(3.1.2) is sufficiently general to cover problems related to n^{th}-order ordinary differential equations subject to nonlocal conditions involving the unknown function and its derivatives until order $n-1$.

The problem (3.1.1)-(3.1.2) can be rewritten in the vector form

$$\begin{cases} u'(t) = f(t, u(t)), \quad \text{a.e. on } [0,1], \\ u(0) = \alpha[u], \end{cases} \tag{3.1.3}$$

where $u = (u_1, u_2, ..., u_n)$, $f = (f_1, f_2, ..., f_n)$ and

$$\alpha[u] = (\alpha_1[u], \alpha_2[u], ..., \alpha_n[u]), \tag{3.1.4}$$

$$\alpha_i\,[u] = \alpha_{i1}[u_1] + \alpha_{i2}[u_2] + ... + \alpha_{in}\,[u_n]\,, \quad i = 1, 2, ..., n. \tag{3.1.5}$$

Note that α is a linear continuous mapping from $C\left([0,1], \mathbb{R}^n\right)$ to $\mathbb{R}^n$.

We assume that the matrix

$$I - \alpha\,[1] \quad \text{is non-singular,} \tag{3.1.6}$$

where I is the unit matrix of order n and by $\alpha\,[1]$ we mean the square matrix $\alpha\,[1] := (\alpha_{ij}\,[1])_{1\leq i,j\leq n}\,.$

Therefore, the problem (3.1.1)-(3.1.2) is equivalent to the integral type equation

$$u\,(t) = (I - \alpha\,[1])^{-1}\,\alpha\left[\int_0^t f\,(s, u\,(s))\,ds\right] + \int_0^t f\,(s, u\,(s))\,ds \tag{3.1.7}$$

in the space $C\left([0,1]\,, \mathbb{R}^n\right).$

Our approach is to seek solutions of the equation (3.1.7) as fixed points of the operator

$$(Tu)\,(t) = (I - \alpha\,[1])^{-1}\,\alpha\left[\int_0^t f\,(s, u\,(s))\,ds\right] + \int_0^t f\,(s, u\,(s))\,ds. \tag{3.1.8}$$

in the space $C\left([0,1]\,, \mathbb{R}^n\right)$. Throughout this chapter we assume that

$$\alpha_{ij}\,[v] = \int_0^{t_0} v\,(s)\,dA_{ij}\,(s)\,, \quad (v \in C\,[0,1])$$

for $i, j = 1, 2, ..., n$, where $t_0 \in [0,1]$. Note that, in this case,

$$\alpha\,[v] = 0 \;\; \text{whenever } v\,(t) \equiv 0 \text{ in } [0, t_0]\,. \tag{3.1.9}$$

Under the assumption (3.1.9), the operator T can be written as sum of two operators, one of Fredholm type and the other of Volterra type,

$$T = T_F + T_v,$$

where

$$(T_F u)\,(t) = \begin{cases} (I - \alpha\,[1])^{-1}\,\alpha\left[\int_0^t f\,(s, u\,(s))\,ds\right] + \int_0^t f\,(s, u\,(s))\,ds, & \text{for } 0 \leq t \leq t_0 \\ (I - \alpha\,[1])^{-1}\,\alpha\left[\int_0^t f\,(s, u\,(s))\,ds\right] + \int_0^{t_0} f\,(s, u\,(s))\,ds, & \text{for } t_0 \leq t \leq 1 \end{cases}$$

$$(T_V u)\,(t) = \begin{cases} 0, & \text{for } 0 \leq t \leq t_0 \\ \int_{t_0}^t f\,(s, u\,(s))\,ds, & \text{for } t_0 \leq t \leq 1. \end{cases}$$

3.2 Existence and uniqueness. The case of Lipschitz nonlinearities

In this section we assume that the functions $f_1,\ f_2,\ ...,\ f_n$ satisfy Lipschitz conditions of the form

$$|f_i(t,u) - f_i(t,v)| \le \begin{cases} a_{i1}(t)\,|u_1 - v_1| + ... + a_{in}(t)\,|u_n - v_n|\,, \text{ for } t \in [0,t_0]\,, \\ b_{i1}(t)\,|u_1 - v_1| + ... + b_{in}(t)\,|u_n - v_n|\,, \text{ for } t \in [t_0,1]\,, \end{cases} \tag{3.2.10}$$

for all $u, v \in \mathbb{R}^n$, $u = (u_1, u_2, ..., u_n)\,,\ v = (v_1, v_2, ..., v_n)\,,$ and some functions $a_{ij} \in L^1((0,1);\mathbb{R}_+)\,,\ b_{ij} \in L^p((0,1);\mathbb{R}_+)$ and $1 < p \le \infty\ \ (1 \le i,j \le n)\,.$

Using vector notations we can rewrite the condition (3.2.10) as follows:

$$\|f(t,u) - f(t,v)\| \le \begin{cases} A(t)\,\|u - v\|\,, & \text{for } t \in [0,t_0]\,, \\ B(t)\,\|u - v\|\,, & \text{for } t \in [t_0,1]\,, \end{cases} \tag{3.2.11}$$

where $A(t)\,, B(t)$ are the matrices of Lipschitz coefficients

$$A(t) = (a_{ij}(t))_{1\le i,j\le n}\,, \quad B(t) = (b_{ij}(t))_{1\le i,j\le n}\,.$$

Clearly $A \in L^1((0,1);\mathcal{M}_n(\mathbb{R}_+))$ and $B \in L^p((0,1);\mathcal{M}_n(\mathbb{R}_+))\,.$

Theorem 3.2.1 *Let $f : [0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function satisfying* (3.2.11) *and let $\alpha : C([0,1],\mathbb{R}^n) \to \mathbb{R}^n$ be linear and continuous satisfying* (3.1.6) *and* (3.1.9)*. If*

$$\textit{the matrix } \left(\left|(I - \alpha[1])^{-1}\right| |\alpha| + I\right) |A|_{L^1(0,t_0)} \textit{ is convergent to zero,} \tag{3.2.12}$$

then the problem (3.1.3) *has a unique solution $u \in W^{1,1}((0,1);\mathbb{R}^n)\,.$*

Denote by M_0, the matrix

$$M_0 := \left|(I - \alpha[1])^{-1}\right| |\alpha| + I. \tag{3.2.13}$$

Example 3.2.2 Consider the nonlocal problem

$$\begin{cases} x' = a\sin x + by + g(t) \equiv f_1(t,x,y), \\ y' = \cos(cx + dy) + h(t) \equiv f_2(t,x,y), \\ x(0) = \frac{1}{4}\int_0^{t_0} (x(s) + y(s))\,ds, \\ y(0) = \frac{1}{4}\int_0^{t_0} (x(s) + y(s))\,ds, \end{cases} \tag{3.2.14}$$

where $t \in [0,1]\,,\ a,b,c,d \in R;\ g,h \in L^1(0,1)$ and $0 \le t_0 \le 1$. We have $\alpha_{ij}[1] = \frac{t_0}{4}$ and $|\alpha_{ij}| = \frac{t_0}{4}$ for $1 \le i,j \le 2$. Then

$$M_0\,|A|_{L^1(0,t_0)} = \frac{t_0}{2(2 - t_0)} \begin{bmatrix} (4 - t_0)\,|a| + t_0\,|c| & (4 - t_0)\,|b| + t_0\,|d| \\ t_0\,|a| + (4 - t_0)\,|c| & t_0\,|b| + (4 - t_0)\,|d| \end{bmatrix}. \tag{3.2.15}$$

Therefore, if the matrix (3.2.15) is convergent to zero, then the problem (3.2.14) has a unique solution.

Particular cases: (a) if $|a| = |c|$ and $|b| = |d|$, then

$$M_0 |A|_{L^1(0,t_0)} = \frac{2t_0}{2-t_0} \begin{bmatrix} |a| & |b| \\ |a| & |b| \end{bmatrix},$$

whose eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = \frac{2t_0}{2-t_0} (|a| + |b|)$. Hence the matrix is convergent to zero if and only if

$$\frac{2t_0}{2-t_0} (|a| + |b|) < 1.$$

(b) if $|a| = |d|$ and $|b| = |c|$, then

$$M_0 |A|_{L^1(0,t_0)} = \frac{t_0}{2(2-t_0)} \begin{bmatrix} (4-t_0)|a| + t_0|b| & t_0|a| + (4-t_0)|b| \\ t_0|a| + (4-t_0)|b| & (4-t_0)|a| + t_0|b| \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = t_0(|a| - |b|)$, $\lambda_2 = \frac{2t_0}{2-t_0} (|a| + |b|)$. Since $|\lambda_1| \le \lambda_2$, the matrix is convergent to zero if and only if $\frac{2t_0}{2-t_0} (|a| + |b|) < 1$.

3.3 Existence. Nonlinearities with growth at most linear

We now assume that the functions $f_1, f_2, ..., f_n$ satisfy instead of Lipschitz conditions, the more relaxed conditions of growth at most linear

$$|f_i(t,u)| \le \begin{cases} a_{i1}(t)|u_1| + ... + a_{in}(t)|u_n| + \overline{a}_i(t), & \text{for } t \in [0,t_0], \\ b_{i1}(t)|u_1| + ... + b_{in}(t)|u_n| + \overline{b}_i(t), & \text{for } t \in [t_0,1], \end{cases} \tag{3.3.16}$$

for $u \in \mathbb{R}^n$, $u = (u_1, u_2, ..., u_n)$ and some functions $a_{ij}, \overline{a}_i, \overline{b}_i \in L^1((0,1);\mathbb{R}_+)$, $b_{ij} \in L^p((0,1);\mathbb{R}_+)$ and $1 < p \le \infty$ $(1 \le i,j \le n)$.

Using vector notations we can rewrite the condition (3.3.16) as follows:

$$\|f(t,u)\| \le \begin{cases} A(t)\|u\| + \overline{a}(t), & \text{for } t \in [0,t_0], \\ B(t)\|u\| + \overline{b}(t), & \text{for } t \in [t_0,1], \end{cases} \tag{3.3.17}$$

where $A(t), B(t)$ are the matrices of coefficients $A(t) = (a_{ij}(t))_{1\le i,j\le n}$, $B(t) = (b_{ij}(t))_{1\le i,j\le n}$ and $\overline{a}(t), \overline{b}(t)$ are the column matrices $\overline{a}(t) = (\overline{a}_i(t))_{1\le i\le n}$, $\overline{b}(t) = \left(\overline{b}_i(t)\right)_{1\le i\le n}$. One has $A \in L^1((0,1);\mathcal{M}_n(\mathbb{R}_+))$, $B \in L^p((0,1);\mathcal{M}_n(\mathbb{R}_+))$ and $\overline{a}, \overline{b} \in L^1\left((0,1);\mathbb{R}^n_+\right)$.

Theorem 3.3.1 *Let $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function satisfying* (3.3.17) *and let $\alpha : C([0,1],\mathbb{R}^n) \to \mathbb{R}^n$ be linear and continuous satisfying* (3.1.6) *and* (3.1.9). *If the condition* (3.2.12) *holds, then the problem* (3.1.3) *has at least one solution $u \in W^{1,1}((0,1);\mathbb{R}^n)$.*

Example 3.3.2 Consider the nonlocal problem

$$\begin{cases} x' = ax\sin\left(\frac{y}{x}\right) + by\sin\left(\frac{x}{y}\right) + g(t) \equiv f_1(t,x,y), \\ y' = cx\sin\left(\frac{y}{x}\right) + dy\sin\left(\frac{x}{y}\right) + h(t) \equiv f_2(t,x,y), \\ x(0) = \frac{1}{4}\int_0^{t_0} (x(s) + y(s))\,ds, \\ y(0) = \frac{1}{4}\int_0^{t_0} (x(s) + y(s))\,ds, \end{cases} \tag{3.3.18}$$

where $t \in [0,1]$, $a,b,c,d \in R$; $g,h \in L^1(0,1)$ and $0 \le t_0 \le 1$. Since

$$|f_1(t,x,y)| \le |a|\,|x| + |b|\,|y| + |g(t)|, \qquad |f_2(t,x,y)| \le |c|\,|x| + |d|\,|y| + |h(t)|$$

we are under the assumptions from Section 3.3. Also, the matrix $M_0\,|A|_{L^1(0,t_0)}$ is that from Example 3.2.2. Therefore, according to Theorem 3.3.1, if that matrix is convergent to zero, then the problem (3.3.18) has at least one solution. Note that the functions $f_1(t,x,y)$, $f_2(t,x,y)$ from this example do not satisfy Lipschitz conditions in x,y and consequently Theorem 3.2.1 does not apply.

3.4 Existence. Nonlinearities with a more general growth

We now assume that the nonlinearities $f_1, f_2, ..., f_n$ satisfy more general growth conditions, namely

$$|f_i(t,u)| \le \begin{cases} \omega_i(t,\|u\|), & \text{if } t \in [0,t_0], \\ \beta(|u|)\gamma_i(t), & \text{if } t \in [t_0,1], \end{cases} \tag{3.4.19}$$

for all $u = (u_1, u_2, ..., u_2) \in \mathbb{R}^n$ $(1 \le i \le n)$. Here ω_i are L^1-Carathéodory functions on $[0,t_0] \times \mathbb{R}^n_+$, nondecreasing in their second argument, $\gamma_i \in L^1((t_0,1);\mathbb{R}_+)$, while $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and $1/\beta \in L^1_{loc}(\mathbb{R}_+)$; here, the symbols $|.|$, $\|.\|$ are used to denote the Euclidean norm and the vector-valued norm on $\mathbb{R}^n$, respectively.

Using vector notation, the condition (3.4.19) can be rewritten as follows:

$$\|f(t,u)\| \le \begin{cases} \omega(t,\|u\|), & \text{if } t \in [0,t_0], \\ \beta(|u|)\gamma(t), & \text{if } t \in [t_0,1], \end{cases} \tag{3.4.20}$$

where $\omega(t,\|u\|)$, $\gamma(t)$ are the column matrices $\omega(t,\|u\|) = (\omega_i(t,\|u\|))_{1\le i\le n}$, $\gamma(t) = (\gamma_i(t))_{1\le i\le n}$.

Theorem 3.4.1 *Assume that the condition* (3.4.20) *holds. In addition assume that there exists a vector $R_0 \in \mathbb{R}^n_+$ and a number $R_1 > 0$ such that*

$$\text{if } \ \rho \in \mathbb{R}^n_+ \ \text{ and } \ M_0 \int_0^{t_0} \omega(t,\rho)dt \ge \rho, \ \text{ then } \ \rho \le R_0 \tag{3.4.21}$$

and

$$\int_{t_0}^1 |\gamma(s)|\,ds = \int_{R_0^*}^{R_1} \frac{1}{\beta(\tau)}d\tau, \tag{3.4.22}$$

where $R_0^* = \left|M_0 \int_0^{t_0} \omega(s,R_0)ds\right|$ and M_0 is given by (3.2.13)*. Then the problem* (3.1.3) *has at least one solution $u \in W^{1,1}((0,1);\mathbb{R}^n)$ such that*

$$\|u\|_{C[0,t_0]} \le R_0 \ \text{ and } \ |u|_{C([t_0,1],\mathbb{R}^n)} \le R_1. \tag{3.4.23}$$

Chapter 4

Existence results for second order three-point boundary value problems

4.1 Overview

This chapter is devoted to the study of second order differential equations and systems with nonlinear three point boundary conditions.

Motivated by paper E.V. Castelani and T. F. Ma [23], in Section 4.2, we study the three-point boundary value problem for second order differential equations:

$$\begin{cases} u'' = f\left(t, u, u'\right), \quad 0 < t < t_0 \\ u(0) = 0, \, u(t_0) = g(u(\eta)) \end{cases} \tag{4.1.1}$$

where $0 < \eta < t_0 < 1$ and f, g are continuous functions. Our tools here are Banach's and Schauder's fixed point theorems.

Then, in Section 4.3, we discuss differential systems of the type

$$\begin{cases} u''(t) = f\left(t, u(t), v(t)\right) \\ v''(t) = g\left(t, u(t), v(t)\right) \\ u(0) = 0, \; u(t_0) = \phi(u(\eta), v(\eta)) \\ v(0) = 0, \; v(t_0) = \psi(u(\eta), v(\eta)) \end{cases} \quad (0 < t < t_0)$$

by using Perov's and Schauder's fixed point theorems and the technique based on convergent to zero matrices and vector-valued norms.

Section 4.4 is devoted to the problem

$$\begin{cases} u'' = f\left(t, u, u'\right), \quad 0 < t < 1 \\ u(0) = 0, \; u(t_0) = g(u(\eta)). \end{cases} \tag{4.1.2}$$

Compared to problem (4.1.1), even if the three-boundary condition is the same, equation (4.1.2) is considered on the larger interval $[0, 1]$ which allows us to combine the operator method with the technique based on Bielecki norm. Finally, in Section 4.5, a similar strategy is applied to a system of two second order differential equations.

4.2 Existence results for equations

Consider problem (4.1.1) with $f : [0,t_0] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ continuous functions. Here are some hypotheses:

(H1) there exist $a, b, c > 0$ such that

$$\begin{cases} |f(t,u,v) - f(t,\overline{u},\overline{v})| \le a\,|u-\overline{u}| + b\,|v-\overline{v}| \\ |g(u) - g(\overline{u})| \le c\,|u-\overline{u}|, \end{cases} \tag{4.2.1}$$

for $t \in [0,t_0]$ and $u, v, \overline{u}, \overline{v} \in \mathbb{R}$.

(H2) there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 > 0$ such that

$$|f(t,u,v)| \le \alpha_1\,|u| + \alpha_2\,|v| + \alpha_3 \;\text{ and }\; |g(u)| \le \beta_1\,|u| + \beta_2, \tag{4.2.2}$$

for $t \in [0,t_0]$ and $u, v \in \mathbb{R}$.

4.2.1 Application of Banach's contraction principle

We begin this section by pointing out that problem (4.1.1) can be written equivalently as

$$u(t) = \int_0^{t_0} G(t,s) f\left(s, u\left(s\right), u'(s)\right) ds + \frac{t}{t_0} g(u(\eta)) \tag{4.2.3}$$

where G is the Green function for $u''(t) = f(t)$ with $u(0) = u(t_0) = 0$, namely

$$G(t,s) = \begin{cases} -\frac{t(t_0-s)}{t_0}, & 0 \le t \le s \le t_0 \\ -\frac{s(t_0-t)}{t_0}, & 0 \le s \le t \le t_0. \end{cases} \tag{4.2.4}$$

We observe that u is a solution of (4.1.1) if and only if u is a fixed point of the operator $T : C^1[0,t_0] \to C^1[0,t_0]$, defined by

$$(Tu)(t) = \int_0^{t_0} G(t,s) f\left(s, u\left(s\right), u'(s)\right) ds + \frac{t}{t_0} g(u(\eta)). \tag{4.2.5}$$

Theorem 4.2.1 *If f, g satisfy* (H1) *with*

$$\frac{a+b}{2} t_0 + \frac{c}{t_0} < 1, \tag{4.2.6}$$

then problem (4.1.1) *has a unique solution. Moreover, this solution can be obtained as limit of the sequence of succesive approximations.*

4.2.2 Application of Schauder's fixed point theorem

Under the weaker hypothesis (H2), we have the following existence result as a consequence of Schauder's fixed point theorem.

Theorem 4.2.2 *Assume that* (H2) *holds with*

$$\frac{\alpha_1 + \alpha_2}{2} t_0 + \frac{\beta_1}{t_0} < 1. \tag{4.2.7}$$

Then problem (4.1.1) *has at least one solution.*

4.2.3 Application of Boyd-Wong's fixed point theorem

Theorem 4.2.3 (Boyd-Wong contraction principle) *Let (X, d) be a complete metric space and suppose $T : X \to X$ satisfies:*

$$d(Tx, Ty) \leq \Psi(d(x, y)) \quad \text{for each } x, y \in X,$$

where $\Psi : [0, \infty) \to [0, \infty)$, $0 \leq \Psi(t) < t$ for $t > 0$ and Ψ is upper semicontinuous from the right, that is, $r_j \searrow r \geq 0$ implies $\limsup_{j \to \infty} \Psi(r_j) \leq \Psi(r)$. Then T has a unique fixed point x^* and $(T^n(x))$ converges to x^* for each $x \in X$.

In this section, instead of the Lipschitz condition on f from (H1), we shall consider more generally conditions of Boyd-Wong type, namely:

(H3) there exist $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ upper semicontinuous from the right and nondecreasing, and $c > 0$ such that

$$\begin{cases} |f(t, u, v) - f(t, \overline{u}, \overline{v})| \leq \psi_1(|u - \overline{u}|) + \psi_2(|v - \overline{v}|) \\ |g(u) - g(\overline{u})| \leq c\, |u - \overline{u}|, \end{cases} \tag{4.2.8}$$

for $t \in [0, t_0]$ and $u, \overline{u}, v, \overline{v} \in \mathbb{R}$.

Theorem 4.2.4 *If f, g satisfy satisfy* (H3) *and*

$$\Psi(t) := \frac{t_0}{2}(\psi_1 + \psi_2)(t) + \frac{c}{t_0} t < t, \tag{4.2.9}$$

for $t > 0$, then problem (4.1.1) *has a unique solution. Moreover, this solution can be obtained as limit of the sequence of succesive approximations.*

4.3 Existence results for systems

We next deal with the three-point boundary value problem for second order differential systems of the type:

$$\begin{cases} u''(t) = f\,(t, u(t), v(t)) \\ v''(t) = g\,(t, u(t), v(t)) \\ u(0) = 0, \;\; u(t_0) = \phi(u(\eta), v(\eta)) \\ v(0) = 0, \;\; v(t_0) = \psi(u(\eta), v(\eta)), \end{cases} \quad 0 < t < t_0 \tag{4.3.10}$$

where $0 < \eta < t_0 < 1$ and f, g, ϕ and ψ are continuous functions.

This can be viewed as a fixed point problem in $C\,[0, t_0]^2$

$$\begin{cases} u = T_1(u, v) \\ v = T_2(u, v) \end{cases}$$

for a completely continuous operator $T=(T_1,T_2)$, $T: C[0,t_0]^2 \to C[0,t_0]^2$ where T_1, T_2 are given by

$$T_1(u,v)(t) = \int_0^{t_0} G(t,s) f(s,u(s),v(s))\,ds + \tfrac{t}{t_0}\phi(u(\eta),v(\eta))$$
$$T_2(u,v)(t) = \int_0^{t_0} G(t,s) g(s,u(s),v(s))\,ds + \tfrac{t}{t_0}\psi(u(\eta),v(\eta)).$$

4.3.1 Nonlinearities with the Lipschitz property. Application of Perov's fixed point theorem

Here the existence of solutions to problem (4.3.10) is established by means of Perov's fixed point theorem. For this, we assume global Lipschitz conditions, that is

$$\begin{cases} |f(t,u,v)-f(t,\overline{u},\overline{v})| \le a_1|u-\overline{u}| + b_1|v-\overline{v}| \\ |g(t,u,v)-g(t,\overline{u},\overline{v})| \le a_2|u-\overline{u}| + b_2|v-\overline{v}| \\ |\phi(u,v)-\phi(\overline{u},\overline{v})| \le c_1|u-\overline{u}| + d_1|v-\overline{v}| \\ |\psi(u,v)-\psi(\overline{u},\overline{v})| \le c_2|u-\overline{u}| + d_2|v-\overline{v}|, \end{cases} \tag{4.3.11}$$

for $t\in[0,t_0]$, $u,v,\overline{u},\overline{v}\in\mathbb{R}$ and some $a_1,b_1,a_2,b_2,c_1,d_1,c_2,d_2 \ge 0$.

Theorem 4.3.1 *Assume that condition* (4.3.11) *holds. If the matrix*

$$M := \begin{bmatrix} \frac{a_1}{8}t_0^2 + c_1 & \frac{b_1}{8}t_0^2 + d_1 \\ \frac{a_2}{8}t_0^2 + c_2 & \frac{b_2}{8}t_0^2 + d_2 \end{bmatrix} \tag{4.3.12}$$

is convergent to zero, then problem (4.3.10) *has a unique solution in* $C[0,t_0]^2$.

4.3.2 Nonlinearities with growth at most linear. Application of Schauder's fixed point theorem

Here the existence of solutions to problem (4.3.10) is established by means of Schauder's fixed point theorem in case that f,g satisfy instead of the Lipschitz condition the more relaxed condition of growth at most linear, that is

$$\begin{cases} |f(t,u,v)| \le a_1|u| + b_1|v| + c_1 \\ |g(t,u,v)| \le a_2|u| + b_2|v| + c_2 \\ |\phi(u,v)| \le a_{01}|u| + b_{01}|v| + c_{01} \\ |\psi(u,v)| \le a_{02}|u| + b_{02}|v| + c_{02}, \end{cases} \tag{4.3.13}$$

for all $t\in[0,t_0]$, $u,v\in\mathbb{R}$ and some $a_i,b_i,c_i,a_{0i},b_{0i},c_{0i}\ge 0, i=1,2$.

Theorem 4.3.2 *If f,g,ϕ,ψ satisfy conditions* (4.3.13) *and the matrix*

$$M := \begin{bmatrix} \frac{a_1}{8}t_0^2 + a_{01} & \frac{b_1}{8}t_0^2 + b_{01} \\ \frac{a_2}{8}t_0^2 + a_{02} & \frac{b_2}{8}t_0^2 + b_{02} \end{bmatrix} \tag{4.3.14}$$

is convergent to zero, then problem (4.3.10) *has at least one solution in* $C[0,t_0]^2$.

4.4 Equations on a larger interval

We present existence results for the three-point boundary value problem:

$$\left\{ \begin{array}{l} u'' = f\left(t, u, u'\right) \\ u(0) = 0,\; u(t_0) = g(u(\eta)) \end{array} \right. , \quad 0 < t < 1 \tag{4.4.15}$$

where $0 < t_0 < \eta < 1$ and f, g are continuous functions. Problem (4.4.15) could be splitted into two parts, one for the subinterval $[0, t_0]$ and the other one for $[t_0, 1]$. More exactly, we look for u such that

$$u(t) = \left\{ \begin{array}{ll} v(t), & \text{if } t \in [0, t_0] \\ w(t), & \text{if } t \in [t_0, 1] \end{array} \right.$$

where v solves

$$\left\{ \begin{array}{l} v'' = f\left(t, v, v'\right) \\ v(0) = 0,\; v(t_0) = g(v(\eta)) \end{array} \right. , \quad 0 < t < t_0 \tag{4.4.16}$$

while w is a solution of

$$\left\{ \begin{array}{l} w'' = f\left(t, w, w'\right) \\ w(t_0) = v(t_0) \\ w'(t_0) = v'(t_0). \end{array} \right. , \quad t_0 < t < 1 \tag{4.4.17}$$

Problem (4.4.16) was already discussed in *Section 4.2*. Here we just point out that it is equivalent to a fixed point problem for the Fredholm type operator $T_F : C^1[0, t_0] \to C^1[0, t_0]$,

$$(T_F v)(t) = \int_0^{t_0} G(t, s) f\left(s, v(s), v'(s)\right) ds + \frac{t}{t_0} g(v(\eta)).$$

For (4.4.17) we construct a Volterra type integral operator $T_V : C^1[t_0, 1] \to C^1[t_0, 1]$ given by

$$(T_V w)(t) = v(t_0) + (t - t_0)v'(t_0) + \int_{t_0}^{t} \int_{t_0}^{\sigma} f\left(s, w(s), w'(s)\right) ds d\sigma. \tag{4.4.18}$$

Notice that w solves (4.4.17) if and only if w is a fixed point of the operator T_V.

4.4.1 Application of Banach's contraction principle

We assume global Lipschitz conditions, that is the existence of $\widetilde{a}_1, \widetilde{b}_1 > 0$ such that

$$|f(t, u, v) - f(t, \overline{u}, \overline{v})| \leq \widetilde{a}_1 |u - \overline{u}| + \widetilde{b}_1 |v - \overline{v}|, \tag{4.4.19}$$

for all $t \in [t_0, 1]$ and $u, v, \overline{u}, \overline{v} \in \mathbb{R}$.

Theorem 4.4.1 *If f satisfies* (4.4.19) *for some numbers $\widetilde{a}_1, \widetilde{b}_1 > 0$, then problem* (4.4.17) *has a unique solution.*

4.4.2 Application of Schauder's fixed point theorem

Assume that f satisfies more general conditions than (4.4.19), namely

$$|f(t,u,v)| \leq \widetilde{\alpha}_1 |u| + \widetilde{\alpha}_2 |v| + \widetilde{\alpha}_3 \tag{4.4.20}$$

for all $t \in [t_0, 1]$ and $u, v \in \mathbb{R}$, where $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 \geq 0$.

Theorem 4.4.2 *If the condition* (4.4.20) *holds for some numbers* $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 \geq 0$, *then problem* (4.4.17) *has at least one solution.*

Putting together the results from Section 4.2.1, Section 4.4.1 and the results from Section 4.2.2, Section 4.4.2 respectively, we obtain the following results for equations on the entire interval $[0,1]$:

Theorem 4.4.3 *If* f, g *satisfy* (H1) *with* (4.2.6) *and condition* (4.4.19), *then problem* (4.4.15) *has a unique solution on* $[0,1]$.

Theorem 4.4.4 *Assume that* (H2) *holds with* (4.2.7). *If, in addition, condition* (4.4.20) *holds, then problem* (4.4.15) *has at least one solution on* $[0,1]$.

4.5 Systems on a larger interval

Here we consider the three point boundary value problems for second order differential systems of the type:

$$\begin{cases} u''(t) = f\left(t, u(t), v(t)\right) \\ v''(t) = g\left(t, u(t), v(t)\right) \\ u(0) = 0, \ u(t_0) = \phi(u(\eta), v(\eta)) \\ v(0) = 0, \ v(t_0) = \psi(u(\eta), v(\eta)), \end{cases} , \quad 0 < t < 1 \tag{4.5.21}$$

where $0 < \eta < t_0 < 1$. These systems can be splitted into two parts, one for the subinterval $[0, t_0]$ and the other one for $[t_0, 1]$, respectively. A similar approach was given for equations in *Section 4.4*. Systems on $[0, t_0]$ were already discussed in *Section 4.3*.

It remains to be considered the system:

$$\begin{cases} x''(t) = f\left(t, x(t), y(t)\right) \\ y''(t) = g\left(t, x(t), y(t)\right) \\ x(t_0) = u_0(t_0), \ x'(t_0) = u_0'(t_0) \\ y(t_0) = v_0(t_0), \ y'(t_0) = v_0'(t_0), \end{cases} , \quad t_0 \leq t \leq 1 \tag{4.5.22}$$

where by (u_0, v_0) we mean the solution of (4.5.21) on the interval $[0, t_0]$.

This can be viewed as a fixed point problem in $C\left[t_0, 1\right]^2$ for the completely continuous operator $T = (T_1, T_2)$, $T : C\left[t_0, 1\right]^2 \to C\left[t_0, 1\right]^2$, where

$$\begin{aligned} T_1(x,y)(t) &= u_0(t_0) + (t - t_0)u_0'(t_0) + \int_{t_0}^{t}\int_{t_0}^{\sigma} f\left(s, x\left(s\right), y(s)\right) ds d\sigma, \\ T_2(x,y)(t) &= v_0(t_0) + (t - t_0)v_0'(t_0) + \int_{t_0}^{t}\int_{t_0}^{\sigma} g\left(s, x\left(s\right), y(s)\right) ds d\sigma. \end{aligned}$$

4.5.1 Nonlinearities with the Lipschitz property. Application of Perov's fixed point theorem

Here we prove that the existence of solutions to problem (4.5.22) by means of Perov's fixed point theorem. For this, we assume global Lipschitz conditions, that is the existence of numbers $\widetilde{a}_1, \widetilde{b}_1, \widetilde{c}_1, \widetilde{d}_1 > 0$ such that:

$$\begin{cases} |f(t,x,y) - f(t,x,y)| \leq \widetilde{a}_1 |x - \overline{x}| + \widetilde{b}_1 |y - \overline{y}| \\ |g(t,x,y) - g(t,x,y)| \leq \widetilde{c}_1 |x - \overline{x}| + \widetilde{d}_1 |y - \overline{y}|, \end{cases} \tag{4.5.23}$$

for $t \in [t_0, 1]$ and $x, y, \overline{x}, \overline{y} \in \mathbb{R}$.

Theorem 4.5.1 *If the conditions* (4.5.23) *hold, then problem* (4.5.22) *has a unique solution in* $C[t_0, 1]^2$.

4.5.2 Nonlinearities with growth at most linear. Application of Schauder's fixed point theorem

Here we show that the existence of solutions to problem (4.5.22) is established by means of Schauder's fixed point theorem in case that f, g satisfy instead of the Lipschitz conditions, the more relaxed conditions of growth at most linear, that is

$$\begin{cases} |f(t,x,y)| \leq \widetilde{a}_1 |x| + \widetilde{b}_1 |y| + \widetilde{c}_1 \\ |g(t,x,y)| \leq \widetilde{a}_2 |x| + \widetilde{b}_2 |y| + \widetilde{c}_2, \end{cases} \tag{4.5.24}$$

for $t \in [t_0, 1]$; $x, y \in \mathbb{R}$ and some $\widetilde{a}_i, \widetilde{b}_i, \widetilde{c}_i \geq 0, i = 1, 2$.

Theorem 4.5.2 *If f, g satisfy conditions* (4.5.24)*, then problem* (4.5.22) *has at least one solution in* $C[t_0, 1]^2$.

Putting together the results from Section 4.3.1, Section 4.5.1 and the results from Section 4.3.2, Section 4.5.2 respectively, we obtain the following results for systems on the entire interval $[0, 1]$:

Theorem 4.5.3 *Assume that conditions* (4.3.11) *and* (4.5.23) *hold. If the matrix* (4.3.12) *is convergent to zero, then problem* (4.5.21) *has a unique solution in* $C[0, 1]^2$.

Theorem 4.5.4 *If f, g satisfy conditions* (4.3.13) *and* (4.5.24) *and if the matrix* (4.3.14) *is convergent to zero, then problem* (4.5.21) *has at least one solution in* $C[0, 1]^2$.

Chapter 5

Nonlinear nonlocal initial value problems

5.1 Overview

The purpose of the present chapter is to study the existence of solutions to initial value problems for first order differential systems with nonlinear nonlocal boundary conditions of functional type.

Therefore, we consider the nonlocal initial value problem for the first order differential system

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)), \\ y'(t) = f_2(t, x(t), y(t)), & \text{on } [0,1], \\ x(0) = \alpha[x,y], \\ y(0) = \beta[x,y]. \end{cases} \tag{5.1.1}$$

Here, $f_1, f_2 : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and $\alpha, \beta : C[0,1]^2 \to \mathbb{R}$ are nonlinear continuous functionals.

Our approach is to rewrite the problem (5.1.1) as system of the form

$$x_a = \left(a + \int_0^t f_1(s, x(s), y(s))\, ds,\ \alpha[x,y]\right),$$
$$y_b = \left(b + \int_0^t f_2(s, x(s), y(s))\, ds,\ \beta[x,y]\right),$$

where by x_a, y_b we mean the pairs $(x,a), (y,b) \in C[0,1]\times\mathbb{R}$.

This, in turn, can be viewed as a fixed point problem in $(C[0,1]\times\mathbb{R})^2$ for the completely continuous operator

$$T = (T_1, T_2) : (C[0,1]\times\mathbb{R})^2 \to (C[0,1]\times\mathbb{R})^2,$$

where T_1 and T_2 are given by

$$T_1[x_a, y_b] = \left(a + \textstyle\int_0^t f_1(s, x(s), y(s))\, ds,\ \alpha[x,y]\right),$$
$$T_2[x_a, y_b] = \left(b + \textstyle\int_0^t f_2(s, x(s), y(s))\, ds,\ \beta[x,y]\right).$$

In this chapter, by $|x|_C$, where $x \in C[0,1]$, we shall mean $|x|_C = \max_{t\in[0,1]} |x(t)|$.

5.2 Existence and uniqueness

In the present section we show that the existence of solutions to problem (5.1.1) follows from Perov's fixed point theorem in case that the nonlinearities f_1, f_2 and also the functionals α, β satisfy Lipschitz conditions of the type:

$$\begin{cases} |f_1(t,x,y) - f_1(t,\overline{x},\overline{y})| \le a_1 |x-\overline{x}| + b_1 |y-\overline{y}| \\ |f_2(t,x,y) - f_2(t,\overline{x},\overline{y})| \le a_2 |x-\overline{x}| + b_2 |y-\overline{y}|, \end{cases} \tag{5.2.1}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$, and

$$\begin{cases} |\alpha[x,y] - \alpha[\overline{x},\overline{y}]| \le A_1 |x-\overline{x}|_C + B_1 |y-\overline{y}|_C \\ |\beta[x,y] - \beta[\overline{x},\overline{y}]| \le A_2 |x-\overline{x}|_C + B_2 |y-\overline{y}|_C, \end{cases} \tag{5.2.2}$$

for all $x, y, \overline{x}, \overline{y} \in C[0,1]$.

For a given number $\theta > 0$, denote

$$\begin{array}{ll} m_{11}(\theta) = \max\left\{\frac{1}{\theta}, a_1 + \theta A_1\right\}, & m_{12}(\theta) = b_1 + \theta B_1, \\ m_{21}(\theta) = a_2 + \theta A_2, & m_{22}(\theta) = \max\left\{\frac{1}{\theta}, b_2 + \theta B_2\right\}. \end{array}$$

Theorem 5.2.1 *If f_1, f_2 satisfy the Lipschitz conditions* (5.2.1), *α, β satisfy conditions* (5.2.2). *In addition assume that for some $\theta > 0$, the matrix*

$$M_\theta = \begin{bmatrix} m_{11}(\theta) & m_{12}(\theta) \\ m_{21}(\theta) & m_{22}(\theta) \end{bmatrix} \tag{5.2.3}$$

is convergent to zero. Then the problem (5.1.1) *has a unique solution.*

Example 5.2.2 Consider the nonlocal problem

$$\begin{cases} x' = \frac{1}{4}\sin x + ay + g(t) \equiv f_1(t,x,y), \\ y' = \cos\left(ax + \frac{1}{4}y\right) + h(t) \equiv f_2(t,x,y), \\ x(0) = \frac{1}{8}\sin\left(x\left(\frac{1}{4}\right) + y\left(\frac{1}{4}\right)\right), \\ y(0) = \frac{1}{8}\cos\left(x\left(\frac{1}{4}\right) + y\left(\frac{1}{4}\right)\right), \end{cases} \tag{5.2.4}$$

where $t \in [0,1]$, $a \in R$ and $g, h \in L^1(0,1)$. We have $a_1 = 1/4$, $b_1 = |a|$, $a_2 = |a|$, $b_2 = 1/4$ and $A_1 = B_1 = A_2 = B_2 = 1/8$. Consider $\theta = 2$. Hence

$$M_\theta = \begin{bmatrix} \frac{1}{2} & |a| + \frac{1}{4} \\ |a| + \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \tag{5.2.5}$$

Since the eigenvalues of M_θ are $\lambda_1 = -|a| + \frac{1}{4}$, $\lambda_2 = |a| + \frac{3}{4}$, the matrix (5.2.5) is convergent to zero if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. It is also known that a matrix of this type is convergent to zero if $|a| + \frac{1}{4} + \frac{1}{2} < 1$ (see R. Precup [65]). Therefore, if $|a| < \frac{1}{4}$, the matrix (5.2.5) is convergent to zero and from Theorem 5.2.1 the problem (5.2.4) has a unique solution.

5.3 Existence results

In the begining of this section, we give an application of Schauder's fixed point theorem. More exactly, we show that the existence of solutions to the problem (5.1.1) follows from Schauder's fixed point theorem in case that f_1, f_2 and also functionals α, β satisfy some relaxed growth conditions of the type:

$$\begin{cases} |f_1(t,x,y)| \leq a_1\,|x| + b_1\,|y| + c_1, \\ |f_2(t,x,y)| \leq a_2\,|x| + b_2\,|y| + c_2, \end{cases} \tag{5.3.1}$$

for all $x, y \in \mathbb{R}$, and

$$\begin{cases} |\alpha\,[x,y]| \leq A_1\,|x|_C + B_1\,|y|_C + C_1, \\ |\beta\,[x,y]| \leq A_2\,|x|_C + B_2\,|y|_C + C_2, \end{cases} \tag{5.3.2}$$

for all $x, y \in C[0,1]$.

Theorem 5.3.1 *If the conditions* (5.3.1), (5.3.2) *hold and the matrix* (5.2.3) *is convergent to zero for some* $\theta > 0$, *then the problem* (5.1.1) *has at least one solution.*

We shall give another numerical example for the above existence result.

Example 5.3.2 Consider the nonlocal problem

$$\begin{cases} x' = \frac{1}{4}x\sin\left(\frac{y}{x}\right) + ay\sin\left(\frac{x}{y}\right) + g(t) \equiv f_1(t,x,y), \\ y' = ax\sin\left(\frac{y}{x}\right) + \frac{1}{4}y\sin\left(\frac{x}{y}\right) + h(t) \equiv f_2(t,x,y), \\ x(0) = \frac{1}{8}\sin\left(x\left(\frac{1}{4}\right) + y\left(\frac{1}{4}\right)\right), \\ y(0) = \frac{1}{8}\cos\left(x\left(\frac{1}{4}\right) + y\left(\frac{1}{4}\right)\right), \end{cases} \tag{5.3.3}$$

where $t \in [0,1]$, $a \in R$ and $g, h \in L^1(0,1)$. Since

$$|f_1(t,x,y)| \leq \frac{1}{4}|x| + |a|\,|y| + |g(t)|, \qquad |f_2(t,x,y)| \leq |a|\,|x| + \frac{1}{4}|y| + |h(t)|$$

we are under the assumptions from the first part of Section 5.3. Also, the matrix M_θ is that from Example 5.2.2 if we consider $\theta = 2$. Therefore, according to Theorem 5.3.1, if that matrix is convergent to zero, then the problem (5.3.3) has at least one solution. Note that the functions $f_1(t,x,y)$, $f_2(t,x,y)$ from this example do not satisfy Lipschitz conditions in x, y and consequently Theorem 5.2.1 does not apply.

In what follows, we give an application of the Leray-Schauder Principle and we assume that the nonlinearities f_1, f_2 and also the nonlinear functionals α, β satisfy more general growth conditions, namely:

$$\begin{cases} |f_1(t,x,y)| \leq \omega_1(t,|x|,|y|), \\ |f_2(t,x,y)| \leq \omega_2(t,|x|,|y|), \end{cases} \tag{5.3.4}$$

for all $x, y \in \mathbb{R}$,

$$\begin{cases} |\alpha\,[x,y]| \leq \omega_3(|x|_C, |y|_C), \\ |\beta\,[x,y]| \leq \omega_4(|x|_C, |y|_C), \end{cases} \tag{5.3.5}$$

for all $x, y \in C[0,1]$. Here ω_1, ω_2 are L^1-Carathéodory functions on $[0,1]\times\mathbb{R}_+^2$, nondecreasing in their second and third arguments, and ω_3, ω_4 are continuous functions on $\mathbb{R}_+^2$, nondecreasing in both variables.

Theorem 5.3.3 *Assume that the conditions* (5.3.4), (5.3.5) *hold. In addition assume that there exists* $R_0 = \left(R_0^1, R_0^2\right) \in (0,\infty)^2$ *such that for* $\rho = (\rho_1, \rho_2) \in (0,\infty)^2$

$$\begin{cases} \int_0^1 \omega_1(s,\rho_1,\rho_2)ds + \omega_3(\rho_1,\rho_2) \geq \rho_1 \\ \int_0^1 \omega_2(s,\rho_1,\rho_2)ds + \omega_4(\rho_1,\rho_2) \geq \rho_2 \end{cases} \quad \textit{implies} \quad \rho \leq R_0. \tag{5.3.6}$$

Then the problem (5.1.1) *has at least one solution.*

Chapter 6

Impulsive systems with nonlocal initial conditions

6.1 Overview

In this chapter we deal with a system of first order differential equations with impulsive terms subject to nonlocal initial value conditions, namely

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) & t \in (0,1),\ t \neq t_0, \\ \quad \Delta x|_{t=t_0} = I_1(x(t_0)), \quad \Delta y|_{t=t_0} = I_2(y(t_0)), \\ \quad x(0) = \alpha_1[x], \qquad\qquad y(0) = \alpha_2[y]. \end{cases} \tag{6.1.1}$$

Here $\Delta v|_{t=t_0}$ denotes the "jump" of the function v in $t = t_0$, that is $\Delta v|_{t=t_0} = v(t_0^+) - vt_0^-)$, where $v(t_0^-), v(t_0^+)$ are the left and the right limits of v in $t = t_0$ and α_i, $i = 1, 2$ are linear continuous functionals on $C[0,1]$ which satisfy the condition (2.3.14).

We rewrite the system (6.1.1) as a system of integral equations

$$\begin{cases} x(t) = \dfrac{1}{1-\alpha_1[1]}\alpha_1[g_1] + g_1(x,y)(t) + G_1(x)(t), \\ y(t) = \dfrac{1}{1-\alpha_2[1]}\alpha_2[g_2] + g_2(x,y)(t) + G_2(y)(t), \end{cases} \tag{6.1.2}$$

where the terms G_i take into account the impulsive effect.

Also, we shall benefit of a careful decomposition similar to the one proposed in A. Boucherif and R. Precup [17] and later used in O. Nica and R. Precup [52], O. Nica [53], also presented in Chapter 2 and Chapter 3. This is the first time that this approach is used in the context of nonlocal impulsive systems.

6.2 An existence and uniqueness result

We work in the Banach space $PC_{t_0}[0,1]^2$, where

$$PC_{t_0}[0,1] := \{x : [0,1] \to \mathbb{R}|,\ x \text{ is continuous for } t \in [0,1]\backslash\{t_0\}, \\ \text{there exist } x(t_0^-) = x(t_0) \text{ and } |x(t_0^+)| < \infty\}.$$

Solving (6.1.1), equivalently (6.1.2), reduces to the existence of a fixed point of the operator

$$T = T_F + T_V + G, \tag{6.2.3}$$

where

$$T_F(x,y)(t) = \begin{pmatrix} T_{F_1}(x,y)(t) \\ T_{F_2}(x,y)(t) \end{pmatrix}, \quad T_V(x,y)(t) = \begin{pmatrix} T_{V_1}(x,y)(t) \\ T_{V_2}(x,y)(t) \end{pmatrix},$$

with for $i = 1, 2$

$$T_{F_i}(x,y)(t) = \begin{cases} \frac{1}{1-\alpha_i[1]}\alpha_i[g_i] + \int_0^t f_i\left(s, x\left(s\right), y(s)\right) ds, & \text{if } t < t_0, \\ \frac{1}{1-\alpha_i[1]}\alpha_i[g_i] + \int_0^{t_0} f_i\left(s, x\left(s\right), y(s)\right) ds, & \text{if } t \geq t_0, \end{cases}$$

$$T_{V_i}(x,y)(t) = \begin{cases} 0, & \text{if } \ t < t_0, \\ \int_{t_0}^t f_i\left(s, x\left(s\right), y(s)\right) ds, & \text{if } \ t \geq t_0, \end{cases}$$

and

$$G(x,y)(t) = \begin{pmatrix} G_1(x)(t) \\ G_2(y)(t) \end{pmatrix}, \quad G_i(v)(t) = \begin{cases} 0, & \text{if } \ t \leq t_0, \\ I_i(v(t_0)), & \text{if } \ t > t_0. \end{cases}$$

First, by means of the fixed point theorem of Perov, we obtain an existence and uniqueness result, provided that f_1, f_2 and also the impulsive terms I_i satisfy the Lipschitz conditions

$$|f_1(t,x,y) - f_1(t,\overline{x},\overline{y})| \leq \begin{cases} a_1\left|x-\overline{x}\right| + b_1\left|y-\overline{y}\right|, & \text{if } t \in [0,t_0] \\ a_2\left|x-\overline{x}\right| + b_2\left|y-\overline{y}\right|, & \text{if } t \in [t_0,1], \end{cases} \tag{6.2.4}$$

$$|f_2(t,x,y) - f_2(t,\overline{x},\overline{y})| \leq \begin{cases} A_1\left|x-\overline{x}\right| + B_1\left|y-\overline{y}\right|, & \text{if } \ t \in [0,t_0] \\ A_2\left|x-\overline{x}\right| + B_2\left|y-\overline{y}\right|, & \text{if } t \in [t_0,1], \end{cases} \tag{6.2.5}$$

and also

$$|I_i(v) - I_i(\overline{v})| \leq d_i\left|v-\overline{v}\right|, \text{ for } i = 1, 2, \tag{6.2.6}$$

for all $x, y, \overline{x}, \overline{y}, v, \overline{v} \in \mathbb{R}$.

In what follows we denote by

$$\begin{aligned} A_{\alpha_i} \ &:= \frac{\|\alpha_i\|}{|1-\alpha_i[1]|} + 1, \ i = 1, 2. \\ M \ &:= t_0 \begin{bmatrix} a_1\left(\frac{\|\alpha_1\|}{|1-\alpha_1[1]|}+1\right) & b_1\left(\frac{\|\alpha_1\|}{|1-\alpha_1[1]|}+1\right) \\ A_1\left(\frac{\|\alpha_2\|}{|1-\alpha_2[1]|}+1\right) & B_1\left(\frac{\|\alpha_2\|}{|1-\alpha_2[1]|}+1\right) \end{bmatrix}, \quad M_I := \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}. \end{aligned}$$

Theorem 6.2.1 *If the conditions* (6.2.4), (6.2.5), (6.2.6) *hold and the matrix*

$$M_0 := M + M_I \tag{6.2.7}$$

is convergent to zero, then the problem (6.1.1) has a unique solution.

Example 6.2.2 We present a modified version of Example 2.2 in R. Precup and D. Trif [66] that takes into account systems and impulsive effects. Consider the initial value problem

$$\left\{\begin{array}{ll}x' = \frac{1}{2}y\left[1+e^{-\frac{4}{5}(x-1)}\right]^{-1} \equiv f_1(x,y), & \\ y' = \frac{1}{10}x\left[1+e^{-\frac{2}{5}(y-1)}\right]^{-1} \equiv f_2(x,y), & \\ \Delta x|_{t=\frac{3}{4}} = \frac{1}{3}\cos\left(x\left(\frac{3}{4}\right)\right), \qquad \Delta y|_{t=\frac{3}{4}} = \frac{1}{5}\sin\left(y\left(\frac{3}{4}\right)\right), & \\ x(0) = \frac{1}{2}\int_0^{\frac{1}{2}} x(s)ds, \qquad y(0) = \frac{1}{2}\int_0^{\frac{1}{2}} y(s)ds. & \end{array}\right., \ t \in [0,1] \tag{6.2.8}$$

Here we have that matrix

$$M_0 = \frac{1}{15}\left(\begin{array}{cc} 6 & 5 \\ 1 & 4 \end{array}\right)$$

is convergent to zero since its eigenvalues satisfy $\lambda_1 = 0.17 < 1, \lambda_2 = 0.5 < 1$. From Theorem 6.2.1, the problem (6.2.8) has a unique solution.

6.3 An existence result

We now show that the existence of solutions for the problem (6.1.1) follows from Schauder's fixed point theorem when f_1, f_2 satisfy some growth conditions of the type: there exist nonnegative coefficients $a_i, b_i, c_i, A_i, B_i, C_i$ such that

$$|f_1(t,x,y)| \leq \left\{\begin{array}{ll} a_1|x| + b_1|y| + c_1, & \text{if } t \in [0,t_0] \\ a_2|x| + b_2|y| + c_2, & \text{if } t \in [t_0,1], \end{array}\right. \tag{6.3.9}$$

$$|f_2(t,x,y)| \leq \left\{\begin{array}{ll} A_1|x| + B_1|y| + C_1, & \text{if } t \in [0,t_0] \\ A_2|x| + B_2|y| + C_2, & \text{if } t \in [t_0,1], \end{array}\right. \tag{6.3.10}$$

for all $x, y \in \mathbb{R}$.

We also assume that the impulses satisfy the growth conditions, i.e. there exist $d_i, e_i \in [0,\infty)$ such that for every $v \in \mathbb{R}$ we have

$$|I_i(v)| \leq d_i|v| + e_i, \quad \text{for } i = 1,2. \tag{6.3.11}$$

Theorem 6.3.1 *If the conditions* (6.3.9), (6.3.10), (6.3.11) *are satisfied and the matrix* (6.2.7) *is convergent to zero, then the problem* (6.1.1) *has at least one solution.*

Example 6.3.2 Consider the initial value problem

$$\left\{\begin{array}{ll}x' = \frac{1}{4}x\sin\left(\frac{y}{x}\right) + \frac{1}{3}y\sin\left(\frac{x}{y}\right) + g(t) \equiv f_1(t,x,y), & \\ y' = \frac{1}{3}x\sin\left(\frac{y}{x}\right) + \frac{1}{6}y\sin\left(\frac{x}{y}\right) + h(t) \equiv f_2(t,x,y), & \\ \Delta x|_{x=\frac{3}{4}} = \frac{1}{3}\sin\left(x\left(\frac{3}{4}\right)\right), \qquad \Delta y|_{y=\frac{3}{4}} = \frac{1}{4}\cos\left(y\left(\frac{3}{4}\right)\right), & \\ x(0) = \frac{1}{2}\int_0^{\frac{1}{2}} x(s)ds, \qquad y(0) = \frac{1}{2}\int_0^{\frac{1}{2}} y(s)\,ds, & \end{array}\right., \ t \in [0,1] \tag{6.3.12}$$

where $g, h \in L^1(0,1)$. We obtain

$$M_0 = \frac{1}{36}\left(\begin{array}{cc} 18 & 8 \\ 8 & 13 \end{array}\right),$$

which is convergent to zero because its eigenvalues (rounded to the third decimal place) are $|\lambda_1| = 0.485 < 1, |\lambda_2| = 0.237 < 1$. From Theorem 6.3.1, the problem (6.3.12) has at least one solution.

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