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MORPHIC OBJECTS IN CATEGORIES

-Ph. D. thesis summary-

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Contents

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Introduction

In the 70's, in two succesive papers, Gertrude Ehrlich, defined and studied a special class of (von Neumann) regular rings, called unit-regular rings. A ring with identity R is unit-regular if for every $a \in R$ there is a unit $u \in U(R)$ such that $a = aua$. Among other properties, Ehrlich proved the following characterization:

Theorem. Let A be a ring with identity and let M be a right $A-$module such that $R = End_A M$ is a regular ring and let $\alpha \in R$. The following statements are equivalent:

(1) α is unit-regular

(2) there is an automorphism $u : M \to M$ such that $Im\alpha \oplus uKer\alpha = M$,

(3) $Ker\alpha \simeq Coker\alpha$.

Hence $\alpha \in R$ is unit-regular if and only if the endomorphism $\alpha : M \to M$ satisfies the condition $M/\mathrm{Im}\alpha \simeq Ker\alpha$. In other words, α is unit-regular if and only if the dual of the Noether isomorphism theorem holds for α.

This was actually the starting point for the studies concerning the dual of E. Noether isomorphism theorem.

A morphic endomorphism is characterized by the following theorem: an endomorphism α is morphic if and only if there exists an endomorphism β such that

$\text{Im}\alpha = Ker\beta$ and $\text{Im}\beta = Ker\alpha$.

Extending the notion of exact sequence defined not only for exact categories, but also for categories with kernels and images, the characterization theorem of the morphic endomorphism is used as definition for the morphic endomorphism in categories with kernels and images.

The definition of morphic object in categories with kernels and images provides the same language for the mathematical theories concerning morphic modules, morphic Abelian groups, morphic groups.

After finding the basic properties, we characterize the morphic objects in $p-$exact categories and in abelian categories. Then we investigate when a direct coproduct of morphic objects is morphic. We study the morphic objects in the category $R{-}Mod{-}S$ and then we introduce in the category $R - Mod$ a new class of objects between the class of morphic objects and the class of Hopfian objects. We also give examples and counterexamples.

In an additive category with kernels and images, the set of endomorphisms of an object has a ring structure. A classical problem in modern algebra is to find correspondences between the properties of an object and the properties of its endomorphisms.

Therefore, given an object M in an additive category with kernels and images we can raise the following two problems:

1) finding the properties of the object M as morphic object

2) finding the properties of the endomorphism ring $End_{\mathcal{C}}(M)$.

This thesis is structured in four chapters numbered from 1 to 4. In its turn, each chapter is subdivided in sections, which are numbered with two numbers: the first is the number of the chapter, the second is the order number of the section in the chapter.

In Preliminaries we establish the language and the notions, recall some known results and extend some notions in categories with kernels and images. The results from here will be used in the following chapters of the paper.

In Chapter 2 we investigate morphic objects in categories with kernels and images. Connections with morphic, unit-regular and regular objects are made.

Chapter 3 is dedicated for the study of morphic modules in the category of bimodules $R - Mod - S$. A ring R is called morphic if the bimodule ${}_RM_R$ is morphic.

In Chapter 4 we study a new class of modules, namely $\delta-$modules.

Chapter 1

Preliminaries

We present in this chapter the mathematical objects (w-exact category, p-exact category, additive category, abelian category) and some suitable notations that will be used along the thesis. Some notions and results are generalized in categories with kernels and images in order to use them in the next chapters.

1.1 w-exact categories

First we present the definition of the w-exact category.

We say that the category $\mathcal{C}$ is w^*-exact if its dual $\mathcal{C}^{op}$ is $w-$exact.

We say that the category $\mathcal{C}$ is $p-$exact if it is $w-$exact and w^*-exact.

A characterization theorem of a p-exact category is given.

The category $\mathcal{C}$ is called abelian if it is additive, $p-$exact and has finite products.

In the sequel a characterization theorem of the abelian (respectively p-exact) category is presented.

1.2 Categories with kernels and images

We give the definition of the category with kernels and images and some immediate properties.

Every $w-$exact category $\mathcal{C}$ has kernels and images.

Every $p-$exact category has images, kernels, coimages and cokernels and so the abelian categories have all of this.

We extend the definition of the exact sequence in a category $\mathcal{C}$ with kernels and images.

It is trivially that:

Example 1.2.1. The sequence $M \stackrel{0_M}{\to} M \stackrel{\alpha}{\to} M \stackrel{0_M}{\to} M$ in $\mathcal{C}$ is exact, where 0_M is the zero endomorphism and α is an automorphism.

Proposition 1.2.2. *If the sequence* $M \stackrel{\alpha}{\to} M \stackrel{\beta}{\to} M \stackrel{\alpha}{\to} M$ *in* $\mathcal{C}$ *is exact, then* $\beta\alpha = 0 = \alpha\beta$.

In the sequel we present theorems concerning kernels and images that will be used in the proofs from chapter 2.

1.3 Endomorphisms

We present examples of endomorphisms that will be used in the following chapters of the thesis.

An endomorphism $\alpha : M \to M$ is called regular (unit-regular) if there exists the endomorphism (automorphism) σ such that $\alpha\sigma\alpha = \alpha$ and is called idempotent if $\alpha^2 = \alpha$.

In an additive category, the set $End_{\mathcal{C}}(M)$ together with the addition and the composition of the morphisms is an associative ring with identity. This ring is called the ring of endomorphisms of the object M.

The unit-regular endomorphism is characterized by the following proposition:

Proposition 1.3.1. *Let $\mathcal{C}$ be an additive category. Then $\alpha \in End_{\mathcal{C}}(M)$ is unit-regular if and only if there exists $\sigma \in Aut_{\mathcal{C}}(M)$ and $\pi \in End_{\mathcal{C}}(M)$ idempotent such that $\alpha = \pi\sigma$.*

The ring $(End_{\mathcal{C}}(M), +, \circ)$ is called regular (unit-regular) if all endomorphisms are regular (unit-regular).

The object M is called regular (unit-regular) if all its endomorphisms are regular (unit-regular).

1.4 Hopfian, co-Hopfian and Dedekind-finite objects

The object M in the category $\mathcal{C}$ is called Hopfian if every epimorphism $\alpha \in End_{\mathcal{C}}(M)$ is an automorphism.

The object M in the category $\mathcal{C}$ is called co-Hopfian if every monomorphism $\alpha \in End_{\mathcal{C}}(M)$ is an automorphism.

The object M is called Dedekind-finite (DF) if $\alpha\beta = 1_M$ implies $\beta\alpha = 1_M$ for every endomorphisms $\alpha, \beta \in End_{\mathcal{C}}(M)$.

Proposition 1.4.1. *If M is a Hopfian (co-Hopfian) object in a category $\mathcal{C}$, then M is Dedekind-finite.*

Proposition 1.4.2. *In $p-$exact category, an infinite product (coproduct) nonequal with zero of the same object (if exists) is not Dedekind finite.*

1.5 Functors. The Mitchell Theorem

Theorem 1.5.1. *(Mitchell) For every small abelian category $\mathcal{C}$ there exists a ring R and a covariant additive, exact, full functor $F : \mathcal{C} \to R - Mod$ embedding to the suitable category of modules.*

Finally we give examples of functors that will be used in the chapters 2 and 3.

1.6 The subobject lattice

Proposition 1.6.1. *i)If $\mathcal{C}$ is a $p-$exact category, then the lattice $(\mathcal{S}(M), \leq)$ is modular.*

ii)If M and N are two objects in a $p-$exact category $\mathcal{C}$, then $\inf(M, N) = M \cap N$ *and* $\sup(M, N) = M \cup N$.

We say that the nonzero object M in $\mathcal{C}$ is simple, if the only subobjects of M are $[M, 1_M]$ and $[O, 0_{OM}]$ and the only factor objects are $[1_M, M], [0_{MO}, O]$.

Proposition 1.6.2. *Let $\mathcal{C}$ be a $w-$exact category. Every nonzero endomorphism of the simple object M is an automorphism.*

The following statement is an immediate consequence of the Proposition 1.6.2:

Corollary 1.6.3. *If M is simple in a $p-$exact, aditive category, then the endomorphism ring of the object M is a field.*

1.7 The isomorphism theorems

The three theorems of isomorphism available in the abelian categories can be extended in $p-$exact categories.

Theorem 1.7.1. *(The first isomorphism Nöether theorem) If $\mathcal{C}$ is a $p-$exact category, then for every morphism $\alpha : M \to N$ we have $M/Ker\alpha \simeq Im\alpha$.*

Proposition 1.7.2. *The following sequences:*

$0 \to M_1 \overset{u_1}{\to} M_1 \oplus M_2 \overset{p_2}{\to} M_2 \to 0$

$0 \to M_2 \overset{u_2}{\to} M_1 \oplus M_2 \overset{p_1}{\to} M_1 \to 0$

are exact.

Therefore $M_1 \oplus M_2/M_1 \simeq M_2$ and $M_1 \oplus M_2/M_2 \simeq M_1$.

Proposition 1.7.3. *If M, N, P are objects in the abelian category $\mathcal{C}$ and $P \subset N$ then $(M \oplus N)/\ (M \oplus P) \simeq N/P$.*

Chapter 2

Morphic objects in categories

In this chapter we define and study the notion of morphic object in categories with kernels and images. For the beginning we define the notion and present the basic properties, then we study as applications the morphic objects in the categories $pEns, Rng$ and the category associated to the ring denoted $\mathcal{C}_R$. We also discuss about the morphic objects in $p-$exact and abelian categories. In this categories, an endomorphism is morphic if and only if the dual of the isomorphism theorem holds and an object is morphic if and only if for every endomorphism the dual of the isomorphism theorem holds.

We also give a latticeal characterization of the morphic object and a generalisation of the Ehrich theorem, that is an endomorphism is unit regular if and only if it is regular and morphic.

We shall state and proof a theorem concerning the behavior of the morphic object via a functor.

2.1 The notion of morphic object. Basic properties

Definition 2.1.1. Let M be an object in a category $\mathcal{C}$ with kernels and images. An endomorphism α of M is called *morphic* if there is an endomorphism β of M such that the following sequence is exact

$$M \xrightarrow{\alpha} M \xrightarrow{\beta} M \xrightarrow{\alpha} M.$$

The object M is itself *morphic* if every endomorphism α of M is morphic.

β will be called a *complementary* (endomorphism) for α.

Remark 2.1.2. 1) If α is morphic and β is the complementary endomorphism then $\alpha\beta = 0 = \beta\alpha$ (by P. 1.2.2). More, if $\mathcal{C}$ is additive, β belongs to both left and right annihilators of α.

2) The translation $t_m, m \in \mathbb{Z}\backslash\{-1, 0, 1\}$ is not a morphic endomorphism of the group $(\mathbb{Z}, +)$ in the category *Ab* (by O 1.2.6.) and $t_m t_0 = 0 = t_0 t_m$. Therefore if $\alpha\beta = 0 = \beta\alpha$ in $\mathcal{C}$ does not generally follows that β is morphic.

Example 2.1.3. i) The morphic objects in the category R-Mod are the morphic modules.

ii) The morphic objects in the category *Ab* are the morphic abelian groups.

iii) The morphic objects in the category *Grp* are the morphic groups .

iv) The object V in the category $K - Vect$ is morphic if and only if it has finite dimension.

v) All objects in the category of vector spaces of finite dimension are morphic.

vi) Let $\mathcal{T}$ be the full subcategory of $pTop$ (pointed topological spaces) whose objects are:

(1) a singleton pointed space $\{*\}$, and

(2) a 3 elements pointed space (T, a), with the open sets: $\varnothing$, T, $\{a\}$ and $\{b, c\}$ if $T = \{a, b, c\}$. Then $\mathcal{T}$ has kernels and cokernels, is normal and conormal (and so, has epimorphic images and coimages) but is not p-exact.

It is easy to verify that (T, a) has only 5 endomorphisms and that all object in $\mathcal{T}$ are morphic.

Remark 2.1.4. 1)The composition of two morphic endomorphism is not generally morphic.

2) The subobjects of the morphic object need not be morphic.

3) The factor objects of the morphic object need not be morphic.

4) An object which is not morphic in a category C may be morphic in a subcategory of C.

5) The zero morphism is morphic (by Exercise 1.2.1.). Every automorphism $\alpha : M \to M$ is morphic since the zero morphism is the complementary endomorphism of α. In particular, the identity endomorphism is morphic. It is clear that if all nonzero endomorphisms of an object are morphic, then the object is morphic.

Example 2.1.5. 1) The zero object is morphic.

2) If $U(End_{\mathcal{C}}(M)) = End_{\mathcal{C}}(M)\backslash\{0\}$ then M is a morphic object in $\mathcal{C}$.

3) Every simple object in a $w-$exact category is morphic.

4) The object $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ in the category Rng is morphic.

Proposition 2.1.6. *Let $\mathcal{C}$ be an additive category with kernels and images and M be an object in $\mathcal{C}$. If $\alpha : M \to M$ is an idempotent endomorphism, then α is morphic.*

Lemma 2.1.7. *Let $\alpha \in End_{\mathcal{C}}(M)$ be a morphic endomorphism in a category $\mathcal{C}$ with kernels and images. If $\sigma : M \to M$ is an automorphism, then $\alpha\sigma$ and $\sigma\alpha$ are both morphic. Moreover, if $\mathcal{C}$ is additive, every unit-regular endomorphism is morphic.*

Example 2.1.8. i) The ring $(\mathbb{Z}_n, +, \cdot), n \in \mathbb{N}, n \geq 2$ is morphic in the category *Rng*.

ii) Let M be an object in the additive category $\mathcal{C}$ with kernels and images. If the ring $End_{\mathcal{C}}(M)$ is unit-regular (i.e. M is a unit-regular object) , then the object M is morphic.

Remark 2.1.9. In general we can say nothing about the morphicity of the projective generator (or the injective cogenerator).

2.2 Examples and applications

i)Let $\mathcal{C}$ and $\mathcal{D}$ be two categories with kernels and images. It is easy to see that when $\mathcal{C}$ is a full subcategory of $\mathcal{D}$ the following holds: if $M \in Ob(\mathcal{C})$ is morphic in $\mathcal{C}$, then M is also morphic in $\mathcal{D}$.

The category *Ab* is a full subcategory of the category *Grp*. Hence, if the group G is morphic in *Ab*, then it is also morphic in *Grp*.

We give examples of categories in which the morphic object can not be defined.

ii) Morphic objects in the category *pEns*

Theorem 2.2.1. *A pointed set (A, a) is morphic if and only if A is finite.*

The following proposition is an immediate consequence of the theorem:

Corollary 2.2.2. *1) The infinite product (coproduct) of the pointed sets is not morphic.*

2) A finite product (coproduct) is morphic if and only if all factors are morphic.

iii) Morphic objects in the category of rings

The following proposition characterizes the morphic endomorphism in the category Rng :

Lemma 2.2.3. *Let $R \in Rng$. The endomorphism $\alpha \in End_{Rng}(R)$ is morphic if and only if* $\mathrm{Im}\alpha$ *is an ideal in R and $R/\mathrm{Im}\alpha \simeq Ker\alpha$.*

The following proposition is an immediate consequence of Lemma 2.2.4.:

Corollary 2.2.4. *The ring R is a morphic object in Rng if and only if* $\mathrm{Im}\alpha$ *is an ideal in R and $R/\mathrm{Im}\alpha \simeq Ker\alpha$ for every $\alpha \in End_{Rng}(R)$.*

The morphic object has a latticial characterization as follows:

Theorem 2.2.5. *The ring R is morphic object in the category Rng if and only if whenever $R/I \simeq J$, where I and J are ideals, then $R/J \simeq I$.*

iv) Morphic objects in the category associated to a ring

Let R be a ring. Denote by $\mathcal{C}_R$ the category associated to the ring $(R, +, \cdot)$.

Let $(R, +, \cdot)$ be an integral domain. Then $(R\backslash\{0\}, \cdot)$ is a monoid with right and left simplification. Hence every $a \in R\backslash\{0\}$ is a bimorphism. The automorphisms of R are the invertable elements of the ring, that is $Aut(R) = U(R, +, \cdot)$. The subobjects of the object R are: $0_{OR}, a \in R\backslash\{0\}$.

It is easy to see that $\mathcal{C}_R$ is a category with kernels and images.

Clearly, the bimorphisms of R that are not automorphisms are not morphic, but all the bimorphisms that are automorphisms are morphic.

Example 2.2.6. Let $(R, +, \cdot)$ be an integral domain.

i) If R is a field, then R is morphic in the category $\mathcal{C}_R$.

ii) If R is not a field, then R is not morphic in the category $\mathcal{C}_R$.

iii) $\mathbb{Z}$ is not morphic in the category $\mathcal{C}_{\mathbb{Z}}$.

iv) If R is finite, then R is morphic in $\mathcal{C}_R$.

v) If R is infinite, but is not a field, then R is not morphic in $\mathcal{C}_R$.

vi) If p is prime, then the field $\mathbb{Z}_p$ is a morphic object in the category $\mathcal{C}_{\mathbb{Z}_p}$.

In the following proposition we shall characterize the monomorphisms and the epimorphisms of the category $\mathcal{C}_R$.

Lemma 2.2.7. *Let R be an associative ring with identity. Then:*

i) $a \in R$ is mono if and only if the right annihilator of a is zero (denote this annihilator by $r(a)$)

ii) $a \in R$ is epic if and only if the left annihilator of a is zero (denote this annihilator by $l(a)$).

Observe that in $\mathcal{C}_R$ the left nonzero divisors are monomorphisms and the right nonzero divisors are epimorphisms. If R is a commutative ring with zero divisors, then $\mathcal{C}_R$ has kernels.

Example 2.2.8. Consider the triangular ring $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$. R is a noncommutative ring with zero divisors. The matrice $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is an epimorphism without being a monomorphism in the category $\mathcal{C}_R$.

2.3 Morphic objects in $p-$exact categories

In a $p-$ exact category the morphic endomorphism is characterized as:

Lemma 2.3.1. *(the dual of the isomorphism theorem) In a $p-$exact category an endomorphism $\alpha : M \to M$ is morphic if and only if $M/\mathrm{Im}\alpha \simeq Ker\alpha$.*

Another characterization of the morphic object is given by the following theorem:

Lemma 2.3.2. *(the latticeal characterization) The following are equivalent for an object M in an exact-additive category:*

1) M is morphic

2) If $M/K \simeq N$ where K and N are subobjects of M, then $M/N \simeq K$.

Proposition 2.3.3. *If M is a morphic object in a $p-$exact category $\mathcal{C}$, then M is Hopfian and co-Hopfian.*

Remark 2.3.4. If M is a morphic object in a $p-$exact category $\mathcal{C}$, then M is Dedekind-finite.

Proposition 2.3.5. *In a $p-$exact category, the infinite coproduct (if it exists) of the same object is not morphic.*

Proposition 2.3.6. *If M is a morphic object and K is a subobject of M, then:*

1. If $M \simeq K$, then the subobjects K and M are the same.

2. If $M \simeq M/K$, then K is the zero object.

An object M is called strongly morphic if for every subobject of M there exists a factor object of M isomorphic to it.

Theorem 2.3.7. *i)In a $p-$exact category, for a morphic object M the following statements are equivalents:*

(1) M is strongly morphic

(2) For every factor object of M there exists a subobject of M isomorphic to it.

ii) Let M be a strongly morphic object. If N and N' are subobjects of $M, M/N \simeq M/N'$ if and only if $N \simeq N'$.

Example 2.3.8. 1) Let $n \in \mathbb{N}, n \geq 2$. The group $(\mathbb{Z}_n, +)$ is strongly morphic in the category Ab.

2) The group $(\mathbb{Q}, +)$ is morphic in Ab, but is not strongly morphic.

3) $(\mathbb{Z}, +)$ is not morphic in Ab.

For $m, n \in \mathbb{N}, n, m \geq 2, m \neq n$ we have: $(n\mathbb{Z}, +) \simeq (m\mathbb{Z}, +)$, and $(\mathbb{Z}/n\mathbb{Z}, +) \not\simeq (\mathbb{Z}/m\mathbb{Z}, +)$.

Definition 2.3.9. A subobject N of the object M in the category $\mathcal{C}$ is called relatively morphic if $M/K \simeq N$ implies $M/N \simeq K$ for every subobject K of M.

We give a characterization of morphic objects using the relatively morphic subobjects:

Theorem 2.3.10. *An object M in an exact category $\mathcal{C}$ is morphic if and only if every subobject of M is relatively morphic.*

Definition 2.3.11. A subobject N of the object M in the category $\mathcal{C}$ is called dual relatively morphic if $M/N \simeq K$ implies $M/K \simeq N$ for every subobject K of M.

An analogous theorem holds.

Remark 2.3.12. i) The zero subobject of the object of the object M is relatively morphic.

ii) The subobject M of the object M is dualy relatively morphic.

2.4 Morphic objects in abelian categories

Theorem 2.4.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a full and faithful exact functor between two abelian categories, and $M \in Ob(\mathcal{C})$. Then M is morphic if and only if $F(M)$ is morphic in $\mathcal{D}$.*

Remark 2.4.2. i) Let $\mathcal{C}$ be an abelian category. Then M is morphic in $\mathcal{C}$ if and only if M^0 is morphic in the dual category $\mathcal{C}^{op}$.

Moreover, this statements also holds when $\mathcal{C}$ is a $p-$exact category.

ii) Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories. If $\mathcal{C}$ is a full subcategory of $\mathcal{D}$, then the object M in $\mathcal{C}$ is morphic if and only if M is morphic in $\mathcal{D}$.

iii) If $\mathcal{C}$ is a small abelian category, then there exists a ring R and a full, exact and faithful functor $F : \mathcal{C} \to R - Mod$ such that $M \in Ob\mathcal{C}$ is morphic if and only if $F(M)$ is a morphic $R-$module.

iv) The abelian group $(M, +)$ is morphic if and only if M is a morphic $\mathbb{Z}-$module.

Theorem 2.4.3. *In an abelian category, every direct summand of a morphic object is again morphic.*

Remark 2.4.4. The class of morphic modules is not closed under taking direct sums.

ii) Numerical groups

The only one numerical group is $(\mathbb{Q}, +)$. The direct finite summ $\mathbb{Q} \oplus ... \oplus \mathbb{Q}$ is morphic. ([9])

Proposition 2.4.5. *If M and N are morphic objects with $Hom_{\mathcal{C}}(M, N) = 0 = Hom_{\mathcal{C}}(N, M)$, then $M \oplus N$ is morphic.*

Proposition 2.4.6. *Let M and N be objects with $Hom_{\mathcal{C}}(M, N) = 0$. If there is an epimorphism $\pi : N \to M$, then the direct summ $S = M \oplus N$ is not morphic.*

Lemma 2.4.7. *In an abelian category let* $\alpha \in EndM$ *and* $M = N \oplus Ker\alpha$. *Then* $\alpha M = \alpha N$.

Remark 2.4.8. Embedding this diagram into a 9 Lemma type diagram (0 pull-back for the left upper square), shows that, $\overline{\alpha} : N \longrightarrow M \longrightarrow I$ is an isomorphism.

The following characterization generalizes a result of Ehrlich:

Theorem 2.4.9. *In any abelian category* $\mathcal{C}$*, an endomorphism* $\alpha \in End_{\mathcal{C}}(M)$ *is unit regular if and only if it is both regular and morphic.*

Corollary 2.4.10. *In an abelian category* $\mathcal{C}$, *the object* M *is unit regular if and only if it is regular and morphic.*

An object M is kernel-direct (image-direct respectively) if for every $\alpha \in \mathrm{End}_{\mathcal{C}}(M)$, $Ker\alpha$ ($Im\alpha$ respectively) is a direct summand of M.

Theorem 2.4.11. *The following are equivalent for an object* M *in an abelian category:*

(1) $End(M)$ *is unit regular.*

(2) M *is morphic and kernel-direct.*

(3) M *is morphic and image-direct.*

Chapter 3

Morphic bimodules and rings

In this chapter the morphic bimodules and rings are defined and studied. We also study special class of rings with involution and modules over skew group rings.

In the beginning we present the notion of the morphic bimodule and the basic properties. Taking in particular this notion in the category $R - Mod - S$, we obtain the notion of morphic ring. We prove that the bimodule ${}_RM_S$ is morphic if and only if the module ${}_{R\otimes S^{op}}M$ is morphic and in particular the ring R is morphic if and only if the module ${}_{R\otimes S^{op}}R$ is morphic. In the sequel we give characterizations for the morphic ring, the rings with involution are studied, respectively the action of a group over a ring, pointing out conditions when such a ring is morphic.

We shall see that for principal, commutive rings the notion of morphic ring is equivalent to the notion of dual ring. We also discuss the conditions when special subrings of a ring are morphic.

3.1 Morphic bimodules. Basic properties

Definition 3.1.1. Let R and S be two associative rings with identity and M be a $(R, S)-$bimodule. We say that an $(R, S)-$*endomorphism* $f \in End_{R,S}(M)$ is morphic if it is a morphic endomorphism in the category $R - Mod - S$. We say that the $(R, S)-$bimodule M is morphic if it is a morphic object in the category $R - Mod - S$.

Exemples of $(R, S)-$ endomorphisms:

Proposition 3.1.2. *If $a \in Z(R)$ and $b \in Z(S)$, then the map $t_{a,b} : {}_RM_S \to {}_RM_S$ given by $t_{a,b}(x) = axb$ is an $(R, S)-$endobimorphism.*

A nonzero element $e \in M$ is called free if *satisfy the condition*: for $r \in R, s \in S$ with $res = 0$ implies $r = 0$ or $s = 0$.

Proposition 3.1.3. *Let M be a $(R, S)-$bimodule with free elements. Let $a \in R, a \neq 0$, $b \in S, b \neq 0$ and $t_{a,b} : M \to M$ given by $t_{a,b}(x) = axb$. The following statements are equivalent:*

1) $t_{a,b} \in End_{R,S}(M)$

2) $a \in Z(R)$ and $b \in Z(S)$

The following proposition holds:

Proposition 3.1.4. *Let M be an $(R, S)-$bimodule and $a, c \in Z(R), b, d \in Z(S)$. The following holds:*

1. *If $t_{a,b}, t_{c,d} \in End_{R,S}(M)$, then $t_{a,b} \circ t_{c,d} = t_{ac,db}$*
2. *If $a \in U(R)$ and $b \in U(S)$, then $t_{a,b}$ is an automorphism.*

Morphic endomorphisms are characterized as follows:

Lemma 3.1.5. *Let $f : M \to M$ be an (R,S)-endomorphism. Then the following statements are equivalent:*

1. The endomorphism $f : M \to M$ is morphic;

2. There exists the endomorphism $g : M \to M$ such that $\mathrm{Im} f = Kerg$ *and* $\mathrm{Im} g = Kerf$*;*

3. $M/\mathrm{Im} f \simeq Kerf$*;*

4. There exists the endomorphism $g : M \to M$ such that $\mathrm{Im} g \simeq Kerf$ *and* $\mathrm{Im} f = Kerg$*.*

Therefore morphic bimodules are characterized (using Lemma 2.3.2) by:

Theorem 3.1.6. *Let M be an (R,S)-bimodule. Then the following statements are equivalent:*

1. M is morphic;

2. For every endomorphism $f : M \to M$ there exists an endomorphism $g : M \to M$ such that $\mathrm{Im} f = \mathrm{Ker} g$ *and* $\mathrm{Im} g = \mathrm{Ker} f$*;*

3. $M/\mathrm{Im} f \simeq \mathrm{Ker} f$*, for every endomorphism $f : M \to M$;*

4. If $M/N \simeq K$ where K and N are subbimodules of M then $M/K \simeq N$.

Consider the functor $F : R\text{-Mod-}S \to R \otimes S^{op}\text{-Mod}$ defined as in 1.5.

Since this functor F is an isomorphism of categories, a bimodule ${}_R M_S$ is morphic if and only if ${}_{R\otimes S^{op}} M$ is morphic.

Example 3.1.7. An (R,S)-bimodule which is morphic as a left R-module, but it is not morphic as a right S-module.

3.2 Morphic rings

Definition 3.2.1. A ring R is called *morphic* if the bimodule ${}_RR_R$ is morphic in the category R-Mod-R.

It is easy to see that a direct product of morphic rings is morphic if and only if each factor is morphic.

Definition 3.2.2. An element $a \in Z(R)$ is called *morphic* if t_a is a morphic endomorphism for the object ${}_RR_R$ in the category R-Mod-R.

Remark 3.2.3. 1) The ring R is morphic if and only if the module ${}_{R\otimes R^{op}}R$ is morphic.

2) $\mathrm{Im} t_a = Ra$ and $Ker t_a = \mathrm{Ann}_R(a)$ for all $a \in Z(R)$.

The morphic elements are characterized by:

Lemma 3.2.4. *For $a \in Z(R)$ the following statements are equivalent:*

1) a is morphic;

2) $R/Ra \simeq Ann_R(a)$;

3) There exists $b \in Z(R)$ such that $Ra = Ann_R(b)$ and $Ann_R(a) = Rb$.

Clearly $\mathrm{End}_{R\otimes R^{op}}(R) = \{t_a | a \in Z(R)\}$.

Proposition 3.2.5. *The module ${}_{R\otimes R^{op}}R$ is morphic if and only if for each $a \in Z(R)$, there exists $b \in Z(R)$ such that $Ra = Ann_R(b)$ and $Rb = Ann_R(a)$.*

Proposition 3.2.6. *Let R be a commutative, principal ring. Then R is morphic if and only if R is dual.*

Proposition 3.2.7. *${}_{R\otimes R^{op}}R$ is image projective if and only if $Za = Ra \cap Z$ for every $a \in Z$ (here Z denotes $Z(R)$).*

Proposition 3.2.8. *If ${}_{R\otimes R^{op}}R$ is morphic and image projective, then the ring $Z = Z(R)$ is morphic.*

Proposition 3.2.9. *If $Z(R)$ is morphic, then the module ${}_{R\otimes R^{op}}R$ is image projective.*

Corollary 3.2.10. *If R is a morphic ring, then $Z(R)$ is morphic if and only if ${}_{R\otimes R^{op}}R$ is image projective.*

Proposition 3.2.11. *If $Z = Z(R)$ is a morphic ring with the property that $Rx = Ann_R(y)$ for every $x, y \in Z$ such that $Zx = Ann_Z(y)$, then ${}_{R\otimes R^{op}}R$ is morphic.*

Proposition 3.2.12. *If ${}_{R\otimes R^{op}}R$ is morphic then $Ra = Ann(Ann(a))$ for every $a \in Z(R)$.*

Recall that a ring R is called *unit-regular* if for every $a \in R$ there is a unit $u \in R$ such that $a = aua$.

Proposition 3.2.13. *If $(Z(R), +, \cdot)$ is a division ring, then R is morphic.*

Example 3.2.14. Let K be a field and $n \geq 2$, $n \in \mathbb{N}$. Then the square matrix ring $(M_n(K), +, \cdot)$ and its center are both morphic.

Proposition 3.2.15. *If $Z(R)$ has no zero divisors in R, then ${}_{R\otimes R^{op}}R$ is semiprojective (i.e. $Ra \cap Z = Za$ for every $a \in Z$).*

Theorem 3.2.16. *If no central element of R is a zero divisor then the following are equivalent:*

1. *R is morphic;*
2. *$Z(R)$ is morphic;*
3. *$Z(R)$ is a field.*

Corollary 3.2.17. *If R has no zero divisors, then the following are equivalent:*

1. *R is morphic;*
2. *$Z(R)$ is morphic;*
3. *$Z(R)$ is a field.*

A similar argument gives

Proposition 3.2.18. *If the ring $Z = Z(R)$ is unit regular then the ring R is morphic.*

3.3 Morphic rings with involutions

Let $Z_0 = \{x \in Z | \rho(x) = x\}$ be the set of *symmetric* elements.

Let $Z_1 = \{x \in Z | \rho(x) = -x\}$ be the set of *antisymmetric* elements.

Proposition 3.3.1. *If $1 + 1 = 2$ is invertible, then $Z = Z_0 \oplus Z_1$ (direct sum of subgroups).*

Proposition 3.3.2. *If $\mathrm{char} R \neq 2$ and the map given by $a \in Z \longmapsto 2a \in Z$ is surjective, then $Z = Z_0 \oplus Z_1$.*

Proposition 3.3.3. $\mathrm{End}_T(R) = \{t_a | a \in Z_0\}$.

Proposition 3.3.4. *The function $\phi : \mathrm{End}_T(R) \to R, \phi(\psi) = \psi(1)$ is an injective morphism and $\mathrm{Im}\psi = Z_0$.*

Remark 3.3.5. $(\mathrm{End}_T(R), +, \circ) \simeq (Z_0, +, \cdot)$.

Proposition 3.3.6. *${}_TR$ is morphic if and only if for every $a \in Z_0 = Z(R, \rho)$, there exists $b \in Z_0$ such that $Ra = Ann_R(b)$ and $Rb = Ann_R(a)$.*

Proposition 3.3.7. *${}_T R$ is image projective if and only if $Z_0 a = Ra \cap Z_0$ for every $a \in Z_0 = Z(R, \rho)$.*

Proposition 3.3.8. *If ${}_T R$ is morphic and image projective, then $Z_0 = Z(R, \rho)$ is morphic.*

Theorem 3.3.9. *Let R be a ring with the involution $\rho : R \to R$ and with the property that the elements in Z_0 are not zero divisors in Z. If the modules ${}_{R \otimes R^{op}} R$, ${}_T R$ are image projective, then R is a morphic ring if and only if ${}_T R$ is morphic.*

Proposition 3.3.10. *If Z_0 is morphic, then ${}_T R$ is image projective.*

Example 3.3.11. Let D be a division ring and $R = M_n(D)$, $n \in \mathbb{N}$, $n \geq 1$ the square matrix ring and the involution given by the transpose. Then R is a morphic T-module.

Proposition 3.3.12. *If the ring $Z_0 = Z(R, \rho)$ is unit regular, then the module ${}_T R$ is morphic.*

3.4 The action of a group of automorphisms over the ring itself

Consider the set $R^G = \{a \in Z(R) / \forall \sigma \in G : \sigma(a) = a\} \subseteq Z(R)$, where σ is an action of the group G over the ring R and the module R over the skew group ring.

Recall that $(\mathrm{End}_{R*G}(R), +, \circ) \simeq (R^G, +, \cdot)$.

Proposition 3.4.1. *${}_{R*G} R$ is morphic if and only if for every $a \in R^G$ there exists $b \in R^G$ such that $Ra = Ann_R(b)$ and $Ann_R(a) = Rb$.*

Proposition 3.4.2. *${}_{R*G}R$ is image projective if and only if we have $R^G a = Ra \cap R^G$ for all $a \in R^G$.*

Proposition 3.4.3. *If the ring R^G is unit regular, then the module ${}_{R*G}R$ is morphic.*

Chapter 4

Between Morphic and Hopfian

In this chapter we define and study the relatively morphic, respectively dual relatively morphic submodules. Then using this notion we define the $\delta-$module (resprectively $\gamma-$module) and study the properties.

We shall see that the class of $\delta-$modules (respectively the class of $\gamma-$modules) is placed between the class of morphic modules and the class of Hopfian modules (respectively co-Hopfian).

Taking in particular the notion of $\delta-$module (respectively $\gamma-$module) we intoduce the notion of $\delta-$group (respectively $\gamma-$group). Since the morphic abelian groups are rare (for example there exists only one numerical morphic group that is $(\mathbb{Q}, +)$) and there are many Hopfian groups (respectively co-Hopfian) findind $\delta-$modules (respectively $\gamma-$modules) is of interest.

We shall study in details the $\delta-$groups (respectively the $\gamma-$groups).

4.1 Relatively morphic submodules

Definition 4.1.1. A submodule N of a module ${}_RM$ will be called *relatively morphic* if whenever $M/K \simeq N$ holds for a submodule K, then $M/H \simeq K$.

While 0 is relatively morphic in any module M, it is readily checked that:

Lemma 4.1.2. *The following statements are equivalent:*

1) M is relatively morphic in M

2) M has no proper factor groups isomorphic to M

3) M is Hopfian.

The submodules of a morphic module are relatively morphic and if M is Hopfian the set of submodules of M consists of trivial submodules.

Example 4.1.3. 1) The proper submodules $N < M$, $N \simeq M$ are not relatively morphic (i.e. $M/0 \simeq N$, but $M/N \not\simeq 0$).

2) Suppose that in a module M, there are no proper submodules N, K with $M/K \simeq N$. Then (trivially) all proper submodules are relatively morphic in M.

Moreover, let A be a rank 1 torsion-free abelian group. All the proper subgroups of A are relatively morphic in A.

3) Not all the subgroups of a free abelian group of finite rank are relatively morphic.

4) *Relatively morphic is not a transitive relation* i.e., if L is relatively morphic in H and H is relatively morphic in M, then L is not generally relatively morphic in M.

5) *The conditions [N is relatively morphic in M] and [M/N is morphic] are logically independent.*

6) *Not all direct summands are relatively morphic.*

Proposition 4.1.4. *N is relatively morphic in M if and only if every endomorphism $\alpha \in \mathrm{End}(M)$ with $\mathrm{Im}\alpha = N$ is morphic.*

4.2 Property (δ)

Proposition 4.2.1. *For modules the following conditions are equivalent:*

(i) all direct summands are relatively morphic, and

(ii) for any direct summand C and submodule K, $M/K \simeq M/C$ implies $K \simeq C$.

Definition 4.2.2. We say that a module is δ ($\delta-$module) if all direct summands are relatively morphic.

The statement: "Any direct summand of the module M is relatively morphic" is called the property δ. It is denoted by (δ).

It is easily seen the following proposition:

Proposition 4.2.3. *An indecomposable module satisfies (δ) (i.e. is $\delta-$module) if and only if it is Hopfian.*

Since $\mathbb{Z}$ and $\mathbb{Q}$ are Hopfian and ${}_{\mathbb{Z}}\mathbb{Z}, {}_{\mathbb{Z}}\mathbb{Q}$ are indecomposable, it follows that $\mathbb{Z}$ and $\mathbb{Q}$ satisfy (δ) (moreover, all rank 1 torsion-free Abelian groups satisfy (δ)).

Proposition 4.2.4. *i) If M is morphic, then M is a $\delta-$module.*

ii) If M is a $\delta-$module, then M is Hopfian.

Therefore we have: morphic $\Rightarrow (\delta) \Rightarrow$ Hopfian.

In general, the converse of the first and second part of the previous proposition does not hold.

The class of Hopfian modules includes the class of δ-modules which includes the class of morphic modules.

Proposition 4.2.5. *If M satisfies (δ), then any direct summand satisfies (δ).*

Proposition 4.2.6. *The following conditions are equivalent:*

(i) M is Hopfian

(ii) If $\mathrm{Im}\alpha = M$ then $Ker\alpha = 0$

(iii) Every epic endomorphism of M is morphic.

Proposition 4.2.7. *A module M has a Dedekind finite endomorphism ring if and only if all retractions (right invertible endomorphisms) in $\mathrm{End}(M)$ are morphic.*

4.3 Abelian $\delta-$groups

An abelian group G is called $\delta-$group if the $\mathbb{Z}$-module G is a $\delta-$module.

i) Divisible δ-groups

Since $\mathbb{Z}_{p^\infty}/\mathbb{Z}_{p^n} \simeq \mathbb{Z}_{p^\infty}$ it follows that $\mathbb{Z}_{p^\infty}$ is not Hopfian, so nor $\delta-$group; $\mathbb{Z}_{p^\infty}$ is a divisible torsion group.

Proposition 4.3.1. *Torsion divisible groups are not Hopfian.*

Remark 4.3.2. 1) Since the divisible torsion groups are not Hopfian, they are not δ, nor morphic.

2) If the divisible group M is Hopfian, then M is torsion-free.

3) If the divisible group M is a $\delta-$group (or morphic), then M is torsion free.

Proposition 4.3.3. *If a torsion-free group is morphic, then it is divisible.*

Remark 4.3.4. The statement in P.4.3.3 is not generally true for $\delta-$groups or Hopfian groups.

ii) Torsion-free $\delta-$groups

The group $\mathbb{Z}$ is torsion-free, satisfy (δ) and is free of rank 1.

Proposition 4.3.5. *Finite rank torsion-free groups are δ-groups.*

Remark 4.3.6. 1) The condition that at least one of the subgroups H and K should be a direct summand was not used. Thus, finite rank free groups share a stronger property than (δ): for every two subgroups H, K in G such that $H \simeq K$ we have $G/H \simeq G/K$.

2) Being finitely generated, the finite rank torsion-free groups are also Hopfian.

3) An infinite rank free group is not Hopfian and so nor (δ).

All finite rank torsion-free groups are Hopfian (this is due to the rank formula for torsion-free groups). However, since nonisomorphic subgroups may have the same rank, this fails for δ-groups.

Proposition 4.3.7. *If* $\mathrm{Hom}(M, N) = 0 = \mathrm{Hom}(N, M)$ *holds for δ-modules M and N, then $M \oplus N$ satisfies (δ).*

Proposition 4.3.8. *Every finite rank completely decomposable (torsion-free) Abelian group with incomparable type summands is a δ-group.*

iii) Torsion δ-groups

Proposition 4.3.9. *A torsion group has property (δ) if and only if all its primary components have property (δ).*

Lemma 4.3.10. *If $m < n$ are positive integers, the direct sum $A = \mathbb{Z}(p^m) \oplus \mathbb{Z}(p^n)$ is not (δ).*

Theorem 4.3.11. *A (reduced) p-group G has property (δ) if and only if it is finite and homogeneous.*

Corollary 4.3.12. *A p-group is (δ) if and only if it is morphic. So torsion δ-groups are exactly the torsion morphic groups.*

Example 4.3.13. The group $\bigoplus_{p\in\mathbf{P}} \mathbb{Z}_p$ is morphic, so (δ) and Hopfian (where **P** is a finite set of prime numbers).

iv) Mixed δ-groups

Proposition 4.3.14. *If G is (δ) then so is its torsion part $T(G)$.*

Remark 4.3.15. In any splitting mixed δ-group, $G = T(G) \oplus F$, the torsion part and $F \simeq G/T(G)$, as direct summands, are also δ-groups.

Lemma 4.3.16. *If the direct sum of R-modules $N \oplus K$ is morphic and there exists a R-linear epimorphism $\lambda : K \longrightarrow N$ then $K \simeq N \oplus Ker\lambda$.*

4.4 Special relatively morphic subgroups

Proposition 4.4.1. *(i) If G is a torsion group with finitely many and Hopfian components, then any primary component is relatively morphic.*

(ii) If G is a torsion $\delta-$group with finitely many components, then any primary component is morphic.

Proposition 4.4.2. *If the divisible part $D(G)$ of a group G has finite rank, then $D(G)$ is relatively morphic.*

Example 4.4.3. $\mathbb{Q}^n$ is relatively morphic in $\mathbb{Z} \oplus \mathbb{Q}^n$ for every positive integer n.

Corollary 4.4.4. *If the reduced part $D(G)$ of a torsion-free group is of rank 1 then $D(G)$ is relatively morphic.*

4.5 Simple morphic modules

Definition 4.5.1. A module M is called simple morphic if it has no proper relatively morphic submodules.

Obviously, a simple module is simple morphic.

Proposition 4.5.2. *If a module M is simple morphic, then it is indecomposable.*

Remark 4.5.3. Abelian indecomposable groups are cocyclic or torsion-free.

Proposition 4.5.4. *The finite cocyclic group $\mathbb{Z}_{p^n}$ is not simple-morphic and the infinite cocyclic group $\mathbb{Z}_{p^\infty}$ is simple-morphic.*

Remark 4.5.5. The cocyclic finite groups are not simple-morphic and the cocyclic infinite groups are simple-morphic. The group $\mathbf{Z}(p^\infty)$ is simple-morphic and it is not simple.

Proposition 4.5.6. *The only rank 1 torsion-free (indecomposable) simple-morphic group is $\mathbb{Z}$.*

4.6 The duality

Definition 4.6.1. We say that a submodule K is dually relatively morphic if whenever $M/N \simeq K$ then $M/K \simeq N$ for every submodule N.

Proposition 4.6.2. *M is dually relatively morphic in any module M.*

Proposition 4.6.3. *Every submodule of a morphic module is relatively morphic and dually relatively morphic.*

Proposition 4.6.4. *K is dually relatively morphic if and only if every endomorphism $\alpha \in \mathrm{End}(M)$ with $Ker\alpha = K$ is morphic.*

Lemma 4.6.5. *For a module M, the following statements are equivalent:*

i) 0 *is dually relatively morphic in M*

ii) M has no proper subgroups isomorphic to M

iii) M is co-Hopfian

Proposition 4.6.6. *For modules the following conditions are equivalent:*

(i) All direct summands of M are dually relatively morphic.

(ii) $K \simeq C$ implies $M/K \simeq M/C$, where C is a direct summand and K is a submodule.

Denote by (γ) the property above.

Definition 4.6.7. We say that M is a $\gamma-$module if all direct summands are morphic.

Example 4.6.8. 1) $\mathbb{Z}$ is not (γ) and not co-Hopfian because 0 is not dually relatively morphic in $\mathbb{Z}$

2) The essential subgroups in a torsion-free Abelian group are dually relatively morphic.

The class of co-Hopfian modules includes the class of $\gamma-$modules which includes the class of morphic modules.

Proposition 4.6.9. *Let $M = H \oplus C$. If M is a δ-module and C is dually relatively morphic then H is relatively morphic. If M is a γ-module and H is relatively morphic then C is dually relatively morphic.*

Proposition 4.6.10. *(γ) is inherited by direct summands in modules with cancellation for direct summands.*

Proposition 4.6.11. *A module M has property (γ) if and only if every kernel-direct endomorphism α of M is morphic.*

Proposition 4.6.12. *The following conditions are equivalent*

(i) M is co-Hopfian

(ii) $Ker\alpha = 0$ implies $\mathrm{Im}\alpha = M$

(iii) The monic endomorphisms of M are morphic.

Proposition 4.6.13.A module M has a Dedekind finite endomorphism ring if and only if all sections in End(M) are morphic.

4.7 Abelian $\gamma-$groups

Proposition 4.7.1. *Torsion-free γ-groups are divisible.*

Corollary 4.7.2. *The only torsion-free γ-groups are the finite direct sums of $\mathbb{Q}$.*

Corollary 4.7.3. *Every splitting mixed γ-group G, decomposes as $G = T(G) \oplus D(G)$, with $D(G)$, a finite direct sum of $\mathbb{Q}$.*

Proposition 4.7.4. *The torsion divisible γ-groups are exactly the divisible torsion groups with finite p-ranks.*

Corollary 4.7.5. *Torsion groups with homogeneous p-components are γ-groups if and only if each p-component is a γ-group.*

Corollary 4.7.6. *Torsion groups with homogeneous p-components of finite rank are γ-groups.*

Lemma 4.7.7. *If $m < n$ are positive integers, the direct sum $A = \mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n}$ is not (γ).*