"Babeş-Bolyai" University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Andrea-Éva Molnár

VARIATIONAL PRINCIPLES WITH APPLICATIONS

Summary of the Doctoral Thesis

Scientific advisor:

Prof. Dr. CSABA GYÖRGY VARGA

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Contents

Introduction

Calculus of variations concerns with the optimization of physical quantities, such as time, area or distance. The typical problem of the calculus of variations is to find a minimum of a functional of the form

$$F(u) = \int_{\Omega} f(x, u(x), Du(x))dx,$$

where u is a real or vector valued function which belongs to some suitable class of functions and the hypotheses on the integrand may vary depending on the situation.

The brachistochrone problem was one of the earliest problems posed in the calculus of variations. However, for the apparition of modern calculus of variations must have been waited until the middle of the 19th century, being a basic tool in the qualitative analysis of models arising in physics. An important milestone in the transition from classical to contemporary physics represents the characterization of phenomena by means of variational principles. Since the 1950's, variational principles occupy an important place in the study of nonlinear partial differential equations and many problems arising in applications. According to Ioffe and Tikhomirov [55], "the term 'variational principle' refers essentially to a group of results showing that a lower semi-continuous, lower bounded function on a complete metric space possesses arbitrarily small perturbations such that the perturbed function will have an absolute (and even strict) minimum."

Ekeland's variational principle, established by Ivar Ekeland in his pioneer work [42] in 1974 (shortly, EVP) is one of the most important and maybe the most fruitful results of the mathematical analysis, being a very useful tool to solve problems in optimization, game theory, optimal control theory, nonlinear equations and in dynamical systems; see for instance J.-P. Aubin, H. Frankowska [12], M. Bianchi et al. [16], I. Ekeland [42], [43], D. G. De Figueiredo [38], etc. This principle provides a minimizing sequence for any lower semicontinuous and bounded from below functional f, where the elements of the minimizing sequence minimize an appropriate sequence of perturbations of f which converges locally uniformly to f.

Since its discovery many generalizations and equivalent formulations of this principle have also appeared (see J. Daneš [36], P.G. Georgiev [48], [49], A.H. Hamel [53], I. Meghea [73], W. Oettli, M. Théra [80], J.-P. Penot [84], L. Yongxing, S. Shuzhong [93] or Section 1.3, respectively). In the other hand, EVP is equivalent with Caristi's and Tarafdar's fixed point theorem (J. Caristi [26], E. Tarafdar [91]).In addition, EVP has also many other equivalent formulations such as Daneš drop theorem (J. Daneš [35]) and the Flower Petal theorem of Penot (J.-P. Penot [84]), just to name a few. Moreover, EVP is equivalent to the Bishop-Phelps theorem in the setting of Banach spaces (J. M. Borwein, Q. J. Zhu [23]).

In the hope of finding more applications, EVP and several results mentioned above have been extended to spaces with more general distances, for example to quasimetric spaces. In the literature, the notion of quasimetric space is used in two different ways: the first concept means asymmetry of the metric (see, for example, S. Al-Homidana et al. [1]), while in the second concept a quasimetric satisfies a relaxed triangle inequality, rather then the usual triangle inequality (see I.A. Bakhtin [13], S. Czerwik [33], J. Heinonen [54]). Obviously, the asymmetry is not equivalent with the relaxed triangle inequality. Very recently S. Al-Homidan, Q.H. Ansari and J.-C. Yao in [1] gave an extended result of Ekeland's variational principle to quasimetric spaces (according to the first concept) and as an application of this result, they presented an existence theorem for solutions of equilibrium problems and fixed points. In this thesis, we focus our attention to the latter quasimetric, called also b-metric. The concept of this space was introduced by I.A. Bakthin in [13] and S. Czerwik in [33]. Since the publication of these works, several papers have appeared which were concerned to these spaces obtaining important results in many fields of mathematics: fixed point theory (study of fixed point theorems for single-valued and multivalued operators), geometry, calculus of variations; see for example the works of N. Bourbaki, I.A. Bakhtin [13], V. Berinde [15], S. Czerwik [33], S.L. Singh et al. [90], etc.

The first purpose of this thesis is to develop the theory of some important and celebrated variational principles such as the Ekeland's variational principle or the generalization of it, i.e. Zhong's variational principle (abbreviated, ZVP) due to C.-K. Zhong ([94], [95]). Since the power of variational principles is motivated by various applications in different fields of analysis, in this thesis we deal to derive some applications of the extended EVP, resp. ZVP to fixed point theory and to equilibrium problems.

One of the most important problems in nonlinear analysis is the so-called equilibrium problem which can be defined in the following way: Let A and B be two nonempty sets and $f : A \times B \to \mathbb{R}$ a given function. The problem consists on finding an element $\overline{x} \in A$ such that

$$(EP) \qquad f(\overline{x}, y) \geq 0, \forall y \in B.$$

Notice that $\overline{x}$ is an equilibrium point of f on $A \times B$.

In the recent past, (EP) becomes an attractive field both in theory and applications (see e.g. [16], [17], [21], [58], [18], [19], [20], [60], [72] and the references therein).

In [9], Q. H. Ansari et al. introduced and investigated systems of equilibrium problems. For the definition of these type of equilibrium problems, the reader is reffered to Section 1.5 from this thesis.

Since the appearances of the (EP) and (SEP), respectively, many authors are interested in extending Ekeland's theorem to the setting of an (EP), resp. (SEP). In the study of equilibrum problems we frequently encounter with such a situation when an equilibrium problem may not have solution even in case when the problem arises from practice. For this issue the EVP offers an accessible solution, since it provides the existence of approximate solutions of minimization problems for lower semicontinuous functions (see, for instance, J.P. Aubin [11]). In the other hand, we know that minimization problems are particular cases of equilibrium problems. For details concerning Ekeland's variational principle and its relation with the equilibrium problems, see, for example, A. Amini-Harandi et al. [3], Q.H. Ansari [5]-[7], Q.H. Ansari, L.-J. Lin [8], Y. Araya et al. [10], M. Bianchi et al. [16] and the references quoted therein.

As we mentioned before, an other goal of this thesis is to extend the EVP to the setting of (EP) and (SEP), and with the aid of the extended principles and of ZVP, respectively, to give some existence, resp. localization results of these type of problems.

Ekeland's variational principle can be applied in critical point theory to prove different existence and multiplicity results. We emphasize here the work of D. G. De Figueiredo [38], where the author shows how to use a consequence of EVP (i.e. a general variational principle of the min-max type) to derive the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [2], as well as the Saddle Point and the Generalized Mountain Pass Theorems investigated by Rabinowitz (see [85] and [86], respectively). For locally Lipschitz functions the Mountain Pass Theorem of Ambrosetti and Rabinowitz was proved by P. Mironescu and V. Rădulescu (see [74], [87]).

The aforementioned types of results are very useful tools in the study of inequalities and systems of inequalities of hemivariational types. Hemivariational inequalities were introduced by P.D. Panagiotopoulos in the early 1980's (see P.D. Panagiotopoulos [81],[82]) as a generalization of variational inequalities. Actually, they are more than simple generalizations, since a hemivariational inequality is not equivalent to a minimum problem. The hemivariational inequality problem becomes a variational inequality problem, if we assume that the involved functionals are convex.

Hemivariational inequalities gave rise to a new branch in nonlinear analysis, the so-called *nonsmooth analysis*, its main tool is the concept of Clarke's generalized gradient of a locally Lipschitz function (see F.H. Clarke [27]-[30]). The theory of hemivariational inequalities has produced important results both in pure and applied mathematics; see the monographs of Z. Naniewicz and P.D. Panagiotopoulos [79], D. Motreanu and P.D. Panagiotopoulos [77], D. Motreanu and V. Rădulescu [78] and the references cited therein.

Nonlinear hemivariational inequalities were introduced recently, in 2010, by N. Costea and V. Rădulescu [31] (see also I. Andrei and N. Costea [4]), while the first paper dealing with systems of nonlinear hemivariational inequalities is due to N. Costea and Cs. Varga [32]. For more details and connections regarding systems of hemivariational inequalities one can consult the works of B.E. Breckner, A. Horváth and Cs. Varga [24], A. Kristály [61], [63] and D. Repovš, Cs. Varga [89].

There are three approaches to the study these type of inequalities: via monotone and pseudo-monotone operators, (see e.g. Z. Liu and D. Motreanu [71], Z. Naniewicz and P.D. Panagiotopoulos [79]), with the aid of critical point theory (F. Gazzola and

V. Rădulescu [47], Z. Naniewicz and P.D. Panagiotopoulos [79], D. Motreanu and P.D. Panagiotopoulos [77], D. Motreanu and V. Rădulescu [78], A. Kristály [62], [64], Z. Dályai and Cs. Varga [34], Cs. Varga [92], F. Faraci et al. [44]) and using fixed point theory results(see P. D. Panagiotopoulos et al. [83], A. Kristály and Cs. Varga [66], V. Rădulescu and D. Repovš [88]). Following the latter approach, in this thesis we study the existence of solutions of some inequalities and systems of inequalities of hemivariational types via fixed point techniques without imposing any monotonicity assumptions, underlining the possibilities of the diverse applicability of the obtained existence results.

The thesis is divided into 4 chapters.

Chapter 1: Preliminaries.

In this first chapter we present some notions and results which we will use in the next chapters of this thesis.

Chapter 2: Variational principles in b-metric spaces with applications to fixed point results.

This chapter deals to present several extensions of the EVP in the setting of complete b-metric spaces for single-valued function and for functions with two variables and to emphasize the applicability of the obtained generalized variational principles to fixed point theory. The main theorem of this chapter is Ekeland's variational principle stated in complete b-metric spaces for single-valued functions. In the first section, inspired by the common generalization of the Ekeland and Borwein-Preiss variational principle due to L. Yongxing and S. Shuzhong [93], we formulate and prove the EVP in the setting of complete b-metric spaces for single-valued mappings and then for functions with two variables. As consequence, we present a weak version of ZVP in complete b-metric spaces. In addition, an extension of ZVP to complete b-Banach spaces is obtained. The second section is dedicated to present some new results of the fixed point theory in b-metric spaces, namely the Caristi-Kirk fixed point theorem for single-valued mappings and for bifunctions, which can be considered also as applications of the versions of EVP and of the generalized form of it, respectively, in the framework of b-metric spaces.

Chapter 3: Variational principles in metric spaces with applications to equilibrium problems.

The aim of this chapter is to extend some results investigated by M. Bianchi et al. in [16] to complete metric spaces without any convexity requirements. In the literature, when dealing with equilibrium problems and the existence of their solutions, the most used assumptions are the convexity of the domain and the generalized convexity and monotonicity, together with some weak continuity assumptions, of the function (see M. Bianchi, R. Pini [18], [19], N. Hadjisavvas et al. [52], G. Kassay, J.Kolumbán [59]). E. Blum, W. Oettli [21] and W. Oettli and M. Théra [80] were the first who established an existence result for a solution of an equilibrium problem in the setting of complete metric spaces. They proved this result with the aid of EVP but without imposing any convexity requirements. Almost ten years later, M. Bianchi, G. Kassay and R. Pini in their afformentioned paper [16] aimed to extend Ekeland's variational principle for (EP) and (SEP) in the setting of Euclidean spaces without any kind of convexity of the functions involved in the formulation of the principle. They also showed that the obtained equilibrium versions of the EVP guarantee the existence of an approximate equilibria for (EP) and (SEP), and using this result it is possible to show the existence of equilibria on general closed sets without any convexity assumptions neither on the sets nor on the functions involved (see Section 1.5). Very recently, A. Amini-Harandi et al. [3] established equilibrium version of EVP on complete metric spaces. The authors only focused on conditions that do not involve any semicontinuity concept for the bifunction involved.

The chapter contains two sections. In the first section the relationship between (EP) and the EVP are discussed, while in the second section we extend all our theorems presented in the previous section for systems. We establish two extensions of Ekeland's variational principle in complete metric spaces (one for (EP), the other for (SEP)) without assuming any kind of convexity of the bifunction, resp. the function system involved in the formulation of the principle or of the set whereon the bifunction, resp. the function system are defined. We also show that these principles ensure the nonemptiness of the solution set of (EP) and (SEP) with compactness assumption on the underlying sets but without any convexity requirements.
Another powerful topic in the study of (EP) is considered the localization of the equilibrium point. Starting from the ZVP, as further applications of the main theorems, we aim to prove localization-type results for (EP) and (SEP). The importance of the

main theorems in finding equilibrium points of differential equations and systems of differential equations is also highlighted.

Chapter 4: Fixed point techniques used in the study of different classes of inequalities and systems of inequalities of hemivariational type.

In this chapter we study a variational-hemivariational inequality problem and a type of a nonlinear hemivariational inequality system problem, respectively, on closed and convex sets. Concerning these problems we establish existence results with the aid of fixed point technics without assuming any kind of monotonicity assumptions.

The first section is devoted to discuss the solvability of a variational-hemivariational inequality problem on a closed and convex set (either bounded or unbounded). We start this paragraph with the assumptions and formulating the problem. Then we present the main theorem of this paragraph which is an existence result regarding the studied system. The proof is based on a version of the well-known fixed theorem of Knaster-Kuratowski-Mazurkiewicz.

In the second section we introduce a new type of inequality systems, which we call nonlinear hemivariational-like inequality system. After imposing the corresponding assumptions and formulating the problem itself, we prove the existence of at least one solution of the studied problem without employing the nonsmooth critical point theory (we apply Lin's fixed point theorem) and without imposing any monotonicity assumptions.

The chapter ends with a section which presents a wide range of applications of the results obtained in the previous sections to Nash generalized derivative points, Schrödinger-type problems and to problems with radially symmetric functions.

Our contributions to this thesis are based on five papers, written in collaboration:

- H. Lisei, A. É. Molnár, Cs. Varga [69] - appeared in Journal of Mathematical Analysis and Applications (2010);

- M. Bota, A. É. Molnár, Cs. Varga [22] - published in Fixed Point Theory (2011);

- Cs. Farkas, A. É. Molnár [45] - appeared in Journal of Optimization Theory and Applications (2013);

- A. É. Molnár, O. Vas [75] - accepted for publication in Studia Universitatis Babeş-Bolyai Mathematica;
- Cs. Farkas, A. É. Molnár [46] - submitted to Studia Universitatis Babeş-Bolyai Mathematica.

We specify here our original results:

Chapter 1:
Lemma 1.1.1.

Chapter 2:
Theorems 2.1.1, 2.1.2, 2.1.3, 2.1.4, 2.2.1, 2.2.2;
Lemmas 2.1.1, 2.1.2;
Corollaries 2.1.1, 2.1.2;
Remarks 2.1.1, 2.1.2, 2.2.1, 2.2.2.

Chapter 3:
Theorems 3.1.1, 3.1.2, 3.1.3, 3.2.1, 3.2.2, 3.2.3;
Remarks: 3.1.1, 3.1.2, 3.1.3, 3.1.4, 3.2.1;
Definitions: 3.1.1, 3.2.1;
Examples: 3.1.1, 3.1.2, 3.1.3.

Chapter 4:
Theorems: 4.1.1, 4.1.2, 4.2.1, 4.3.1, 4.3.2;
Lemma 4.1.1;
Corollaries: 4.3.1, 4.3.2;
Proposition 4.2.1;
Remarks: 4.1.1, 4.2.1, 4.2.2, 4.3.1, 4.3.2;
Examples: 4.1.1, 4.1.2, 4.2.1, 4.2.2.

Finally, we mention here two other papers which also contains original results, but these results are not included in this thesis, since these results could not be bounded directly to the topic of the thesis, and they would have destructed the unity of it. One of these papers [A. É. Molnár, *A Nonsmooth Sublinear Elliptic Problem in* $\mathbb{R}^N$ *with Perturbations*, Stud. Univ. Babeş-Bolyai Math. 57 (1) (2012), 61-68.] deals with a differential inclusion problem in $\mathbb{R}^{\mathbb{N}}$ involving the p-Laplace operator and a $(p-1)$-sublinear term, $p > N > 1$. This problem was also studied by Kristály, Marzantowicz and Varga [J. Global Optim. 46 (1) (2010), 49-62.]. The aim of the aforementioned paper is to show that under the same assumptions, a more precise conclusion can be concluded by exploiting a recent result of Iannizzotto (see [Set-Valued and Variational Analysis, 19 (2) (2011), 311-327].). In addition, the non-sensitivity of the studied problem with respect to small perturbations is showed.

In the second paper [Cs. Farkas, A. É. Molnár, Cs. Varga, *Multiple symmetric solutions of the semilinear elliptic problem*, manuscript] we study a semilinear elliptic differential inclusion problem coupled with a homogeneous Dirichlet boundary condition on the unit ball, depending on a positive parameter λ. We proved that for large values of λ, problem $(\mathcal{P}_\lambda)$ has at least two non-zero symmetric weak solutions. The proof of the multiplicity result is based on the general minimax theorem for locally Lipschitz functionals, stated in the same paper, and on a symmetric version of Ekeland's variational principles due to M. Squassina [J. London Math. Soc. 85 (2012), 323-348.]

Keywords: Ekeland-type variational principles, Zhong-type variational principles, Caristi-type fixed point theorems, b-metric space, Palais-Smale condition, equilibrium problem, system of equilibrium problems, approximate solution, fixed point theory, locally Lipschitz functionals, variational-hemivariational inequality, system of nonlinear hemivariational-like inequality, Nash equilibrium point, Schrödinger-type problem, radially symmetric functions, set-valued operator, nonsmooth functions

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Chapter 1

Preliminaries

The aim of this chapter is to recall some basic notions and results needed in the next chapters of this work, allowing us to present the results of this Ph.D. thesis. We mainly follow the works of H. Brézis [25]; F.H. Clarke [27]-[30]; Z. Denkowski, S. Migórski, N. S. Papageorgiou, [40], [41]; A. Kristály, V. Rădulescu, Cs. Varga, [65]; A. Kristály, Cs. Varga, [67]; D. Motreanu, P.D. Panagiotopoulos, [77].

1.1 Basic notions and properties in several metric spaces

In this section we develop the most basic facts about the metric and the b-metric spaces, respectively, which will be crucial tools in the following. The last part of this paragraph is devoted to present the main notions and properties related to normed and b-normed spaces, respectively.

Definition 1.1.1 *(I.A. Bakthin [13], S. Czerwik [33]) Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if and only if for all points $x, y, z \in X$ the following conditions are satisfied:*

1. $d(x, y) = 0$ *if and only if* $x = y$*;*

2. $d(x, y) = d(y, x)$*;*

3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A b-metric space consist of a set X and a b-metric d.

Lemma 1.1.1 *(M. Bota, A. É. Molnár, Cs. Varga, [22]) Let (X, d) be a b-metric space. Suppose that (X, d) is complete. Then, for every descending sequence $\{F_n\}_{n\geq 1}$ of nonempty closed subsets of X, that is,*

$$F_1 \supset F_2 \supset F_3 \supset ... \supset F_n \supset ... \tag{1.1.1}$$

such that

$$\text{diam}(F_n) \to 0 \text{ as } n \to \infty, \tag{1.1.2}$$

we have that the intersection $\bigcap_{n=1}^{\infty} F_n$ contains one and only one point.

1.2 Elements of topological and convex analysis

We continue with presenting some useful notions and results of topological and convex analysis.

1.3 Variational principles

For the sake of completeness, this section is devoted to present some well-known variational principles, most of them being the starting point in our research.

1.4 Fixed point theorems

In this section we present some fixed point results needed in the sequel.

1.5 Equilibrium points and problems

This section is devoted to present some useful notions and properties concerning equilibrium problems.

1.6 Differentiability. Elements of the theory developed by Clarke

In this section we recall some useful notions and facts about differentiability.

1.7 L^p and Sobolev spaces

The last section treats with different function spaces such as L^p and Sobolev spaces. We start with L^p spaces.

Chapter 2

Variational principles in b-metric spaces with applications to fixed point results

The purpose of this chapter is to establish some extensions of the celebrated Ekeland's variational principle (abbreviated EVP) in the setting of b-metric spaces for single-valued functions and for functions with two variables and to emphasize the applicability of the obtained generalized variational principles to fixed point theory.

This capitol is based on the following two papers: M. Bota, **A. É. Molnár**, Cs. Varga [22]; Cs. Farkas, **A. É. Molnár** [46].

2.1 Variational principles in b-metric spaces

In this paragraph we present the extended versions of some well-known variational principles (such as Ekeland's variational principle or Zhong's variational principle) in the setting of complete b-metric spaces for single-valued maps and for bifunctions.

In order to obtain our results, let us consider the complete b-metric space (X, d) such that the b-metric d is continuous. We impose the following condition:

(**F**) Let $f : X \to \overline{\mathbb{R}}(= \mathbb{R} \cup \{+\infty\})$ be a lower semicontinuous, proper and lower bounded mapping.

We need also to introduce the notation below:

$$\mathcal{F}[x;m] = f(x) + \sum_{n=0}^{m} \frac{1}{s^n} \cdot d(x, x_n).$$

In possession of the suitable conditions and terms we can formulate our first main result.

Theorem 2.1.1 *(M. Bota, A. É. Molnár, Cs. Varga, [22]) Let (X,d) be a complete b-metric space with a continuous b-metric d. Suppose that the function $f : X \to \overline{\mathbb{R}}$ satisfies the assumption $(\mathbf{F})$. Then, for every $x_0 \in X$ and $\varepsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon,$$

there exists a sequence $\{x_n\} \subset X$ and $x_\varepsilon \in X$ such that

$$x_n \to x_\varepsilon, \text{ as } n \to \infty \tag{2.1.1}$$

$$d(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n}, \quad n \in \mathbb{N} \tag{2.1.2}$$

$$\mathcal{F}[x_\varepsilon; +\infty] \leq f(x_0) \tag{2.1.3}$$

and for every $x \neq x_\varepsilon$, we have

$$\mathcal{F}[x; +\infty] > \mathcal{F}[x_\varepsilon; +\infty]. \tag{2.1.4}$$

Remark 2.1.1 *Theorem 2.1.1 is the extended version of Ekeland's variational principle to b-metric spaces. We note, that if $s = 1$ we get back the original version of Ekeland's variational principle, stated in metric spaces.*

The following result is a consequence of the previous theorem.

Corollary 2.1.1 *(M. Bota, A. É. Molnár, Cs. Varga, [22]) Consider the complete b-metric space (X,d) such that the b-metric d is continuous. Suppose that the function $f : X \to \overline{\mathbb{R}}$ satisfies the condition $(\mathbf{F})$. Then, for every $\varepsilon > 0$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ and $x_\varepsilon \in X$ such that*

$$x_n \to x_\varepsilon, \quad \text{as } n \to \infty, \tag{2.1.5}$$

$$\mathcal{F}[x_\varepsilon; +\infty] \leq \inf_{x \in X} f(x) + \varepsilon \tag{2.1.6}$$

and for any $x \in X$ we have

$$\mathcal{F}[x; +\infty] \geq \mathcal{F}[x_\varepsilon; +\infty]. \tag{2.1.7}$$

In the sequel, as a generalization of Theorem 2.1.1, we establish a generalized variational principle for b-metric spaces. In order to obtain this result, we need to impose some further assumptions. We suppose again that (X, d) is a complete b-metric space (the b-metric d is continuous) and $f : X \to \overline{\mathbb{R}}$ satisfies the condition (**F**). Let us also consider the continuous non-increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ and the non-negative number sequence $\delta_n \subset \mathbb{R}_+$ such that $\delta_0 > 0$. We also assume the following:

(**R**) The function $\rho : X \times X \to \overline{\mathbb{R}}_+$ satisfies:

- (i) for each $x \in X$, we have $\rho(x, x) = 0$;
- (ii) for each $(y_n, z_n) \in X \times X$ such that $\rho(y_n, z_n) \to 0$ as $n \to \infty$, we have $d(y_n, z_n) \to 0$ as $n \to \infty$;
- (iii) for each $z \in X$, the function $y \mapsto \rho(y, z)$ is lower semicontinuous.

In addition, we introduce the following notation:

$$\mathcal{F}_h[x; m] = f(x) + h(d(x_0, x)) \sum_{n=0}^{m} \delta_n \rho(x, x_n), m \in \overline{\mathbb{N}}.$$

A generalized form of Theorem 2.1.1 is given below.

Theorem 2.1.2 *(Cs. Farkas, A. É. Molnár, [46]) Suppose that (X, d) is a complete b-metric space, d is continuous, and the mappings $f : X \to \overline{\mathbb{R}}$ and $\rho : X \times X \to \overline{\mathbb{R}}_+$ satisfy the assumptions* (**F**) *and* (**R**)*, respectively. Then for every $x_0 \in X$ and $\varepsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon,$$

we assume the existence of a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ and an element $x_\varepsilon \in X$ such that

$$x_n \to x_\varepsilon, \text{ whenever } n \to \infty \tag{2.1.8}$$

$$h(d(x_0, x_\varepsilon))\rho(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n \delta_0}, \text{ for all } n \in \mathbb{N}. \tag{2.1.9}$$

If $\delta_n > 0$ for infinitely many $n \in \mathbb{N}$, then we have

$$\mathcal{F}_h[x_\varepsilon; +\infty] \leq f(x_0), \tag{2.1.10}$$

and for $x \neq x_\varepsilon$ we have

$$\mathcal{F}_h[x;+\infty] > \mathcal{F}_h[x_\varepsilon;+\infty]. \tag{2.1.11}$$

If $\delta_k > 0$ for some $k \in \mathbb{N}^*$ and $\delta_j = 0$ for every $j > k$, then for each $x \neq x_\varepsilon$ there exists $m \in \mathbb{N}$, $m \geq k$ such that

$$\mathcal{F}_h[x;k-1]+h(d(x_0,x))\delta_k\rho(x,x_m) > \mathcal{F}_h[x_\varepsilon;k-1]+h(d(x_0,x_\varepsilon))\delta_k\rho(x_\varepsilon,x_m). \tag{2.1.12}$$

Remark 2.1.2 *If in the previous theorem we choose $h(x) \equiv 1$ and $\delta_n = \dfrac{1}{s^n}$, we get back our first result (Theorem 2.1.1). In other words, for every $x \neq x_\varepsilon$ we obtain the inequality* (2.1.4), *that is,*

$$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} d(x,x_n) > f(x_\varepsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} d(x_\varepsilon,x_n). \tag{2.1.13}$$

The rest of this section is dedicated to the presentation of the extended forms of Theorems 2.1.1 and 2.1.2, respectively, to bifunctions. As in above, we introduce some new symbols and notations which we will use in the rest of this paragraph. Let C be a closed subset of the complete b-metric space (X,d) (suppose that d is continuous), $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function and $\rho : X \times X \to \overline{\mathbb{R}}_+$ be a function which satisfies the assumption (**R**).

Instead of the condition (**F**), we shall assume:

(**F2**) For the mapping $f : C \times C \to \mathbb{R}$ the following assertions are fulfilled:

- (i) $f(x,\cdot)$ is lower bounded and lower semicontinuous, for every $x \in C$;
- (ii) $f(z,z) = 0$, for each $z \in C$;
- (iii) $f(z,x) \leq f(z,y) + f(y,x)$, for all $x,y,z \in C$.

We will also use the following notation:

$$\overline{\mathcal{F}}_h[y,x;m] = f(y,x) + h(d(x_0,x)) \sum_{n=0}^{m} \delta_n \rho(x,x_n).$$

Theorem 2.1.3 *(Cs. Farkas, A. É. Molnár, [46]) Let us consider the complete b-metric space (X,d) with the continuous b-metric and the mapping $f : C \times C \to \mathbb{R}$*

which satisfies the assumption (**F2**). *Then, for every* $x_0 \in X$ *and* $\varepsilon > 0$, *there exists a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset C$ *which converges to some* $x_\varepsilon \in X$, *i. e.*

$$x_n \to x_\varepsilon, \text{ as } n \to \infty \tag{2.1.14}$$

such that

$$h(d(x_0, x_\varepsilon))\rho(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n \delta_0}, \ n \in \mathbb{N}, \tag{2.1.15}$$

$$\overline{\mathcal{F}}_h[x_0, x_\varepsilon; 0] \leq 0. \tag{2.1.16}$$

Moreover, for all $x \neq x_\varepsilon$ *we have*

$$\overline{\mathcal{F}}_h[x_\varepsilon, x; +\infty] - h(d(x_0, x_\varepsilon)) \sum_{n=0}^{\infty} \delta_n \rho(x_\varepsilon, x_n) > 0. \tag{2.1.17}$$

One of the particular cases of Theorem 2.1.3 we can obtain if we set: $h \equiv 1$, $\delta_n = \frac{1}{s^n}$ and $\rho = d$. But at the same time, this result can be viewed as a generalization of Ekeland's variational principle in the setting of complete b-metric spaces for single-valued maps (see Theorem 2.1.1).

Theorem 2.1.4 *(Cs. Farkas, A. É. Molnár, [46]) Let* (X, d) *be a complete* b*-metric space with* $s > 1$, *where* d *is continuous,* $C \subset X$ *be a closed set and* $f : C \times C \to \mathbb{R}$ *be a function which satisfies the assertion* (**F2**). *Then, for every* $x_0 \in X$ *and* $\varepsilon > 0$ *there exists a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset C$ *and an element* $x_\varepsilon \in C$ *such that* $x_n \to x_\varepsilon$, *as* $n \to \infty$ *and*

$$d(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n}, \quad n \in \mathbb{N}, \tag{2.1.18}$$

$$f(x_0, x_\varepsilon) + d(x_\varepsilon, x_0) \leq 0, \tag{2.1.19}$$

and for every we have $x \neq x_\varepsilon$

$$f(x_\varepsilon, x) + \sum_{i=0}^{\infty} \frac{1}{s^i} d(x, x_i) - \sum_{i=0}^{\infty} \frac{1}{s^i} d(x_\varepsilon, x_i) > 0. \tag{2.1.20}$$

We close this paragraph with a special case of Theorem 2.1.2. Actually, this result can be considered as a weak Zhong-type variational principle (for the statement of the original Zhong variational principle, see [94], [95]). Before the presentation of this result, at first we establish two technical lemmas which play an important role in the demonstration of this principle.

Lemma 2.1.1 *(Cs. Farkas, A. É. Molnár, [46]) If $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function, and $x \notin B(x_0; d(x_0, x_\varepsilon))$ then we have*

$$\frac{d(x_0, x)}{1 + g(d(x_0, x))} - s \cdot \frac{d(x_0, x_\varepsilon)}{1 + g(d(x_\varepsilon, x_0))} \leq s \cdot \frac{d(x, x_\varepsilon)}{1 + g(d(x_\varepsilon, x_0))}. \tag{2.1.21}$$

Lemma 2.1.2 *(Cs. Farkas, A. É. Molnár, [46]) If $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function, and $\frac{g(x)}{x}$ is decreasing on $(0, d(x_0, x_\varepsilon)]$ then we have*

$$\frac{d(x_0, x)}{1 + g(d(x_0, x))} - s \cdot \frac{d(x_0, x_\varepsilon)}{1 + g(d(x_\varepsilon, x_0))} \leq s \cdot \frac{d(x, x_\varepsilon)}{1 + g(d(x_\varepsilon, x_0))}.$$

Next we show that in a special case of the Theorem 2.1.2 we get the original version of Zhong's variational principle (see for instance C.-K. Zhong [94, 95]). In order to obtain this result let us choose the sequence δ_n and the functions h, ρ as follows. For every $n > 0$, let $\delta_0 = 1$ and $\delta_n = 0$. Furthermore, we take $\varepsilon, \lambda > 0$ and $h(t) = \dfrac{\varepsilon}{\lambda(1 + g(t))}$, where $g : [0, \infty) \to [0, \infty)$ is a continuous non-decreasing function. In this case, we have

$$\sum_{n=0}^{\infty} \delta_n \rho(x, x_n) = \delta_0 \rho(x, x_0) = \rho(x, x_0).$$

If we put $\rho = d$, then we can consider the following form of Theorem 2.1.2:

$$f(x) \geq f(x_\varepsilon) + \frac{\varepsilon}{\lambda(1 + g(d(x_0, x_\varepsilon)))} d(x_\varepsilon, x_0) - \frac{\varepsilon}{\lambda(1 + g(d(x_0, x)))} d(x, x_0). \tag{2.1.22}$$

Therefore, by Lemmas 2.1.1 and 2.1.2, we obtain the following weak Zhong-type variational principle in complete b-metric space.

Corollary 2.1.2 *(Cs. Farkas, A. É. Molnár, [46]) Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous non-decreasing function. Let (X, d) be a complete b-metric space (the b-metric d is continuous) and $f : X \to \mathbb{R}$ be a function which satisfies the assumption* $(\mathbf{F})$. *Then for every $x_0 \in X$ and $\varepsilon > 0$ with*

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon,$$

we assume the existence of a sequence $\{x_n\} \subset X$ which converges to some $x_\varepsilon \in X$ such that

$$h(d(x_0, x_n)) d(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n}, \; n \in \mathbb{N}. \tag{2.1.23}$$

Then we can distinguish two cases:
1. If $x \notin B(x_0, d(x_0, x_\varepsilon))$, then we have

$$f(x) \geq f(x_\varepsilon) - s \cdot \frac{\varepsilon}{\lambda(1 + g(d(x_0, x_\varepsilon)))} d(x, x_\varepsilon). \tag{2.1.24}$$

2. If $\frac{g(x)}{x}$ is decreasing on $(0, d(x_0, x_\varepsilon)]$, then for all we have $x \neq x_\varepsilon$

$$f(x) \geq f(x_\varepsilon) - s \cdot \frac{\varepsilon}{\lambda(1 + g(d(x_0, x_\varepsilon)))} d(x, x_\varepsilon).$$

In the following, we assume that $g : [0, +\infty) \to [0, +\infty)$ is a continuous non-decreasing function, the mapping $\frac{g(x)}{x}$ is decreasing, and $f : X \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous, Gâteaux differentiable function and not identically with $+\infty$. In this case we can extend Theorem 2.1 from Zhong [94]. More precisely, if f is bounded from below, then for $\varepsilon > 0$, every $y \in X$ such that

$$f(y) \leq \inf_{x \in X} f(x) + \varepsilon, \tag{2.1.25}$$

and every $\lambda > 0$, there exists a $z \in X$ such that

$$f(z) \leq f(y),$$

$$\|f'(z)\| \leq s \cdot \frac{\varepsilon}{\lambda(1 + g(\|z\|))}.$$

From the above, one can extend the notion of weak (PS)-condition from [94], and we can prove that a function, which is bounded below, and satisfies the weak (PS)-condition has a minimal point.

2.2 Applications to fixed point theory

This paragraph is devoted to highlight the importance of the extended versions of EVP presented in the previous section establishing some applications of them to fixed point theorems in the setting of b-metric spaces. We will show that Ekeland's variational principle in the setting of complete b-metric spaces for single-valued functions and for functions with two variables, respectively, can be also applied to prove fixed point theorems within the framework of complete b-metric spaces. This usefulness is demonstrated by two versions of Caristi's fixed point theorem, both of them stated in complete b-metric spaces, one for single-valued functions and one for bifunctions, since the proofs of these results become very simple if we invoke the extended versions of the well-known variational principle.

Throughout this paragraph we will use the same notations as in the previous section. We suppose again that (X, d) is a complete b-metric space, where d is continuous.

We start our discussion with the extension of Caristi's fixed point theorem in the context of complete b-metric spaces for single-valued functions. Me mention that Corollary 2.1.1 is the main ingredient for the proof of this result.

Theorem 2.2.1 *(M. Bota, A. É. Molnár, Cs. Varga, [22]) Let (X, d) be a complete b-metric space (with $s > 1$), such that the b-metric d is continuous. Let $\varphi : X \to X$ be an operator for which there exists a lower semicontinuous mapping $f : X \to \overline{\mathbb{R}}$, such that*

$$d(u, v) + s\, d(u, \varphi(u)) \geq d(\varphi(u), v) \tag{2.2.1}$$

$$\frac{s^2}{s-1}\, d(u, \varphi(u)) \leq f(u) - f(\varphi(u)), \forall u, v \in X \tag{2.2.2}$$

Then, φ has at least one fixed point.

Remark 2.2.1 *If $s = 1$, then we get back Caristi's fixed point theorem in complete metric spaces (see [26], [43]).*

Similarly to the above theorem, we can formulate the following Caristi-type fixed point theorem in complete b-metric spaces for bifunctions. We mention here that ξ denotes the sum of the convergent series $\sum_{n=0}^{\infty} \delta_n$, i.e. $\xi := \sum_{n=0}^{\infty} \delta_n$.

Theorem 2.2.2 *(Cs. Farkas, A. É. Molnár, [46]) Let (X, d) be a complete b-metric space such that d is a continuous b-metric and $\rho : X \times X \to \mathbb{R}_+$ be a continuous function. Let us consider the operator $\varphi : X \to X$ such that there exists a lower semicontinuous mapping $f : X \to \mathbb{R}_+$ satisfying the following assumptions:*

$$h(d(x_0, \varphi(x)))\rho(\varphi(x), y) - h(d(x_0, x))\rho(x, y) \leq \rho(x, \varphi(x)), \tag{2.2.3}$$

$$\xi\rho(u, \varphi(u)) \leq f(u) - f(\varphi(u)). \tag{2.2.4}$$

Then φ has at least one fixed point.

Remark 2.2.2 *We notice that Theorem 2.2.1 is a particular case of Theorem 2.2.2, since if in Theorem 2.2.2 we choose adequately the functions ρ and h and, respectively the sequence δ_n, i.e. $h \equiv 1, \rho = d$, and $\delta_n = \dfrac{1}{s^n}$, then we obtain Theorem 2.2.1.*

Chapter 3

Variational principles in metric spaces with applications to equilibrium problems

The aim of this chapter is to study the relationship between (EP), respectively (SEP), and the celebrated Ekeland's variational principle. Inspired by the paper of M. Bianchi, G. Kassay and R. Pini [16], we establish two versions of Ekeland's variational principle in complete metric spaces (one **for/to** (EP), the other **for/to** (SEP)) without assuming any kind of convexity of the bifunction involved in the formulation of the principle or of the set whereon the bifunction, resp. the function systems are defined. As applications of these principles, we derive existence results for a solution of (EP) and (SEP), respectively, with compactness assumption on the underlying sets.

As further applications of the main theorems, we aim to prove localization-type results for (EP) and (SEP). The importance of the main theorems in finding equilibrium points of differential inequations and systems of differential inequations is also motivated.

The results of this chapter are contained in Cs. Farkas, **A. É. Molnár**, [45].

3.1 On the existence and localization of equilibrium points for bifunctions

Let (X, d) be a complete metric space, C a closed subset of X, and $f : C \times C \to \mathbb{R}$ be a function. Let $h : [0, \infty[\to [0, \infty[$ be a non-increasingfunction. Throughout this section we shall use the same notations as in Chapter 2.

We begin this paragraph with the definition of a new equilibrium point, namely the (x_0, h)-equilibrium point. We notice that this notation stems from the fact that the underlined equilibrium point depends on an element $x_0 \in C$ and on a function h defined as above.

Definition 3.1.1 *(Cs. Farkas, A. É. Molnár, [45]) Let $f : C \times C \to \mathbb{R}$ be a function and $x_0 \in C$ an element. $\tilde{x} \in C$ is said to be an (x_0, h)-equilibrium point of f, if and only if*

$$f(\tilde{x}, y) + h(d(x_0, y))d(\tilde{x}, y) \geq 0 \ , \ \forall y \in C. \tag{3.1.1}$$

The main result of this section is given below.

Theorem 3.1.1 *(Cs. Farkas, A. É. Molnár, [45]) Let (X, d) be a complete metric space, $C \subset X$ be a closed set, and $f : C \times C \to \mathbb{R}$ be a mapping. Let us also consider the continuous non-increasing function $h : [0, \infty[\to]0, \infty[$ and a point $x_0 \in X$. Suppose that f satisfies the assumption (**F2**). Then, there exists a $\tilde{x} \in C$ such that*

(a) $f(x_0, \tilde{x}) + h(d(x_0, \tilde{x}))d(x_0, \tilde{x}) \leq 0$,

(b) $f(\tilde{x}, x) + h(d(x_0, x))d(x, \tilde{x}) > 0$, $\forall x \in C$, $x \neq \tilde{x}$.

Now we show that there exists a function which satisfies the condition (**F2**), but it does not have any equilibrium point in the classical sense.

Example 3.1.1 *(Cs. Farkas, A. É. Molnár, [45]) Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $f(x, y) = \frac{x}{1+x} - \frac{y}{y+1}$. It is easy to see that this function satisfies all the assumptions of the Theorem 3.1.1. If there exists an $x_0 \in \mathbb{R}_+$ such that $f(x_0, y) \geq 0, \forall y \in \mathbb{R}_+$, then we get $x_0 \geq y, \forall y \in \mathbb{R}_+$, which is a contradiction. So the solution set of the problem (EP) is empty, but by Theorem 3.1.1, we can guarantee that there exists an (x_0, h)-equilibrium point of f.*

Consequently, we could assert that we obtained an "almost" equilibrium point by perturbing the initial bifunction by a small perturbation.

Next we give some examples where the defined functions have equilibrium points.

Example 3.1.2 *(Cs. Farkas, A. É. Molnár, [45]) Let* $X = \mathbb{R}$. *Let* C *be a closed and bounded subset of* $\mathbb{R}$ *and* $h : [0, \infty[\to]0, \infty[$ *be a function, defined by* $h(x) = 1 + e^{-x}$, *and let* $x_0 = 0$. *We consider* $F : C \times C \to \mathbb{R}$, $F(x, y) = y - x + (1 + e^{-y})|x - y|$ *and* $f : C \times C \to \mathbb{R}$, $f(x, y) = y - x$. *By Theorem 3.1.1, it follows that* F *has an equilibrium point.*

Example 3.1.3 *(Cs. Farkas, A. É. Molnár, [45]) Let* $X = \mathbb{R} \times \mathbb{R}$ *and let* C *be a closed and bounded subset of* $\mathbb{R}$. *Let us consider the function* $h : [0, \infty[\to]0, \infty[$, $h(x) = \frac{1}{1+x}$, *and let* $x_0 = 0_{\mathbb{R}^2}$. *We consider* $F : C \times C \to \mathbb{R}$, $F(x, y) = y^N - x^N + \frac{1}{1+\|y\|}\|x - y\|$ *and* $f : C \times C \to \mathbb{R}$, $f(x, y) = y^N - x^N$, *where* $N \geq 1$ *is a natural number. It is easy to see that, by Theorem 3.1.1,* F *has an equilibrium point.*

Remark 3.1.1 *(Cs. Farkas, A. É. Molnár, [45]) One might wonder whether a bifunction* f, *satisfying all the assumptions of Theorem 3.1.1, should be of the form* $g(y) - g(x)$. *In this case, Theorem 3.1.1 would reduce to an improved version of the classical Ekeland's principle.*

It is not the case, as the example below shows: Let $f : \mathbb{R}^2 \to \mathbb{R}$ *be a function, defined by*

$$f(x, y) = \begin{cases} e^{-|x-y|} + 1 + \sin(y) - \sin(x), & x \neq y, \\ 0, & x = y, \end{cases}$$

and let $h : [0, \infty[\to]0, \infty[$ *be defined by* $h(x) = \frac{1}{1+x}$. *By Theorem 3.1.1, it follows that*

$$F(x, y) = f(x, y) + \frac{1}{1 + \|y\|}\|x - y\|$$

has an (x_0, h)*-equilibrium point, but clearly* f *cannot be represented in the above mentioned form. We notice here, that in this case* $C = \mathbb{R}$, *which is a closed set.*

Remark 3.1.2 *(Cs. Farkas, A. É. Molnár, [45]) Let* $X = \mathbb{R}^n$ *with the Euclidean norm, i.e.* $d(x, y) = ||x - y||$. *In the framework of Theorem 3.1.1, we get*

(i) $f(x_0, \tilde{x}) + h(\|x_0 - \tilde{x}\|)||x_0 - \tilde{x}|| \leq 0$,

(ii) $f(\tilde{x}, x) + h(\|x_0 - x\|)||x - \tilde{x}|| > 0$, $\forall x \in C$, $x \neq \tilde{x}$.

The importance of Theorem 3.1.1 marks out from the fact that with the aid of it we can derive the following existence result, ensuring the nonemptiness of the solution set of (EP) on compact sets, without any convexity requirement. To this purpose, we need to impose the following additional assumption:

($\widetilde{\mathbf{F2}}$) The mapping $f : C \times C \to \mathbb{R}$ satisfies the following condition

(*iv*) $f(\cdot, y)$ is upper semicontinuous, for every $y \in C$.

Theorem 3.1.2 *(Cs. Farkas, A. É. Molnár, [45]) Let C be a compact (not necessarily convex) subset of an Euclidean space, and $f : C \times C \to \mathbb{R}$ be a function. If the conditions* ($\mathbf{F2}$) *and* ($\widetilde{\mathbf{F2}}$) *are satisfied, then the set of solutions of* (EP) *is nonempty, i.e. there exists $\tilde{x} \in C$ such that $f(\tilde{x}, y) \geq 0, \forall y \in C$.*

Remark 3.1.3 *(Cs. Farkas, A. É. Molnár, [45]) We notice that if we take $h \equiv \varepsilon$ in Theorem 3.1.1, we get back [16, Theorem 2.1], established by M. Bianchi et al. This underlines the fact that our main result is a generalization of [16, Theorem 2.1].*

The next remark emphasize even more the importance of Theorem 3.1.1, showing that this result is a very useful technique in finding equilibrium points of differential inequations (in the case of normed spaces).

Remark 3.1.4 *(Cs. Farkas, A. É. Molnár, [45]) Let X be a normed space. Let $h : [0, \infty[\to]0, \infty[$ be a continuous non-increasing function and let $f : X \times X \to \mathbb{R}$ be a function satisfying the assumptions from Theorem 3.1.1. If in addition we assume that f is Gâteaux differentiable in the second variable, then we have by Theorem 3.1.1*

$$f(\tilde{x}, x) + h(d(x_0, x))d(\tilde{x}, x) > 0.$$

Let $\phi \in X$, $t > 0$, and $x = \tilde{x} + t\phi$, then the above inequality can be rewritten as

$$f(\tilde{x}, \tilde{x} + t\phi) + h(||\tilde{x} + t\phi||)t||\phi|| > 0,$$

therefore

$$\frac{f(\tilde{x}, \tilde{x} + t\phi)}{t} + h(||\tilde{x} + t\phi||)||\phi|| > 0.$$

By $(\mathbf{F2})-(\mathbf{ii})$*, we have* $f(\tilde{x},\tilde{x})=0$*, therefore*

$$\frac{f(\tilde{x},\tilde{x}+t\phi)-f(\tilde{x},\tilde{x})}{t}+h(||\tilde{x}+t\phi||)||\phi||>0.$$

If $t\to 0$*, we get*

$$\partial_2 f(\tilde{x},\tilde{x})(\phi)+h(||\tilde{x}||)||\phi||\geq 0 \text{ for all } \phi\in X. \tag{3.1.2}$$

This means that there exists an equilibrium point, in the sense of Definition 3.1.1, for differential inequations having the above form.

Finally, as a consequence of the Theorem 3.1.1 we present a Zhong-type variational principle for bifunctions (for the original ZVP, see [94],[95]). This result may be important from algorithmic point of view, because it localizes the position of the corresponding equilibrium point (i.e. the location of this equilibrium point is a sphere).

Theorem 3.1.3 *(Cs. Farkas, A. É. Molnár, [45]) Let* (X,d) *be a complete metric space,* $C\subset X$ *be a closed set, and* $f:C\times C\to\mathbb{R}$ *be a mapping, satisfying* $(\mathbf{F2})$*. Let* $g:[0,\infty[\to[0,\infty[$ *be a continuous non-decreasing function such that*

$$\int_0^\infty \frac{1}{1+g(r)}dr=\infty. \tag{3.1.3}$$

Let $x_0\in C$ *be fixed. Then, for every* $\varepsilon>0$ *and* $y\in C$ *for which we have*

$$\inf_{z\in C} f(y,z)>-\varepsilon, \tag{3.1.4}$$

and for every $\lambda>0$*, there exists* $x_\varepsilon\in C$ *such that*

(a) $d(x_0,x_\varepsilon)<r_0+\overline{r}$

(b) $f(x_0,x_\varepsilon)+\frac{\varepsilon}{\lambda(1+g(d(x_0,x_\varepsilon)))}d(x_0,x_\varepsilon)\leq 0,$

(c) $f(x_\varepsilon,x)+\frac{\varepsilon}{\lambda(1+g(d(x_0,x)))}d(x,x_\varepsilon)>0,\ \forall x\in C,\ x\neq x_\varepsilon,$

where $r_0=d(x_0,y)$ *and* $\overline{r}$ *are chosen such that*

$$\int_{r_0}^{r_0+\overline{r}} \frac{1}{1+g(r)}dr\geq\lambda.$$

3.2 On the existence and localization of system of equilibrium points for systems of functions

In this section we aim to extend the results investigated in the previous section for a system of equilibrium problems. Throughout this paragraph will be kept the notations introduced in Chapter 2 and in the previous section, respectively.

Let m be a positive integer and $I = \{1, 2, ..., m\}$. Let $C = \prod_{j\in I} C_j$ and $C_j \subset X_j$ be a closed subset of the complete metric space (X_j, d_j). Consider a system of open balls in X_j: (B_j), $j \in I$ and denote the j^{th} open ball with center x_j^0 and radius r_j, (for each $j \in I$) by $B_j(x_j^0, r_i) = \{z_j \in X | d(x_j^0, z_j) < r_j\}$. Furthermore, consider the functions $f_j : C \times C_j \to \mathbb{R}$, $j \in I$. An element of the set $C^j = \prod_{j\neq i} C_i$ will be represented by x^j; therefore, $x \in C$ can be written as $x = (x^j, x_j) \in C^j \times C_j$. If $x \in \prod X_i$, the symbol $d^*(x_0, x)$ will denote the Tchebychev distance of x from x_0, i.e.,

$$d^*(x_0, x) = \max_j d_j(x_0^j, x_j),$$

and we shall consider the metric space $\prod X_i$ endowed with this metric.

Let $h^i : [0, \infty[\to]0, \infty[$ be a continuous non-increasing function, for each $i \in I$, and let $h : \mathbb{R}_+^m \to \mathbb{R}_+^m$ be defined by $h = (h^1, ..., h^m)$. Let us consider the following definition of an (x_0, h)-equilibrium point for a system of equilibrium problems:

Definition 3.2.1 *(Cs. Farkas, A. É. Molnár, [45]) Let C_i, $i \in I$, be a subset of a certain metric space and put $C = \prod_{i\in I} C_i$. Given $f_i : C \times C_i \to \mathbb{R}, i \in I$, then $\tilde{x} \in C$ is said to be an (x_0, h)-equilibrium point of $\{f_1, ..., f_m\}$, if and only if*

$$f_i(\tilde{x}, y_i) + h^i(d_i(x_i^0, y_i))d(\tilde{x}^i, y_i)_i \geq 0, \forall y_i \in C_i, \text{ and } i \in I.$$

We assume that the following conditions hold:

($\mathbf{F2_i}$) *Consider the functions $f_i : C \times C_i \to \mathbb{R}$, $i \in I$ such that*

- *(i) $f_i(x, \cdot) : C_i \to \mathbb{R}$ is lower bounded and lower semicontinuous, for every $i \in I$;*
- *(ii) $f_i(x, x_i) = 0$, for every $i \in I$, and $x = (x_1, ..., x_m) \in C$;*
- *(iii) $f_i(z, x_i) \leq f_i(z, y_i) + f_i(y, x_i)$, for every $x, y, z \in C$ where $y = (y^i, y_i)$, and for every $i \in I$.*

The main result of this section is:

Theorem 3.2.1 *(Cs. Farkas, A. É. Molnár, [45]) Let* $h_i : [0, \infty[\to]0, \infty[$ *be a non-increasing function, for each* $i \in I$*, and let* $h : \mathbb{R}_+^m \to \mathbb{R}_+^m$ *be a function, defined by* $h = (h^1, ..., h^m)$*. Consider the functions* $f_i : C \times C_i \to \mathbb{R}$*,* $i \in I$*, where* $C = \prod_{i \in I} C_i$ *and* $C_i \subset X_i$ *is a closed subset of the complete metric space* (X_i, d_i)*. Suppose that the functions* f_i *satisfy the condition* $(\mathbf{F2_i})$*, for all* $i \in I$*. Let* $x^0 = (x_1^0, ..., x_m^0)$ *be a fixed element in* C*. Then, there exist* $\tilde{x} \in C$ *such that:*

1. $f_i(x^0, \tilde{x}^i) + h^i(\tilde{x}^i) d_i(x_i^0, \tilde{x}^i) \leq 0,$

2. $f_i(\tilde{x}, x_i) + h^i(x_i) d_i(\tilde{x}^i, x_i) > 0,$ *whenever* $x_i \neq \tilde{x}_i$*,*

for every $i \in I$*.*

As application of Theorem 3.2.1, we derive the following existence result, guaranteeing the nonemptiness of the solution set of (SEP) on compact sets, without any convexity requirement. For this, we assume the assumption above:

$(\widetilde{\mathbf{F2_i}})$ The function $f_i : C \times C_i \to \mathbb{R}$, $i \in I$ satisfies the following assumption

(iv) $f_i(\cdot, y_i)$ is upper semicontinuous, for every $y_i \in C_i$.

Theorem 3.2.2 *(Cs. Farkas, A. É. Molnár, [45]) If in addition to the assumptions of Theorem 3.2.1, for every* $i \in I$*,* C_i *is compact and* $f_i : C \times C_i \to \mathbb{R}$ *satisfies the assumption* $(\widetilde{\mathbf{F2_i}})$*, then the following system of equilibrium problems*

$$f_i(\tilde{x}, y_i) \geq 0 \ \forall i \in I, \ \forall y_i \in C_i,$$

has a solution $\tilde{x} = (\tilde{x}^1, ..., \tilde{x}^m) \in C$*.*

Taking into consideration Remark 3.1.4, we can derive a similar, but more general conclusion regarding differential inequation systems.

Remark 3.2.1 *(Cs. Farkas, A. É. Molnár, [45]) Let* $i \in I$*. Let* X_i *be a normed space and* $h^i : [0, \infty[\to]0, \infty[$ *be a continuous non-increasing function and let* $x_i^0 = 0_{X_i}$*. We also consider that* $f_i : X \times X \to \mathbb{R}$ *is a function which satisfies the assumptions* *from Theorem 3.2.1. In addition, we assume that f_i are Gâteaux differentiable in the second variable (for every $i \in I$). Then, we have by Theorem 3.2.1*

$$f_i(\tilde{x}, x_i) + h(d_i(x_i^0, x_i))d(\tilde{x}_i, x_i) > 0.$$

Let $\phi_i \in X_i$ be an arbitrary element, $t > 0$, and let $x_i = \tilde{x}_i + t\phi_i$. The above inequality can be rewritten as follows

$$f_i(\tilde{x}, \tilde{x}_i + t\phi_i) + h^i(||\tilde{x}_i + t\phi_i||_i)t||\phi_i||_i > 0.$$

Therefore

$$\frac{f_i(\tilde{x}, \tilde{x}_i + t\phi_i)}{t} + h^i(||\tilde{x}_i + t\phi_i||_i)||\phi_i||_i > 0.$$

By $(\widetilde{\mathbf{F2_i}}) - (\mathbf{ii})$, we have $f_i(\tilde{x}, \tilde{x}_i) = 0$, hence

$$\frac{f_i(\tilde{x}, \tilde{x}_i + t\phi_i) - f_i(\tilde{x}, \tilde{x}_i)}{t} + h^i(||\tilde{x}_i + t\phi_i||_i)||\phi_i||_i > 0.$$

If $t \to 0$, we get

$$\partial_2 f_i(\tilde{x}, \tilde{x})(\phi_i) + h^i(||\tilde{x}_i||_i)||\phi_i||_i > 0 \text{ for all } \phi_i \in X_i. \text{ for all } i \in I. \tag{3.2.1}$$

This means that there exists an equilibrium point, in the sense of Definition 3.2.1, for a differential inequation system having the above form.

We conclude this paragraph by stating a localization result for the point $\tilde{x}$ constructed in Theorem 3.2.1.

Theorem 3.2.3 *(Cs. Farkas, A. É. Molnár, [45]) Let $g_i : [0, \infty[\to]0, \infty[$ be a non-decreasing function, for all $i \in I$, let $g : \mathbb{R}_+^m \to \mathbb{R}_+^m$ be defined by $g = (g_1, ..., g_m)$ and*

$$\int_0^\infty \frac{1}{1 + g^i(r)} dr = \infty \text{ for all } i \in I. \tag{3.2.2}$$

Consider the functions $f_i : C \times C_i \to \mathbb{R}$, $i \in I$ where $C = \prod_{i \in I} C_i$ and $C_i \subset X_i$ is a closed subset of the complete metric space (X_i, d_i). Assume that f_i satisfies the condition $(\mathbf{F2_i})$ for all $i \in I$. Let $x^0 = (x_1^0, ..., x_m^0) \in C$ be an arbitrary fixed element. Then, for every $\varepsilon_i > 0$, $i = \overline{1..m}$ and $y \in C$ for which we have

$$\inf_{z_i \in C_i} f_i(y, z_i) > -\varepsilon_i, \tag{3.2.3}$$

and for every $\lambda_i > 0$ there exists $\tilde{x} \in C$ such that

1. $d_i(x_i^0, \tilde{x}_i) \leq r_0^i + \overline{r}_i$

2. $f_i(x^0, \tilde{x}^i) + \frac{\varepsilon_i}{\lambda_i(1+g^i(d_i(x_i^0, \tilde{x}_i)))} d_i(x_i^0, \tilde{x}_i) \leq 0,$

3. $f_i(\tilde{x}, x_i) + \frac{\varepsilon_i}{\lambda_i(1+g^i(d_i(x_i^0, x_i)))} d_i(\tilde{x}_i, x_i) > 0,\ \forall i \in I$

where $r_i^0 = d_i(x_i^0, y_i)$ *and* $\overline{r}_i$ *are choosen such that*

$$\int_{r_i^0}^{r_i^0 + \overline{r}_i} \frac{1}{1 + g^i(r)} dr \geq \lambda_i.$$

Chapter 4

Fixed point techniques used in the study of different classes of inequalities and systems of inequalities of hemivariational type

The theory of hemivariational inequalities has produced important results both in pure and applied mathematics and it is very useful to understand several problems of mechanics and engineering for nonconvex, nonsmooth energy functionals.

There are three different approaches to the study of hemivariational inequalities and of systems of hemivariational inequalities: via monotone operators (see e.g. Z. Naniewicz, P.D. Panagiotopoulos [79]), with the aid of critical point theory (see D. Motreanu and P.D. Panagiotopoulos [77], D. Motreanu and V. Rădulescu [78]) and using fixed point technics (see P. D. Panagiotopoulos, M. Fundo and V. Rădulescu [83], A. Kristály and Cs. Varga [66], V. Rădulescu and D. Repovš [88]). In Chapter 4 we shall follow the latter approach: we study some inequalities and systems of inequalities of hemivariational types via fixed point techniques without imposing any monotonicity assumptions. In the first two sections we describe the abstract framework in which we work, we formulate our problems and the main results, respectively. The last section presents the wide range of applications of the results obtained in the previous sections.

The results of this chapter are published in the papers of H. Lisei, **A. É. Molnár**, Cs. Varga [69], and of **A. É. Molnár**, O. Vas, [75], the latter being a result of our cooperation with the Institute of Mathematics from the University of Debrecen (Hungary).

4.1 On a variational-hemivariational inequality problem with lack of compactness

The purpose of this paragraph is to discuss the solvability of a variational-hemivariational inequality problem on a closed and convex set (either bounded or unbounded) without using critical point theory.

Let $(X, \|\cdot\|)$ be a Banach space and X^* its topological dual. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X and X^*. Let $\Omega \subseteq \mathbb{R}^n$ be an unbounded domain, let p be such that $1 < p < n$ and set $p^* = \dfrac{np}{n-p}$. We investigate the following problem:
(V-HI) Find $u \in K$ such that, for every $v \in K$, it holds

$$\langle Au, v-u\rangle + \int_\Omega f(x,u(x))(v(x)-u(x))dx + \int_\Omega j^0(x,u(x);v(x)-u(x))dx \geq 0,$$

where $K \subseteq X$ is a set and A, f, j are given functionals, which satisfy certain conditions.

In order to establish the existence of at least one solution for the aforementioned variational-hemivariational problem **(V-HI)**, we impose the following hypotheses:

(CT) Assume that for $s \in [p, p^*]$ the embedding $X \hookrightarrow L^s(\Omega)$ is *continuous*, i.e. there exists a constant $C \geq 0$ such that $\|x\|_{L^s(\Omega)} \leq C\|x\|, \forall x \in X$.

(CP) Suppose that for $s \in (p, p^*)$ the embedding $X \hookrightarrow L^s(\Omega)$ is *compact*, this means that there exists a linear and compact operator $T : X \to L^s(\Omega)$, i.e. for each bounded sequence $\{x_n\}$ in X there exists a subsequence $\{Tx_n\}$ which is convergent in $L^s(\Omega)$. For simplicity we will write x_n instead of Tx_n.

(A1) Let $A : X \to X^*$ be an operator with the following property: for any sequence $\{u_n\}_n$ in X which converges weakly to $u \in X$ it holds

$$\langle Au, u-w\rangle \leq \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle, \text{ for all } w \in X.$$

(A2) There exists $\lambda := \inf_{u \in X \setminus \{0\}} \frac{\langle Au, u \rangle}{\|u\|^p} > 0$.

(f1) Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, such that for some $\alpha > 0$ it holds

$$|f(x,y)| \leq \alpha |y|^{p-1} + \beta(x),$$

for a.e. $x \in \Omega$ and all $y \in \mathbb{R}$, where $\beta \in L^{\frac{p}{p-1}}(\Omega)$.

(f2) We assume that the constants from **(f1)** and **(A1)** satisfy $\alpha C_p^p < \lambda$.

(j1) Assume that $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is locally Lipschitz with respect to the second variable, and there exists $c > 0, r \in [p, p^*)$ such that

$$|\xi| \leq c(|y|^{p-1} + |y|^{r-1})$$

for a.e. $x \in \Omega$, all $y \in \mathbb{R}$ and for all $\xi \in \partial j(x,y)$, where $\partial j(x,y)$ denotes the generalized gradient of $j(x, \cdot)$ at $y \in \mathbb{R}$;

(j2) There exists $k \in L^{\frac{p}{p-1}}(\Omega)$ such that

$$|j^0(x, y; -y)| \leq k(x)|y| \text{ for all } x \in \Omega, y \in \mathbb{R},$$

where $j^0(x, u; z)$ is the generalized directional derivative of $j(x, \cdot)$ at the point $u \in \mathbb{R}$ in the direction $z \in \mathbb{R}$.

Now we present two examples for operators which satisfy the condition **(A1)**.

Example 4.1.1 *(H. Lisei, A. É. Molnár, Cs. Varga, [69]) Let $A' : X \to X^*$ be a linear and continuous operator, which is positive, i.e. $\langle A'u, u \rangle \geq 0$ for all $u \in X$. These assumptions imply that A' is weakly sequentially continuous and that* ***(A1)*** *is satisfied.*

Example 4.1.2 *(H. Lisei, A. É. Molnár, Cs. Varga, [69]) We assume that the bilinear form $a : X \times X \to \mathbb{R}$ is compact, which means that for any sequences $\{u_n\}_n$ and $\{v_n\}_n$ from X such that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ $(u, v \in X)$ it follows that $a(u_n, v_n) \to a(u, v)$. Under this condition, the operator $A'' : X \to X^*$ defined by $\langle A''u, v \rangle = a(u, v)$ for all $u, v \in X$ (for this representation see the Lax-Milgram Theorem [25, Corollary 5.8.]) satisfies assumption* ***(A1)***.

In the sequel we state two existence results for the solutions of problem **(V-HI)**, whose proofs are based on the Fan-Knaster-Kuratowski-Mazurkiewicz theorem (see [50, Theorem 8.2.]). We note that instead of this theorem we can apply an elementary principle of KKM-mappings given by A. Granas and M. Lassonde in [51, Theorem 5.2] in the framework of super-reflexive Banach spaces. Beside the Ky Fan version of the well-known Knaster-Kuratowski-Mazurkiewicz theorem, the following lemma will be needed in proving our main results.

Lemma 4.1.1 *(H. Lisei, A. É. Molnár, Cs. Varga, [69]) Suppose that X is a Banach space.*

(1) *Assume that **(j1)** is satisfied and X_1 and X_2 are nonempty subsets of X.*

  (1a) *If the embedding $X \hookrightarrow L^s(\Omega)$ is continuous for each $s \in [p, p^*]$, then the mapping*
$$(u, v) \in X_1 \times X_2 \mapsto \int_\Omega j^0(x, u(x); v(x))dx \in \mathbb{R}$$
is upper semicontinuous.

  (1b) *Moreover, if $X \hookrightarrow L^s(\Omega)$ is compact for every $s \in [p, p^*)$, then the above mapping is weakly upper semicontinuous.*

(2) *Assume that **(f1)** holds and that $X \hookrightarrow L^p(\Omega)$ is compact. Then, for each $v \in X$ the mapping*
$$u \in X \mapsto \int_\Omega f(x, u(x))(v(x) - u(x))dx \in \mathbb{R}$$
is weakly sequentially continuous.

One of our existence theorems is stated below.

Theorem 4.1.1 *(H. Lisei, A. É. Molnár, Cs. Varga, [69]) Suppose that X is a reflexive Banach space and that $K \subseteq X$ is a nonempty, closed, convex and bounded set and that the hypotheses **(CT)**, **(CP)**, **(A1)**, **(f1)**, **(j1)** are fulfilled. Then, problem **(V-HI)** admits at least one solution.*

In the following we will discuss the case when the set K is unbounded. Let us assume, without loss of generality, that $0 \in K$ and for any positive integer n let us set $K_n := \{w \in K : \|w_n\| \leq n\}$. Thus, $0 \in K_n$ for all $n \in \mathbb{N}$.

Let us fix an $n \in \mathbb{N}$. Since K is a nonempty, closed, convex and unbounded subset of X, by applying Theorem 4.1.1 there exists $u_n \in K_n$ such that for all $v \in K_n$ it holds

$$\begin{aligned} \langle Au_n, v - u_n \rangle \ &+ \ \int_\Omega f(x, u_n(x))(v(x) - u_n(x))dx \\ &+ \ \int_\Omega j^0(x, u_n(x); v(x) - u_n(x))dx \geq 0. \end{aligned} \tag{4.1.1}$$

Theorem 4.1.2 *(H. Lisei, A. É. Molnár, Cs. Varga, [69]) Suppose that X is a reflexive Banach space and $K \subseteq X$ is a nonempty, closed, and convex set and that the hypotheses **(CT)**, **(CP)**, **(A1)**, **(A2)**, **(f1)**, **(f2)**, **(j1)**, **(j2)** are fulfilled. Let us also consider a sequence $u_n \in K_n$ such that the inequality* (4.1.1) *is fulfilled for each $n \geq 1$. Then, problem **(V-HI)** has at least one solution.*

Remark 4.1.1 *(H. Lisei, A. É. Molnár, Cs. Varga, [69])* If $0 \in K$ and

$$\langle A0, v \rangle + \int_\Omega f(x, 0)v(x)dx + \int_\Omega j^0(x, 0; v(x))dx \geq 0 \text{ for every } v \in K, \tag{4.1.2}$$

then obviously zero is a solution of the inequality (4.1.2). If Theorem 4.1.1 and Theorem 4.1.2 are applied to (4.1.2), the existence of a nontrivial solution may not be assured without specific additional assumptions.

4.2 Existence of solutions for a nonlinear system of hemivariational-like inequalities

The aim of this section is to study inequality problems in a general and unified framework (as nonlinear hemivariational-like inequalities can be reduced to variational-like inequalities of standard hemivariational inequalities) and to prove the existence of at least one solution for the studied system on a closed and convex set (either bounded or unbounded) without imposing any monotonicity assumptions or using nonsmooth critical point theory.

Let $X_1, \ldots, X_n$ be reflexive Banach spaces, $Y_1, \ldots, Y_n$ be Banach spaces and $D_i \subseteq X_i$ for $i \in \{1, \ldots, n\}$ be bounded, closed and convex sets, where $n \in \mathbb{Z}_+$. In what follows, X_i^* and Y_i^* will denote the topological dual spaces of X_i and Y_i, respectively, for every $i \in \{1, \ldots, n\}$. We suppose that for $i \in \{1, \ldots, n\}$ there exist linear compact operators $A_i : X_i \to Y_i$, the nonlinear functionals $\phi_i : X_1 \times \ldots \times X_i \times \ldots \times X_n \times X_i \to \mathbb{R}$ and the single-valued functions $\eta_i : X_i \times X_i \to X_i$. We also assume that $J : Y_1 \times \ldots \times Y_n \to \mathbb{R}$ is a regular locally Lipschitz functional. Throughout this paragraph, we will use the following notations:

- $X = X_1 \times \ldots \times X_n$, $Y = Y_1 \times \ldots \times Y_n$ and $D = D_1 \times \ldots \times D_n$;
- $\overline{u}_i = A_i(u_i)$, $\overline{\eta}_i(u_i, v_i) = A_i(\eta_i(u_i, v_i))$, for each $i = \overline{1, n}$;
- $u = (u_1, \ldots, u_n)$ and $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$;
- $\eta(u, v) = (\eta_1(u_1, v_1), \ldots, \eta_n(u_n, v_n))$ and $\overline{\eta}(u, v) = (\overline{\eta}_1(u_1, v_1), \ldots, \overline{\eta}_n(u_n, v_n))$;
- $\Phi : X \times X \to \mathbb{R}, \quad \Phi(u, v) = \sum_{i=1}^{n} \phi_i(u_1, \ldots, u_i, \ldots, u_n, \eta_i(u_i, v_i))$.

We impose the following assumptions:

(H) For each $i \in \{1, \ldots, n\}$ the mapping $\eta_i(\cdot, \cdot) : X_i \times X_i \to X_i$ satisfies the following conditions:

- **(i)** $\eta_i(u_i, u_i) = 0$, for all $u_i \in X_i$;
- **(ii)** $\eta_i(u_i, \cdot)$ is linear operator for each $u_i \in X_i$;
- **(iii)** for each $v_i \in X_i, \eta_i(u_i^m, v_i) \rightharpoonup \eta(u_i, v_i)$ whenever $u_i^m \rightharpoonup u_i$.

($\boldsymbol{\Phi}$) For every $i \in \{1, \ldots, n\}$, the functional $\phi_i : X_1 \times \ldots \times X_i \times \ldots \times X_n \times X_i \to \mathbb{R}$ satisfies

- **(i)** $\phi_i(u_1, \ldots, u_i, \ldots, u_n, 0) = 0$ for all $u_i \in X_i$;
- **(ii)** for all $v_i \in X_i$ the mapping $(u_1, \ldots, u_n) \rightsquigarrow \phi_i(u_1, \ldots, u_n; \eta_i(u_i, v_i))$ is weakly upper semicontinuous;
- **(iii)** the mapping $v_i \rightsquigarrow \displaystyle\sum_{i=1}^{n} \phi_i(u_1, \ldots, u_n; \eta_i(u_i, v_i))$ is convex for each $(u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n$.

Remark 4.2.1 *(A. É. Molnár, O. Vas, [75]) Because $J^0_{,i}(u_1, \ldots, u_n; v_i)$ is convex and $\eta_i(u_i, \cdot)$ is linear for each $i \in \{1, \ldots, n\}$ and for each $(u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n$, it follows that the mapping $v_i \rightsquigarrow J^0_{,i}(u_1, \ldots, u_n; \eta_i(u_i, v_i))$ is convex.*

Using the following two examples, we show that there exists a function which satisfies the conditions **(i)-(iii)** from **(H)**.

Example 4.2.1 *(A. É. Molnár, O. Vas, [75]) For $i \in \{1, \ldots, n\}$ let us choose the function $\eta_i : X_i \times X_i \to X_i$ as follows:*

$$\eta_i(u_i, v_i) = v_i - u_i, \text{ for all } u_i, v_i \in X_i.$$

*In this case, $\eta_i(u_i, v_i)$ satisfies the assumptions **(i)-(iii)** from **(H)**. Notice that, with this choice we get back the problem formulated in D. Repovš and Cs. Varga's paper [89].*

Example 4.2.2 *(A. É. Molnár, O. Vas, [75]) Let $B_i : X_i \to X_i$ be a linear compact operator, $\alpha_i > 0, \beta_i \in X_i$, $i \in \{1, \ldots, n\}$. Define a function $f : X_i \to X_i$ by $f_i(x) = \alpha_i B_i(x) + \beta_i$. If we take the function $\eta_i : X_i \times X_i \to X_i$ as follows:*

$$\eta_i(u_i, v_i) = f_i(v_i) - f_i(u_i), \text{ for all } u_i, v_i \in X_i, i \in \{1, \ldots, n\},$$

*then it is clear that the conditions **(H) (i)-(iii)** hold for $\eta_i(u_i, v_i)$.*

In this section, our intention is to investigate the existence of at least one solution for the following nonlinear hemivariational-like system of inequalities:

(NHLIS) Find $(u_1, \ldots, u_n) \in D_1 \times \ldots \times D_n$ such that

$$\begin{cases} \phi_1(u_1, \ldots, u_n, \eta_1(u_1, v_1)) + J^0_{,1}(\overline{u}_1, \ldots, \overline{u}_n; \overline{\eta}_1(u_1, v_1)) \geq 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \phi_n(u_1, \ldots, u_n, \eta_n(u_n, v_n)) + J^0_{,n}(\overline{u}_1, \ldots, \overline{u}_n; \overline{\eta}_n(u_n, v_n)) \geq 0 \end{cases}$$

for all $(v_1, \ldots, v_n) \in D_1 \times \ldots \times D_n$.

Recently, in [89], D. Repovš and Cs. Varga established an existence result for a general class of hemivariational system of inequalities. The next theorem extends this result and provides sufficient conditions for the existence of the solutions of problem **(NHLIS)**. We point out the fact that, in [89], the authors provided two proofs: one using Ky Fan's version of the Knaster-Kuratowsky-Mazurkiewicz theorem - the theorem that we also applied in the previous section, and one using Tarafdar's fixed point theorem for set-valued maps (see E. Tarafdar [91]). Our approach is slightly different, as we use Lin's fixed point theorem (see [68, Theorem 1.]).

Now we present the main result of this section. We deal with the case when the sets D_i are nonempty, bounded, closed and convex.

Theorem 4.2.1 *(A. É. Molnár, O. Vas [75]) Let us consider the nonempty, bounded, closed and convex sets $D_i \subset X_i$ for each $i \in \{1,\ldots,n\}$. If the conditions **(H)** and **(Φ)** are fulfilled, then the system of nonlinear hemivariational-like inequalities* **(NHLIS)** *admits at least one solution.*

In order to prove this result we need to formulate the following hemivariational inequality:
(VHI) Find $u \in D$ such that for all $v \in D$ we have

$$\Phi(u,v) + J^0(\overline{u};\overline{\eta}(u,v)) \geq 0.$$

The next proposition proves that problem **(VHI)** is closely linked to problem **(NHLIS)**:

Proposition 4.2.1 *(A. É. Molnár, O. Vas, [75]) If **(H)**-(i) and **(Φ)**-(i) hold and $u^0 = (u_1^0,\ldots,u_n^0) \in D_1 \times \ldots \times D_n$ is a solution of the inequality* **(VHI)***, then u^0 is also a solution of the system* **(NHLIS)**.

We observe that if we take into consideration Proposition 4.2.1, it is enough to prove that problem **(VHI)** has at least one solution.

Remark 4.2.2 *(A. É. Molnár, O. Vas, [75]) It is known that the solutions of systems of inequalities of hemivariational type on unbounded domains exist, if we extend the assumptions for the bounded domains with a coercivity condition. So, if we impose coercivity conditions, Theorem 4.2.1 will also hold when the sets D_i are unbounded (for details, see [89, Remark 3.3]).*

4.3 Applications

Concerning the applicability of our abstract results, presented in Sections 4.1 and 4.2, we provide some possible applications.

4.3.1 Nash generalized derivative points

Let us consider the Banach spaces $E_1,\ldots,E_n$ for each $i \in \{1,\ldots,n\}$ and the nonempty set $D_i \subset E_i$. Let $D_i' \subset E_i$ be an open set such that for each $i \in \{1,\ldots,n\}$

we have that $D_i \subset D_i'$. Let $f_i : D_1 \times \ldots \times D_i' \times \ldots \times D_n \to \mathbb{R}$ be a functional such that for all $i \in \{1, \ldots, n\}$ the mapping $u_i \rightsquigarrow f_i(u_1, \ldots, u_i, \ldots, u_n)$ is continuous and locally Lipschitz.

In the following, we show the applicability of Theorem 4.2.1. To achieve this purpose, let us take in Theorem 4.2.1

$$\phi_i(u_1, \ldots, u_i, \ldots, u_n, \eta_i(u_i, v_i)) = f_{i,i}^0(u_1, \ldots, u_i, \ldots, u_n; \eta_i(u_i, v_i)), \text{ for } i \in \{1, \ldots, n\}$$

and $J = 0$. Moreover, let us suppose that the function $(u_1, \ldots, u_i, \ldots, u_n; v_i) \rightsquigarrow f_{i,i}(u_1, \ldots, u_i, \ldots, u_n; \eta_i(u_i, v_i))$ is weakly upper semicontinuous for each $v_i \in D_i$, $i = \overline{1, n}$. Under these conditions, we obtain the following existence result for a type of Nash generalized derivative points:

Theorem 4.3.1 *(A. É. Molnár, O. Vas [75]) For each $i \in \{1, \ldots, n\}$ let $D_i \subset X_i$ be a nonempty, bounded, closed and convex set. Let us assume that conditions **(H)** and **(Φ)** hold true. Then, there exists a point $(u_1^0, \ldots, u_i^0, \ldots, u_n^0) \in D_1 \times \cdots \times D_n$ such that for all $(u_1, \ldots, u_n) \in D_1 \times \cdots \times D_n$ and $i \in \{1, \ldots, n\}$ we have*

$$g_{i,i}^0(u_1^0, \ldots, u_i^0, \ldots, u_n^0; \eta_i(u_i^0, v_i)) \geq 0.$$

The following remark highlights the importance of the previous result:

Remark 4.3.1 *(A. É. Molnár, O. Vas, [75]) If we choose $\eta(u_i^0, v_i) = v_i - u_i^0$ for $i \in \{1, \ldots, n\}$, then we give back the existence result for Nash generalized derivative points from D. Repovš and Cs. Varga's paper [89].*

To get the second application, in Theorem 4.2.1 we require that the functionals $\phi_i : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \times Y_i \to \mathbb{R}$ are differentiable in the i^{th} variable for $i \in \{1, \ldots, n\}$ and their derivatives $\phi_i' : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \times Y_i \to \mathbb{R}$ are continuous for $i \in \{1, \ldots, n\}$. The following result shows that there exists at least one solution for a general nonlinear system of hemivariational-like inequalities:

Corollary 4.3.1 *(A. É. Molnár, O. Vas, [75]) Consider the regular, locally Lipschitz function $J : Y_1 \times Y_2 \times \cdots \times Y_i \times \cdots \times Y_n \to \mathbb{R}$ and the nonlinear functionals $\phi_i : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \times Y_i \to \mathbb{R}$ with their continuous derivatives $\phi_i' : Y_1 \times \cdots \times Y_i \times \cdots \times Y_n \times Y_i \to \mathbb{R}$ for $i \in \{1, \ldots, n\}$. Let also $D_i \subset X_i$ be bounded, closed and convex sets for $i \in \{1, \ldots, n\}$ and let us assume that conditions (H) and*

(Φ) hold true. Then, there exists a point $u^0 = (u_1^0, \dots, u_n^0) \in D_1 \times \cdots \times D_n$ *such that*

$$\Phi_i'(\overline{u}^0, \overline{\eta_i}(u_i^0, u_i)) + J_i^0(\overline{u}^0; \overline{\eta_i}(u_i^0, u_i)) \geq 0,$$

for each $u = (u_1, \dots, u_n) \in D_1 \times \cdots \times D_n$ *and* $i \in \{1, \dots, n\}$.

4.3.2 Schrödinger type problems

The following three examples show the applicability of our results obtained in Sections 4.1. and 4.2:

a) Let $a_1, a_2 : \mathbb{R}^n \to \mathbb{R}$ $(n > 2)$ be two continuous functions which satisfy the conditions below:

- $\inf\limits_{x \in \mathbb{R}^n} a_i(x) > 0$, $i = 1, 2$;
- $\text{meas}(\{x \in \mathbb{R}^n : a_i(x) \leq M_i\}) < \infty$, for every $M_i > 0$, $i = 1, 2$.

The spaces

$$X_1 := \left\{ u \in W^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(|\nabla u(x)|^2 + a_1(x) u^2(x) \right) dx < \infty \right\}$$

and

$$X_2 := \left\{ u \in W^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(|\nabla u(x)|^2 + a_2(x) u^2(x) \right) dx < \infty \right\}$$

with the inner products

$$\langle u, v \rangle_{X_1} = \int_{\mathbb{R}^n} \left[\nabla u(x) \nabla v(x) + a_1(x) u(x) v(x) \right] dx$$

and

$$\langle u, v \rangle_{X_2} = \int_{\mathbb{R}^n} \left[\nabla u(x) \nabla v(x) + a_2(x) u(x) v(x) \right] dx,$$

respectively, are Hilbert spaces.

It is known, that the space $W^{1,2}(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n), p \in [2, 2^*]$. Hence, in the case of $q, r \in [2, 2^*]$, the product space $X_1 \times X_2$ is continuously embedded in the space $L^q(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$. On the other hand, T. Bartsch and Z.-Q. Wang proved in [14] that for $q, r \in [2, 2^*)$, X_1 and X_2 are compactly embedded into $L^q(\mathbb{R}^n)$ and $L^r(\mathbb{R}^n)$, respectively. Therefore, $X_1 \times X_2$ is compactly embedded in $L^q(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$.

Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a regular, locally Lipschitz-function satisfying the following condition:

(G1) There exist $c > 0$ and $q \in (2, 2^*), r \in (2, 2^*)$ such that

$$|w_u| \leq c(|u| + |v| + |u|^{q-1}),$$

$$|w_v| \leq c(|v| + |u| + |v|^{r-1}),$$

for all $(u, v) \in \mathbb{R}^2, w_u \in \partial_1 G(u, v), w_v \in \partial_2 G(u, v), i = \overline{1, 2}$, where $\partial_1 G(u, v)$ and $\partial_2 G(u, v)$ denote the (partial) generalized gradient of $G(\cdot, v)$ at the point u and v, respectively, while $2^* = \frac{2N}{N-2}, N > 2$ is the Sobolev critical exponent.

Let K_1, K_2 be two nonempty, bounded, closed and convex subsets of X_1 and X_2, respectively. We shall denote by $G_1^0(u(x), v(x); w_1(x))$ and $G_2^0(u(x), v(x); w_2(x))$ the directional derivatives of G in the first and the second variable along the direction w_1 and w_2, respectively. Our first Schrödinger-type problem is formulated as follows:

(Sch-S) Find $(u_1, u_2) \in K_1 \times K_2$ such that for every $(v_1, v_2) \in K_1 \times K_2$

$$\begin{cases} \langle \overline{u_1}, \overline{\eta}_1(u_1, v_1)\rangle + \displaystyle\int_{\mathbb{R}^n} G_1^0(\overline{u_1}(x), \overline{v_1}(x), \overline{\eta_1}(u_1(x), v_1(x)))dx \geq 0, \forall v_1 \in K_1; \\ \langle \overline{u_2}, \overline{\eta}_2(u_2, v_2)\rangle + \displaystyle\int_{\mathbb{R}^n} G_2^0(u_2(x), v_2(x), \overline{\eta_2}(\overline{u_2}(x), \overline{v_2}(x)))dx \geq 0, \forall v_2 \in K_2. \end{cases}$$

As another application of Theorem 4.2.1, we show the existence of at least one solution of this problem.

Corollary 4.3.2 *(A. É. Molnár, O. Vas [75]) If $K_1 \subset X_1$ and $K_2 \subset X_2$ are two nonempty, convex, closed and bounded subsets and η_1, η_2 satisfy the condition **(H)** and G satisfies **(G1)**, respectively, then the problem **(Sch-S)** has at least one solution.*

b) Let $n > 2$. Consider the function $a_1 : \mathbb{R}^n \to \mathbb{R}$ and the space X_1, equipped with the inner product $\langle u, v\rangle_{X_1}$ and, defined in the same way as in point a). Clearly, assumptions **(CT)** and **(CP)** are satisfied for $p = 2$.

Suppose that the function $A : X_1 \to X_1$ is defined as follows: $\langle Au, v\rangle := (u, v)_{X_1}$. By the properties of the norm and of the weak convergence, it follows that **(A1)** and **(A2)** are satisfied. In this case, if we assume that $K \subseteq X_1$ is chosen properly and f and j satisfy the same conditions as in Theorem 4.1.1 and Theorem 4.1.2, respectively, then we can apply these main results, concluding that problem **(V-HI)** has at least one solution.

c) Analogously to the previous example, it is possible to formulate another Schrödinger type problem, if we consider for $n > 2$ the Hilbert space

$$H := \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} [|\nabla u(x)|^2 + |x|^2 u^2(x)]dx < \infty\}$$

endowed with the inner product

$$(u, v) = \int_{\mathbb{R}^n} [\nabla u(x)\nabla v(x) + |x|^2 u(x)v(x)]dx.$$

Note that $H \hookrightarrow L^s(\mathbb{R}^n)$ is compact for $s \in [2, 2^*)$ (see O. Kavian, [57]). Hence the conditions **(CT)** and **(CP)** hold. So, if we consider the space H instead of the space X_1 from the point **b)**, we can easily derive the results of Theorem 4.1.1 and Theorem 4.1.2.

4.3.3 A problem with radially symmetric functions

In Theorem 4.1.1 and Theorem 4.1.2 it is very important that the assumptions **(CP)**, **(j1)** and **(j2)** are satisfied. In the sequel, we focus our attention to the case when we replace the conditions **(j1)** and **(j2)** with certain conditions and we shall prove that in this case the assertions of Theorem 4.1.1 and Theorem 4.1.2 remain true.

Let $a : \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}$ $(L \geq 2)$ be a non-negative continuous function satisfying the following assumptions:

(a_1) $a(x, y) \geq a_0 > 0$ if $|(x, y)| \geq R$ for a large $R > 0$;

(a_2) $a(x, y) \to +\infty$, when $|y| \to +\infty$ uniformly for $x \in \mathbb{R}^L$;

(a_3) $a(x, y) = a(x', y)$ for all $x, x' \in \mathbb{R}^L$ with $|x| = |x'|$ and all $y \in \mathbb{R}^M$.

Consider the following subspaces of $W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M)$

$$\tilde{E} = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : u(s, t) = u(s', t) \ \forall \ s, s' \in \mathbb{R}^L, |s| = |s'|, \forall t \in \mathbb{R}^M\},$$

$$E = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : \int_{\mathbb{R}^{L+M}} a(x)|u(x)|^p dx < \infty\},$$

$$X := \tilde{E} \cap E = \{u \in \tilde{E} : \int_{\mathbb{R}^{L+M}} a(x)|u(x)|^p dx < \infty\}$$

equipped with the norm

$$\|u\|^p = \int_{\mathbb{R}^{L+M}} |\nabla u(x)|^p dx + \int_{\mathbb{R}^{L+M}} a(x)|u(x)|^p dx.$$

D. C. de Morais Filho, M. A. S. Souto, J. Marcos Do proved in [76] the following result: *X is continuously embedded in $L^s(\mathbb{R}^L \times \mathbb{R}^M)$ if $s \in [p, p^*]$, and compactly embedded if $s \in (p, p^*)$.*

Let

$$\Gamma = \left\{ g : E \to E : g(v) = v \circ \begin{pmatrix} R & 0 \\ 0 & Id_{\mathbb{R}^M} \end{pmatrix}, R \in O(\mathbb{R}^L) \right\},$$

where $O(\mathbb{R}^L)$ is the set of all rotations on $\mathbb{R}^L$ and $Id_{\mathbb{R}^M}$ denotes the $M \times M$ identity matrix.

Next, we assume that $j : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is locally Lipschitz in the real variable (i.e. the second variable) and satisfies the following conditions:

(j1') $j(x, 0) = 0$, and there exist $c > 0$ and $q \in (p, p^\star)$ such that

$$|\xi| \le c(|y|^{p-1} + |y|^{q-1}),\ \forall \xi \in \partial j(x, y),\ (x, y) \in \mathbb{R}^{L+M} \times \mathbb{R};$$

(j3) $\displaystyle\lim_{y \to 0} \frac{\max\{|\xi| : \xi \in \partial j(x, y)\}}{|y|^{p-1}} = 0$ uniformly for every $x \in \mathbb{R}^{L+M}$;

(j4) $j(\cdot, y)$ is Γ-invariant for all $y \in \mathbb{R}$.

In order to derive a new existence result, we use the following proposition instead of Lemma 4.1.1:

Proposition 4.3.1 *(H. Lisei, Cs. Varga, [70]) If $j : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ verifies the conditions (j1'), (j3) and (j4) then*

$$u \in X \mapsto \int_{\mathbb{R}^{L+M}} j(x, u(x))dx$$

is weakly sequentially continuous.

Now we are in the position to state the following existence result.

Theorem 4.3.2 *(H. Lisei, A. É. Molnár, Cs. Varga, [69])*

(i) *Let $K \subset X$ be a nonempty, closed, convex and bounded set. Let $A : E \to E^*$ be an operator satisfying* ***(A1)****. Assume that j satisfies* ***(j1'), (j3)*** *and* ***(j4)****. Then, there exists $u \in K$ such that*

$$\langle Au, v-u\rangle + \int_{\mathbb{R}^{L+M}} j^0(x,u(x);v(x)-u(x))dx \geq 0 \quad \text{for all } v \in K. \tag{4.3.1}$$

(ii) *Moreover, if $K \subset X$ is a nonempty, closed and convex set and $A : X \to X^*$ is an operator satisfying* ***(A1)****,* ***(A2)*** *and if we assume that j satisfies* ***(j1'), (j2), (j3)*** *and* ***(j4)****, then there exists $u \in K$ such that* (4.3.1) *holds.*

Remark 4.3.2 *(H. Lisei, A. É. Molnár, Cs. Varga, [69]) In [70, Theorem 3.1] the authors proved by using a Mountain Pass Theorem combined with the principle of symmetric criticality for Motreanu-Panagiotopoulos type functionals the existence of nontrivial positive solutions for* (4.3.1)*, when K is the cone $\{v \in E : v \geq 0$ a.e. in $\mathbb{R}^L \times \mathbb{R}^M\}$ and $A : X \to X^\star$ is the duality mapping.*

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