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FUZZY APPROXIMATION OPERATORS

DOCTORAL THESIS SUMMARY

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Contents

Introduction

This thesis contains some of the main results that I have obtained in the topic of the approximation of fuzzy numbers. This topic has had an impressive development over the last decade (see Section 3.1 for details). We will discuss two types of approximations. Firstly, we will discuss about approximations of fuzzy number by fuzzy numbers with simpler form such as the trapezoidal fuzzy numbers or parametric fuzzy numbers. This type of approximation should be regarded as an alternative to represent fuzzy numbers in a more simple way especially in applications where not all the information carried by a fuzzy number is necessary. For this reason, maybe the most important part of this topic is the approximation of fuzzy numbers under some additional conditions such as the preserving of the expected interval or the preserving of the ambiguity and value. These characteristics are very important especially in statistical problems or in the ranking of fuzzy numbers. Therefore, we need to find simpler representations for fuzzy numbers and such that one or more of these characteristics are preserved. There are numerous metrical structures defined on the space of fuzzy numbers (see Section 1.9 for details). Some of them are important in solving fuzzy equations, others are important in statistical problems while some of them can be used to rank fuzzy numbers. The present work deals with the problem of the approximation of fuzzy numbers. It seems that the suitable metric for this type of problems is an extension of the Euclidean distance introduced in the paper [58]. Generalizations of this L_2-type metric can be found in the papers [85] and [88] as they are usually called weighted L_2-type metrics. For this reason our research is focusing around these kinds of metrics.

The second type of approximation is the approximation of fuzzy numbers by using approximation operators on the space of fuzzy numbers. I think that this kind of approximation is important since the space of fuzzy numbers can be perceived as a space of functions with certain properties. Another interpretation is that this kind of topic is in the completeness of the previous one where the motivation was that in some applications we need only a part of the information associated with a fuzzy number. But sometimes we deal with situations when all the important informations regarding fuzzy numbers has to be preserved. This is why approximation operators can be the solution. But we need approximation operators which besides the convergence properties they also own important shape preserving properties. As it will be seen in the last chapter of this thesis, we will refer to the Bernstein operators of max-product kind (introduced in the book [55]) as operators with such properties.

This thesis is structured as follows. The first chapter presents briefly the theory of fuzzy sets and in detail some basics about fuzzy numbers. In Chapter 2 we present many important results which will be useful later in the proving of the main results from Chapters 3-4. Apart of that, the last 4 sections are dedicated to the important topic of the ranking of fuzzy numbers. In Chapter 3 we discuss the problem of the approximation of fuzzy numbers by fuzzy numbers with simpler form. Firstly, existence results are provided under some general types of metrics. Then we discuss particular cases. Firstly, we prove the existence and uniqueness of the parametric in particular trapezoidal approximation with respect to weighted L_2-type metrics. Then we prove the existence and the uniqueness of the parametric approximation preserving the expected interval and of the parametric approximation preserving the value and ambiguity. As applications we deduce the algorithms to determine (they always exist and they are unique) trapezoidal approximations preserving the expected interval, trapezoidal approximations preserving the value and the ambiguity and, trapezoidal approximations preserving the ambiguity, all of them with respect to the Euclidean metric. Then we discuss about weighted trapezoidal approximations preserving the weighted expected interval and weighted trapezoidal approximations preserving the core of a fuzzy number. Finally, the approximation operators are tested on some numerical examples. Because the quality of an approximation operator is important nevertheless, Chapter 4 of the thesis is dedicated to the investigation of some basic properties of fuzzy approximation operators. At first, general results on the scale invariance and translation invariance of fuzzy approximation operators are provided. Then considering the continuity property as one of the most important criteria that an approximation operator should possess, we present a detailed study on this mater. Firstly, we prove that fuzzy approximation operators without additional conditions are nonexpansives with respect to L_2-type metrics. Then in the case of the trapezoidal approximation operator preserving the expected interval as well as in the case of the trapezoidal approximation operator preserving the value and ambiguity or preserving only the ambiguity respectively, we prove that all these operators are Lipschitz continuous and in the case of the first two operators even the best Lipschitz constants are obtained. As a negative result it is proved that any trapezoidal fuzzy number valued operator preserving the core has discontinuity points. Another important issue is the additivity. Since most of the approximation operators are non-additive, we will find estimations for the defect of additivity of an approximation operator in the sense given by Ban and Gal in [29]. In the case of the trapezoidal approximation operator preserving the expected interval the best possible estimation is obtained. In the last section of the chapter we discuss trapezoidal approximation in relation with aggregation, another important topic in present days with many applications in fuzzy analysis. In the last chapter of this thesis we discuss the approximation of fuzzy numbers by using the Bernstein operators of max-product kind. It seems that they are convenient when it comes to approximate fuzzy numbers. Besides the convergence in the uniform norm in the case of continuous fuzzy numbers, they also have important shape preserving properties, namely they preserve the support and they are convergent with respect to the core. Finally, it is worth mentioning that in the case of the important characteristics of fuzzy numbers such as the expected interval, ambiguity or value, again, we have convergence properties in the approximation by Bernstein max-product operators. The thesis ends with a conclusion, summarizing the results obtained and proposing further research.

Finally, I would like to mention that this thesis contains original contributions from the papers or manuscripts [17]-[19], [21]-[28], [30], [40]-[41], [44], [46]-[48]. With the exception of Sections 1.1-1.9 (which are generalities), 3.3, 3.5 and 3.9 respectively, the remaining sections are based almost entirely on original contributions. In addition, Section 3.3 is based actually on an original approach. The original contributions are indicated at the beginning of each chapter and section respectively and then adequate references are used inside each section. Also please note that paper [47] is an extended version of paper [46] which will appear in a special issue of the journal Fuzzy Sets and Systems dedicated to the theory of fuzzy numbers, where some papers (including our contribution) are selected from the EUSFLAT-LFA Conference held in 2011 in Aix-Les-Bains. But actually all the main results in [47] are better (or more complete) comparing to those from [46] and moreover many other theoretical results are proposed such as approximations wit respect to L_1-type metrics or convergence results with respect to the important characteristics of a fuzzy number. In addition, the thesis contains original unpublished results and also many results from the thesis improve the published versions.

Keywords: fuzzy number, trapezoidal fuzzy number, parametric fuzzy number, L_2- type metrics,

expected interval, ambiguity, value, reduction function, defuzifier, extended fuzzy number, normed spaces, Hilbert spaces, trapezoidal approximation, parametric approximation, extended trapezoidal approximation, extended parametric approximation, Lipschitz-continuity, defect of additivity, aggregation, Bernstein operator of max-product kind.

Chapter 1

Fuzzy sets and fuzzy numbers

The main concepts on fuzzy numbers discussed in this Chapter can be found in numerous recent papers investigating on the approximations of fuzzy numbers or on the ranking of fuzzy numbers (see e.g. [5], [13], [58], [85]). In addition, Sections 1.10-1.12 contain some of my original contributions taken from papers or manuscripts [19], [21], [25], [28], [48].

1.1 The definition of a fuzzy set

In many practical situation we can precisely verify if a certain object belongs or not to a given set. For example let us consider the set X of people with age less than or equal to 40. If $\mu_X : X \to \{0,1\}$ is the characteristic function of X,

$$\mu_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \notin X, \end{cases}$$

then we can say without any doubt that $\mu_X(x) = 1$ if x is a person with age under 40 and $\mu_X(x) = 0$ if x is a person with age over 40.

But there are situation when we cannot say for sure if an object belongs to a set, especially in the case when we can describe rather ambiguously this object. The following two examples are relevant for such situations.

1) The opposite of the word "young" is the word "old". Therefore, the classical logic encourages us to split people in two categories: old people and young people. However, it is difficult to decide in which category should be a person who's age is 35 or say 52.

2) (Zadeh's example) We consider the set of real numbers which are much greater then 1. Clearly, we can say that the number 100 belongs to this set but again we are in doubt to decide whether the number 10 belongs or not to the set of numbers much greater then 1.

Zadeh observed that there are situation when the classical logic cannot be applied in certain practical situations and to overcome this shortcoming he introduced in the paper [87] the notion of a fuzzy set.

Definition 1.1.1 *Let X be a universe of objects. A fuzzy set A in X is characterized by a membership (characteristic) function $\mu_A : X \to [0,1]$, which assigns to each object $x \in X$ a real number in the interval $[0,1]$, with the value $\mu_A(x)$ representing the grade of membership of x in A.*

Keeping the notations as in the above definition, the fuzzy set A will be given under the explicit form

$$A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0,1]\}.$$

The set of all fuzzy sets of a set X is denoted with $\widetilde{\mathcal{P}}(X)$. If for a fuzzy set $A \in \widetilde{\mathcal{P}}(X)$ we have $\mu_A = 0$, then we say that A is an empty set and we write as usual $A = \varnothing$. If the set $\{x \in X : \mu_A(x) > 0\}$ is finite then the fuzzy set A is called a discrete fuzzy set. In this case the fuzzy set A is given by neglecting the elements $x \in X$ such that $\mu_A(x) = 0$. For example $A \in \widetilde{\mathcal{P}}(\mathbb{Z})$, $A = \{(-3, 0.2), (0, 0.5), (2, 1), (5, 0.7), (6, 0.3)\}$ is an example of a discrete fuzzy set. The interpretation of the value $\mu_A(x)$ is very natural. If $\mu_A(x)$ is very close to 1 then the grade of membership of x in A is very high, while in the case when $\mu_A(x)$ is very close to 0 then the grade of membership of x in A is very low. In the case when $\mu_A(x) = 1$ then we have total membership property and we state that $x \in A$ and in the case when $\mu_A(x) = 0$ then we have nonmembership property and we can state that $x \notin A$.

In the case when $\mu_A(x) \in \{0,1\}$ for all $x \in X$ then the fuzzy set A is reduced to a set in the classical meaning.

1.2 Operations on fuzzy sets

The basic operations on ordinary sets such as: equality, complementation, inclusion, union or intersection, can be extended in a natural way for the case when we are dealing with fuzzy sets. In what follows we list the definitions of these basic operations as they were given by Zadeh in [87].

Definition 1.2.1 *Let A and B denote two fuzzy subsets of the same set X. If $\mu_A(x) = \mu_B(x)$, for all $x \in X$ then we say that A and B are equal and we write $A = B$.*

Definition 1.2.2 *Let A and B denote two fuzzy subsets of the same set X. If $\mu_A(x) \le \mu_B(x)$, for all $x \in X$ then we say that A is included in B and we write $A \subseteq B$.*

Definition 1.2.3 *Let A be a fuzzy set on X. The complement of A denoted $\overline{A}$ is characterized by the membership function $\mu_{\overline{A}} : X \to [0,1]$, $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$, for all $x \in X$.*

Definition 1.2.4 *If A and B are two fuzzy subsets of the same set X then the union of A and B, denoted $A \cup B$, is characterized by the membership function $\mu_{A\cup B} : X \to [0,1]$, $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.*

Definition 1.2.5 *If A and B are two fuzzy subsets of the same set X then the intersection of A and B, denoted $A \cap B$, is characterized by the membership function $\mu_{A\cap B} : X \to [0,1]$, $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.*

1.3 The height, core, support and $\alpha-$cut of a fuzzy set

The height of a fuzzy set $A \in \widetilde{\mathcal{P}}(X)$, is the value $hgt(A) = \sup_{x\in X} \mu_A(x)$. From the definition of a fuzzy set it is immediate that $hgt(A) \le 1$. If there exists $x_0 \in X$ such that $hgt(A) = \mu_A(x_0) = 1$, then the fuzzy set A is called normal.

The core of a fuzzy set $A \in \widetilde{\mathcal{P}}(X)$ is denoted with $core(A)$ and it is given by $core(A) = \{x \in X : \mu_A(x) = 1\}$. It is immediate that $core(A) \ne \varnothing$ if and only if A is normal.

The support of a fuzzy set $A \in \widetilde{\mathcal{P}}(X)$ is denoted with $supp(A)$ and represents the set of all elements of X with a nonzero degree of membership, that is $supp(A) = \{x \in X : \mu_A(x) > 0\}$. It is easy to check that $A \neq \varnothing$ if and only if $supp(A) \neq \varnothing$.

For $\alpha \in [0,1]$, the $\alpha-$cut of a fuzzy set $A \in \widetilde{\mathcal{P}}(X)$ will be denoted in this thesis with A_α and it is given by the equality $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$. It is immediate that $A_0 = X$ and $A_1 = core(A)$. We will see in Section 1.6 that in the case of a fuzzy number the $0-$cut will be defined with a slight modification.

1.4 Convex fuzzy sets

In his celebrated paper, Zadeh introduced the notion of convexity for fuzzy sets in a way which allows to preserve the properties of the ordinary convex sets.

Definition 1.4.1 *Let X be a convex subset of a real vector space. We say that the fuzzy set $A \in \widetilde{\mathcal{P}}(X)$ is convex if for any $\alpha \in [0,1]$ the set $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$ is a convex subset of X.*

We present now a concept which generalizes the concept of monotonicity and convexity as well, concept which will help us to give an equivalent definition of a convex fuzzy set.

Definition 1.4.2 *Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. One says that the function f is:*

(i) quasi-convex on I if it satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, x, y \in I, \lambda \in [0,1];$$

(ii) quasi-concave on I if it satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}, x, y \in I, \lambda \in [0,1].$$

It is known that quasi-convex and quasi-concave functions generalize the concept of monotonicity because monotonous functions are particular cases of quasi-convex and quasi-concave functions. Then, it is known that convex functions are also quasi-convex and that concave functions are also quasi-concave functions.

Proposition 1.4.3 *Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an interval. Then f is quasi-convex (quasi-concave) if and only if for any $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x) \leq \lambda\}$ ($\{x \in X : f(x) \geq \lambda\}$) is a convex subset of $\mathbb{R}$.*

We note that in a more general context a quasi-convex (quasi-concave) function is a function like those from Definition 1.4.2 but with the domain being an arbitrary convex subset of a vector space.

From the above definition it follows that the fuzzy set A is convex if and only if all possible $\alpha-$cuts of A are ordinary convex sets.

Noting the definition of the $\alpha-$cut of a fuzzy set and taking into account Proposition 1.4.3 and the comment afterwords, we have the following equivalent definition of the convexity of a fuzzy set.

Definition 1.4.4 *Let X be a convex subset of a real vector space. We say that the fuzzy set $A \in \widetilde{\mathcal{P}}(X)$ is convex if the membership function μ_A is a quasi-concave function.*

From the above definition it follows that $A \in \widetilde{\mathcal{P}}(X)$ is convex if and only if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$ we have $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\}$.

Remark 1.4.5 *In the particular case when $X = \mathbb{R}$ and the fuzzy set $A \in \widetilde{\mathcal{P}}(\mathbb{R})$ has a continuous membership function and $supp(A)$ is bounded, it follows (see the discussion which follows after Definition 1.4.2) that A is convex if and only if there exist $a, b, c \in \mathbb{R}$, $a \leq c \leq b$ such that:*

1) $\mu_A = 0$ outside the interval $[a, b]$;

2) μ_A is nondecreasing on the interval $[a, c]$;

3) μ_A is nonincreasing on the interval $[c, b]$.

In this way the interpretation of a convex fuzzy set is more clear and in addition we can easily find examples of fuzzy sets that are non-convex.

1.5 The extension principle

The extension principle introduced by Zadeh in [87] allows us to extend the basic mathematical concepts for fuzzy quantities. We have the following definition of the extension principle (see e. g. [64] pp. 41).

Definition 1.5.1 *Let X_1, X_2, ..., X_n, Z, be non-empty sets and let us consider the function $F : X \to Z$ where X is the product space $X = X_1 \times X_2 \times ... \times X_n$. Furthermore, we consider the fuzzy sets $\{A_i\}_{i \in \{1,2,...,n\}}$ such that $A_i \in \widetilde{\mathcal{P}}(X_i)$ for every $i \in \{1, 2, ..., n\}$. Taking use of the function F we can define the fuzzy set $F(A_1, A_2, ...A_n) \in \widetilde{\mathcal{P}}(Z)$, characterized by the membership function $\mu_{F(A_1,A_2,...A_n)} \to [0, 1]$,*

$$\mu_{F(A_1,A_2,...A_n)}(z) = \begin{cases} \sup\limits_{(x_1,x_2,...,x_n) \in F^{-1}(z)} \min\{\mu_{A_1}(x), .., \mu_{A_n}(x)\}, \text{ if } z \in F(X), \\ 0, \text{ otherwise.} \end{cases} \tag{1.1}$$

By particularizing the function F we can define algebraic operations between fuzzy sets as the following examples will prove it.

Example 1.5.2 *We consider the function $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $F(x, y) = x + y$ and the fuzzy sets A, $B \in \widetilde{\mathcal{P}}(\mathbb{Z})$,*

$$\begin{aligned} A &= \{(-1, 0.2),\ (0, 0.5),\ (2, 1),\ (3, 0.6),\ (6, 0.2)\}, \\ B &= \{(-2, 0.3),\ (-1, 0.5),\ (0, 0.8),\ (1, 1),\ (3, 0.7),\ (5, 0.4)\}. \end{aligned}$$

Applying formula (1.1) we get $F(A, B) = A + B$ where

$$\begin{aligned} A + B = \{&(-3, 0.2),\ (-2, 0.3),\ (-1, 0.5),\ (0, 0.5),\ (1, 0.5),\ (2, 0.8),\ (3, 1), \\ &(4, 0.6),\ (5, 0.7),\ (6, 0.6),\ (7, 0.4),\ (8, 0.4),\ (9, 0.2),\ (11, 0.2)\}. \end{aligned}$$

The extension principle stays at the basis of operations between fuzzy numbers and this will be seen in Section 1.7.

1.6 The definition of a fuzzy number. L-R representation and L-U representation

When Dubois and Prade introduced the notion of a fuzzy number they were inspired by many practical situations when uncertain parameters were involved. For this reason, in their opinion a fuzzy number

u is a fuzzy subset of the real line characterized by a continuous membership function $\mu_u : \mathbb{R} \rightarrow [0,1]$, satisfying the following properties:

i) there exist $c, d \in \overline{\mathbb{R}}$, $c \leq d$ such that $\mu_u(x) = 0$ outside the interval $[c, d]$.

ii) there exist the reals $a, b \in \mathbb{R}$, $c \leq a \leq b \leq d$ such that:

ii_1) μ_u is strictly increasing on the interval $[c, a]$;

ii_2) $\mu_u(x) = 1$ for all $x \in [a, b]$;

ii_3) μ_u is strictly decreasing on the interval $[b, d]$.

This definition has suffered some small modifications dictated by practical reasons. In this thesis we adopt the following definition of a fuzzy number, definition which is widely accepted between researchers.

Definition 1.6.1 *A fuzzy number u is characterized by a membership function $\mu_u : \mathbb{R} \rightarrow [0,1]$ of the form:*

$$\mu_u(x) = \begin{cases} 0, & if \quad x \leq a_1, \\ l_u(x), & if \quad a_1 \leq x \leq a_2, \\ 1 & if \quad a_2 \leq x \leq a_3, \\ r_u(x), & if \quad a_3 \leq x \leq a_4, \\ 0, & if \quad a_4 \leq x, \end{cases} \tag{1.2}$$

where $a_1,\ a_2,\ a_3,\ a_4, \in \mathbb{R}$, $l_u : [a_1, a_2] \longrightarrow [0,1]$ is a nondecreasing upper semicontinuous function, $l_u(a_1) = 0$, $l_u(a_2) = 1$, called the left side of the fuzzy number and $r_u : [a_3, a_4] \longrightarrow [0,1]$ is a nonincreasing upper semicontinuous function, $r_u(a_3) = 1$, $r_u(a_4) = 0$, called the right side of the fuzzy number.

For simplicity, from now one we will use the same notation for a fuzzy number and for its membership function. A fuzzy number is said to be continuous if its membership function is a continuous function. The following notions are defined in a similar manner as in the case of the general context of fuzzy sets.

The $\alpha-$cut, $\alpha \in (0,1]$, of a fuzzy number u is a crisp set defined as $u_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$. The support or $0-$cut u_0 of the fuzzy number u is defined as $u_0 = cl\left(\{x \in \mathbb{R} : u(x) > 0\}\right)$.We often use the notation $u_0 = supp(u)$.

Comparing the support of a fuzzy number with the support of a fuzzy set we observe that in the case of fuzzy sets, the support is not taken under the closure operator. Since a fuzzy number is actually a fuzzy set it results that the support can be defined in 2 ways. However, in this thesis we adopt the formula from this section, which in the present is extensively used by researchers.

The core or the $1-$cut u_1, of the fuzzy number u will be denoted from now one with $core(u)$. If $core(u)$ is reduced to a single point then u is called a unimodal fuzzy number and in this case the value $core(u)$ is called modal value.

From Definition 1.6.1, every $\alpha-$cut $\alpha \in [0,1]$, of a fuzzy number u is a closed interval $u_\alpha = [u_L(\alpha), u_U(\alpha)]$, where $u_L(\alpha) = \inf\{x \in \mathbb{R} : u(x) \geq \alpha\}$ and $u_U(\alpha) = \sup\{x \in \mathbb{R} : u(x) \geq \alpha\}$, for any $\alpha \in (0,1]$. If the sides of the fuzzy number u are strictly monotone then one can see easily that u_L and u_U are inverse functions of l_u and r_u respectively. Moreover, it can be proved that the functions u_L and u_U are left continuous.

Using the facts from above it follows that we can define a fuzzy number by using its α-cut representation. Consequently, we obtain the following equivalent definition of a fuzzy number, introduced by Goetschel and Voxman in the paper [56].

Definition 1.6.2 *A fuzzy number u is an ordered pair of left continuous functions $[u_L(\alpha), u_U(\alpha)]$, $0 \leq \alpha \leq 1$, which satisfy the following requirements:*

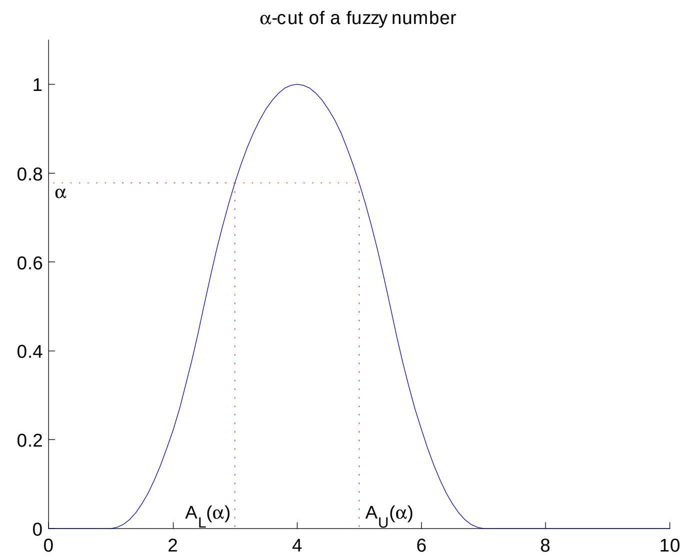

Fig. 1.1

i) u_L *is nondecreasing on* $[0,1]$;
ii) u_U *is nonincreasing on* $[0,1]$;
iii) $u_L(1) \leq u_U(1)$.
We use the notation $u = (u_L, u_U)$.

If a fuzzy number u is defined using Definition 1.6.1, we say that u is given in $L-R$ form. Otherwise, if u is defined using Definition 1.6.2, we say that u is given in $L-U$ form.

From now one, in this thesis we adopt the notation $F(\mathbb{R})$ for the space of fuzzy numbers. We also use the notation $UF(\mathbb{R})$ for the space of unimodal fuzzy numbers.

An important class of fuzzy numbers is the class of symmetric fuzzy numbers. Symmetric fuzzy numbers are often used in practice. They are defined as follows.

Definition 1.6.3 *A fuzzy number u is called a symmetric fuzzy number if* $u_L(1) - u_L(\alpha) = u_U(\alpha) - u_U(1)$, *for all* $\alpha \in [0,1]$.

We denote with $F^S(\mathbb{R})$ the set of all symmetric fuzzy numbers.

At the end of this section we will discuss about the equality of two fuzzy numbers. Due to the fact that most of the main results of the thesis are in relation with L_p-type metrics we adopt the following definition.

Definition 1.6.4 *We say that fuzzy numbers A and B are equal and we denote $A = B$, if $A_L = B_L$ and $A_U = B_U$ almost everywhere* $\alpha \in [0,1]$.

The above definition should count only when we work with L_p-type metrics on the space of fuzzy numbers.

1.7 Basic operations between fuzzy numbers

Since fuzzy numbers extend real numbers, it is natural to introduce on the space of fuzzy numbers the basic operations such as addition, subtraction, multiplication or division. These operations are derived from the Zadeh's extension principle presented briefly in a previous section.

If u and v are two fuzzy numbers then $u+v$ denotes the addition of u and v, where

$$(u+v)(x) = \sup_{y\in\mathbb{R}}\{u(y) \wedge v(x-y)\}, x \in \mathbb{R},$$

where $\wedge$ means minimum. It is immediate that if $u_\alpha = [u_L(\alpha), u_U(\alpha)]$ and $v_\alpha = [v_L(\alpha), v_U(\alpha)]$, $\alpha \in [0,1]$, then

$$(u+v)_\alpha = u_\alpha + v_\alpha = [u_L(\alpha) + v_L(\alpha), u_U(\alpha) + v_U(\alpha)],$$

for all $\alpha \in [0,1]$.

If $\lambda = 0$ then by definition we put $\lambda \cdot u = 0$, for every $u \in F(\mathbb{R})$. Here 0 denotes the neutral element of $F(\mathbb{R})$ with respect to the addition, that is $0(0) = 1$ and $0(x) = 0$ otherwise.

If $\lambda \in \mathbb{R} \setminus \{0\}$ and $u \in F(\mathbb{R})$ then $\lambda \cdot u$ denotes the scalar multiplication of λ with u where

$$(\lambda \cdot u)(x) = u(x/\lambda),\ x \in \mathbb{R}.$$

It is immediate that if $u_\alpha = [u_L(\alpha), u_U(\alpha)]$, $\alpha \in [0,1]$, then

$$(\lambda \cdot A)_\alpha = \lambda A_\alpha = \begin{cases} [\lambda A_L(\alpha), \lambda A_U(\alpha)], & \text{if } \lambda \geq 0, \\ [\lambda A_U(\alpha), \lambda A_L(\alpha)], & \text{if } \lambda < 0, \end{cases}$$

for all $\alpha \in [0,1]$.

The most important properties of the addition of fuzzy numbers and of the scalar multiplication are listed below.

Proposition 1.7.1 *We have:*

i) $u + v = v + u$, $(\forall)\ u, v \in F(\mathbb{R})$;
ii) $(u + v) + w = u + (v + w)$, $(\forall)\ u, v, w \in F(\mathbb{R})$;
iii) for any $u, v, w \in F(\mathbb{R})$ *such that* $u + w = v + w$ *we have* $u = v$;
iv) $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$, $(\forall)\ u, v \in F(\mathbb{R})$, $(\forall)\ \lambda \in \mathbb{R}_+$;
v) $(\lambda + \beta) \cdot u = \lambda \cdot u + \beta \cdot u$, $(\forall)\ u \in F(\mathbb{R})$, $(\forall)\ \lambda, \beta \in \mathbb{R}_+$;
vi) $\lambda \cdot (\beta \cdot u) = \beta \cdot (\lambda \cdot u) = (\lambda\beta) \cdot u$, $(\forall)\ \lambda, \beta \in \mathbb{R}_+$, $(\forall)\ u \in F(\mathbb{R})$;
vii) $1 \cdot u = u$, $(\forall)\ u \in F(\mathbb{R})$.

It is easy to check that the only fuzzy numbers having opposite elements with respect to the addition are fuzzy numbers with equally constant side functions. We will see in the next section that such fuzzy numbers are identified with classical real (crisp) numbers. Actually, it is evident that in general the property $u + (-u) = 0$ does not hold for fuzzy numbers. Therefore, the triplet $(F(\mathbb{R}), +, \cdot)$ is not a vector space and we will refer to it as a semilinear space since this syntax is often used in the literature.

1.8 Remarkable classes of fuzzy numbers

We say that the fuzzy number u is a crisp fuzzy number if there exists $c \in [0,1]$ such that $u(c) = 1$ and $u(x) = 0$ for all $x \in \mathbb{R} \setminus \{c\}$. It is immediate that $u_L = u_U = c$. For simplicity, if u is a crisp fuzzy number then the constant value of the membership function will be denoted with u.

A fuzzy number u is called an interval if there exist the reals $a, b \in \mathbb{R}$, $a \leq b$, such that $u(x) = 1$ for all $x \in [a, b]$ and $u(x) = 0$ for all $x \in \mathbb{R} \setminus [a, b]$. It is immediate that $u_L = a$ and $u_U = b$. We denote $u = [a, b]$.

A fuzzy number A is called a triangular fuzzy number if there exist $t_1 \leq t_2 \leq t_3$ such that

$$A_\alpha = [t_1 + (t_2 - t_1)\alpha, t_3 - (t_3 - t_2)\alpha], \alpha \in [0, 1].$$

We use the notation $A = (t_1, t_2, t_3)$. The family of all triangular fuzzy numbers will be denoted by $F^{\Delta}(\mathbb{R})$ and the family of symmetric triangular fuzzy number will be denoted with $F^{S\Delta}(\mathbb{R})$.

A generalization of the triangular fuzzy number is the trapezoidal fuzzy number. A trapezoidal fuzzy number T is completely determined by four real parameters $t_1 \leq t_2 \leq t_3 \leq t_4$ such that

$$T_\alpha = [t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha], \alpha \in [0, 1]. \tag{1.3}$$

We use the notation $T = (t_1, t_2, t_3, t_4)$. When $t_2 = t_3$, T becomes a triangular fuzzy number. When $t_2 - t_1 = t_4 - t_3$ we obtain a symmetric trapezoidal fuzzy number. The family of all trapezoidal fuzzy numbers will be denoted with $F^T(\mathbb{R})$ and the family of all symmetric trapezoidal fuzzy numbers will be denoted with $F^{ST}(\mathbb{R})$. It is immediate that if T is a trapezoidal fuzzy number then the functions l_T and r_T are linear functions.

Parametric fuzzy numbers were introduced in the paper [68] mainly to generalize the trapezoidal approximation problem. A parametric fuzzy number of type (s_L, s_R) or simply an (s_L, s_R) fuzzy number is a fuzzy number A with $A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0, 1]$, given by

$$A_L(\alpha) = a - \sigma(1 - \alpha)^{1/s_L} \text{ and } A_U(\alpha) = b + \beta(1 - \alpha)^{1/s_R}, \alpha \in [0, 1],$$

where $a, b, \sigma, \beta, s_L, s_R \in \mathbb{R}$, $a \leq b$, $\sigma \geq 0$, $\beta \geq 0$, $s_L > 0$, $s_R > 0$. Note that the condition $a \leq b$ is imposed in order to obtain a proper fuzzy number. We use the notation $A = (a, b, \sigma, \beta)_{s_L, s_R}$.

When $s_L = s_R = 1$ then A becomes a trapezoidal fuzzy number. The family of all (s_L, s_R) fuzzy numbers will be denoted with $F^{s_L, s_R}(\mathbb{R})$. In Fig. 1.2 we consider different kinds of parametric fuzzy numbers. We end the discussion about parametric fuzzy numbers by mentioning that recently (see [84]) parametric fuzzy numbers have been also called semi-trapezoidal fuzzy numbers. However, in this thesis we refer to them only as parametric fuzzy numbers.

Another important class of fuzzy numbers were introduced in [36] as follows. Let $a_1, a_2, a_3, a_4 \in \mathbb{R}$ be such that $a_1 \leq a_2 \leq a_3 \leq a_4$. A fuzzy number A given by

$$A_\alpha = [A_L(\alpha), A_U(\alpha)] = [a_1 + \alpha^{1/r}(a_2 - a_1), a_4 - \alpha^{1/r}(a_4 - a_3)], \alpha \in [0, 1], \tag{1.4}$$

where $r > 0$, is denoted $A = (a_1, a_2, a_3, a_4)_r$.

There are many other important types of fuzzy numbers. We mention the Gaussian fuzzy numbers or the quadratic fuzzy numbers that are successfully used in engineering sciences. For more details we refer again to the book of Hanss ([64]) where these kinds of problems are extensively studied.

1.9 Metrics on the space of fuzzy numbers

Since the space of fuzzy numbers can be regarded as a space of functions with bounded support, the first type of distance that comes in our mind is the distance generated by the uniform norm which is a Chebyshev type metric. Let us denote this metric with D_C. Then we have

$$D_C(A, B) = \sup_{x \in \mathbb{R}} |A(x) - B(x)|, A, B \in F(\mathbb{R}). \tag{1.5}$$

Fuzzy numbers of type (a,b,c,d) r,s

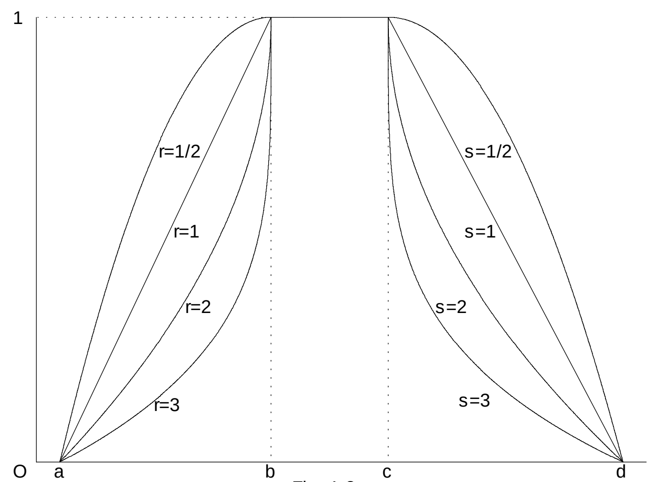

Fig. 1.2

Even if we cannot consider the pair $(F(\mathbb{R}),D_C)$ a normed space, to simplify on the notations sometimes we may denote $\|A-B\|_C = \sup_{x\in\mathbb{R}} |A(x)-B(x)|$. In particular we have

$$\|A\|_C = \sup_{x\in\mathbb{R}} |A(x)| \,. \tag{1.6}$$

Grzegorzewski ([58]) observed that for a fuzzy number A, the functions A_L and A_U are L_p-integrable. He introduced the metric $\delta_{p,q}$ given by

$$\delta_{p,q}(A,B) = \left[(1-q)\int_0^1 |A_L(\alpha)-B_L(\alpha)|^p \,d\alpha + q\int_0^1 |A_U(\alpha)-B_U(\alpha)|^p \,d\alpha\right]^{1/p},$$

where $1 \le p < \infty$ and $0 < q < 1$. When $p=2$ and $q=1/2$ then using the notation $d=(1/2)^{-1/2}\delta_{p,q}$ we obtain the so called Euclidean distance given by

$$d(A,B) = \left[\int_0^1 (A_L(\alpha)-B_L(\alpha))^2 \,d\alpha + \int_0^1 (A_U(\alpha)-B_U(\alpha))^2 \,d\alpha\right]^{1/2}. \tag{1.7}$$

As an application, it is immediate that if $T=(t_1,t_2,t_3,t_4)$ and $T'=(t_1',t_2',t_3',t_4')$, then after elementary calculations we obtain

$$\begin{aligned} d^2(T,T') &= \frac{1}{3}(t_1-t_1')^2 + \frac{1}{3}(t_2-t_2')^2 + \frac{1}{3}(t_1-t_1')(t_2-t_2') \\ &\quad + \frac{1}{3}(t_3-t_3')^2 + \frac{1}{3}(t_4-t_4')^2 + \frac{1}{3}(t_3-t_3')(t_4-t_4'). \end{aligned} \tag{1.8}$$

More generally, Yeh ([85]) proposed the weighted L_2-type distance d_λ,

$$d_\lambda(A,B) = \left[\int_0^1 (A_L(\alpha) - B_L(\alpha))^2 \lambda_L(\alpha) d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 \lambda_U(\alpha) d\alpha \right]^{1/2}, \quad (1.9)$$

where, in order to obtain indeed a metric, we suppose that $\lambda_L, \lambda_U : [0,1] \to \mathbb{R}$ are strictly positive almost everywhere on $[0,1]$ and integrable weight functions. We use the notation $\lambda = (\lambda_L, \lambda_U)$.

More generally, considering $p \geq 1$ and a weight $\lambda = (\lambda_L, \lambda_U)$, then the weighted L_p-type distance $\delta_{p,\lambda}$ is given by

$$\delta_{p,\lambda}(A,B) = \left[\int_0^1 |(A_L(\alpha) - B_L(\alpha))|^p \lambda_L(\alpha) d\alpha + \int_0^1 |(A_U(\alpha) - B_U(\alpha))|^p \lambda_U(\alpha) d\alpha \right]^{1/p}. \quad (1.10)$$

When $\lambda_L(\alpha) = \lambda_U(\alpha) = 1$, $\alpha \in [0,1]$, we prefer the notation

$$d_p(A,B) = \left[\int_0^1 |(A_L(\alpha) - B_L(\alpha))|^p d\alpha + \int_0^1 |(A_U(\alpha) - B_U(\alpha))|^p d\alpha \right]^{1/p}. \quad (1.11)$$

There are many other types of metrics defined on the space of fuzzy numbers such as the Hausdorff type metrics for example but since we will not use them in this thesis, we do not go into details.

1.10 Extended fuzzy numbers

This section contains original contributions from the paper [21]. Also, Definition 1.10.1 is new to my knowledge.

Yeh introduced for the first time the concept of an extended fuzzy number. In the paper [82] he introduced the so-called extended trapezoidal fuzzy numbers because he observed that with the use of them simpler algorithms to determine trapezoidal approximations of fuzzy numbers with respect to the Euclidean metric could be performed. Also, extended trapezoidal fuzzy numbers were used by Yeh in the proving of the continuity of trapezoidal and triangular approximation operators. To include all the types of extended fuzzy numbers that will be used in this thesis we give the following definition.

Definition 1.10.1 *An ordered pair of left continuous functions $A = (A_L, A_U)$, is called an extended fuzzy number if it satisfies the requirements:*

i) A_L is nondecreasing on $[0,1]$;

ii) A_U is nonincreasing on $[0,1]$.

We denote with $F_e(\mathbb{R})$ the space of extended fuzzy numbers.

As in the case of ordinary fuzzy numbers we use the notation $A_\alpha = [A_L(\alpha), A_U(\alpha)]$,$\alpha \in [0,1]$. Note that A_α may fail to be an interval for some $\alpha \in [0,1]$. Unless otherwise specified, addition and scalar multiplication on $F_e(\mathbb{R})$ are similarly defined as on $F(\mathbb{R})$.

Comparing the above definition with the parametric representation of a fuzzy number (see Definition 1.6.2) it is immediate that $F(\mathbb{R}) \subset F_e(\mathbb{R})$. Moreover, one can easily check that all the L_p-type metrics from the previous section can be extended to the space $F_e(\mathbb{R})$. For example, if we report to the

Euclidean metric, the distance between two extended fuzzy numbers or between an extended fuzzy number and a fuzzy number will be given by formula (1.7).

If

$$A_L(\alpha) = t_1 + (t_2 - t_1)\alpha \text{ and } A_U(\alpha) = t_4 - (t_4 - t_3)\alpha, \alpha \in [0,1],$$

with $t_1, t_2, t_3, t_4 \in \mathbb{R}$, then A becomes an extended trapezoidal fuzzy number and it will be denoted with $A = (t_1, t_2, t_3, t_4)$ as in the case of classical trapezoidal fuzzy numbers. The set of all extended trapezoidal fuzzy numbers is denoted with $F_e^T(\mathbb{R})$. Of course we have $F^T(\mathbb{R}) \subset F_e^T(\mathbb{R})$.

If

$$A_L(\alpha) = a - \sigma(1-\alpha)^{1/s_L} \text{ and } A_U(\alpha) = b + \beta(1-\alpha)^{1/s_R}, \alpha \in [0,1],$$

where $a, b, \sigma, \beta, s_L, s_R \in \mathbb{R}$, $\sigma \geq 0$, $\beta \geq 0$, $s_L > 0$, $s_R > 0$, then A becomes an extended parametric fuzzy number of type (s_L, s_R). Extended parametric fuzzy numbers were introduced for the first time in the paper [21]. The set of all extended parametric fuzzy numbers will be denoted with $F_e^{s_L,s_R}(\mathbb{R})$.

1.11 Other notations for extended fuzzy numbers

This section contains original contributions from the paper [21].

The notations for the trapezoidal fuzzy numbers are consecrated since they are used by numerous researchers. Still, sometimes other notations are more suitable for example when we work with L_2-type metrics. In this section we use for trapezoidal fuzzy numbers new notations introduced by Yeh in the papers [82] and [85]. Then for parametric fuzzy numbers of type (s_L, s_R) we use new notations introduced by Ban and Coroianu in the paper [21]. We start with notations for trapezoidal fuzzy numbers which are suitable with the Euclidean metric d as it will be seen later. One can easily verify that an extended trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$ can be written in the form

$$T_\alpha = \left[l + x\left(\alpha - \frac{1}{2}\right), u - y\left(\alpha - \frac{1}{2}\right) \right], \alpha \in [0,1], \tag{1.12}$$

where from relation (1.3) we easily get that

$$l = \frac{t_1 + t_2}{2}, \quad u = \frac{t_3 + t_4}{2}, \tag{1.13}$$

$$x = t_2 - t_1, \quad y = t_4 - t_3, \tag{1.14}$$

or, equivalently

$$t_1 = \frac{2l - x}{2}, \quad t_2 = \frac{2l + x}{2}, \tag{1.15}$$

$$t_3 = \frac{2u - y}{2}, \quad t_4 = \frac{2u + y}{2}. \tag{1.16}$$

An extended trapezoidal fuzzy number T given by (1.12) will be denoted $T = [l, u, x, y]$. From the above considerations it is immediate that T is a trapezoidal fuzzy number if and only if we have

$$x \geq 0, y \geq 0, 2u - 2l \geq x + y.$$

Now, if $T = [l, u, x, y]$ and $T' = [l', u', x', y']$ then from (1.7) and from (1.12), after some simple calculations we get

$$d^2(T, T') = (l - l')^2 + (u - u')^2 + \frac{1}{12}(x - x')^2 + \frac{1}{12}(y - y')^2. \tag{1.17}$$

Clearly, the above expression of the Euclidean distance between two extended trapezoidal fuzzy numbers is more convenient than formula (1.8). Other benefits will be seen in Chapter 3 where we will investigate on the approximation of fuzzy numbers by trapezoidal fuzzy numbers.

Now, let us consider the space of fuzzy numbers endowed with a weighted metric d_λ given by formula (1.9). Let us introduce the following notations:

$$a=\int_0^1 \lambda_L(\alpha)d\alpha, \quad b=\int_0^1 \lambda_U(\alpha)d\alpha, \tag{1.18}$$

$$\omega_L=\frac{1}{a}\int_0^1 \alpha\lambda_L(\alpha)d\alpha, \quad \omega_U=\frac{1}{b}\int_0^1 \alpha\lambda_U(\alpha)d\alpha, \tag{1.19}$$

$$c=\int_0^1 (\alpha-\omega_L)^2\lambda_L(\alpha)d\alpha, \quad d=\int_0^1 (\alpha-\omega_U)^2\lambda_U(\alpha)d\alpha. \tag{1.20}$$

One can easily prove that all the integrals from above are strictly positive. Next, let T be an extended trapezoidal fuzzy number given by

$$T_\alpha=[l+x(\alpha-\omega_L),u-y(\alpha-\omega_U)],\alpha\in[0,1]. \tag{1.21}$$

Such an extended trapezoidal fuzzy number will be denoted for simplicity with $T=[l,u,x,y]_\lambda$ (λ is a generic notation for the pair (λ_L,λ_U)). If $T=[l,u,x,y]_\lambda$ and $T'=[l',u',x',y']_\lambda$, the weighted distance between T and T' becomes (see Proposition 2.2 in [85])

$$d_\lambda^2(T,T')=a(l-l')^2+b(u-u')^2+c(x-x')^2+d(y-y')^2. \tag{1.22}$$

In what follows we present new notations for extended parametric fuzzy numbers of type (s_L,s_R), notations introduced in the paper [21]. For this purpose let $A=(a,b,\sigma,\beta)_{s_L,s_R}$ denotes an extended parametric fuzzy number of type (s_L,s_R). It follows that

$$A_L(\alpha)=a-\sigma(1-\alpha)^{1/s_L} \text{ and } A_U(\alpha)=b+\beta(1-\alpha)^{1/s_R},\alpha\in[0,1].$$

Because the functions A_L and A_U can be written in the form

$$\begin{aligned} A_L(\alpha) &= a-\sigma\frac{s_L}{s_L+1}-\sigma\left((1-\alpha)^{1/s_L}-\frac{s_L}{s_L+1}\right), \\ A_U(\alpha) &= b+\beta\frac{s_R}{s_R+1}+\beta\left((1-\alpha)^{1/s_R}-\frac{s_R}{s_R+1}\right), \end{aligned}$$

we obtain

$$A_L(\alpha) = l-x\left((1-\alpha)^{1/s_L}-\frac{s_L}{s_L+1}\right), \tag{1.23}$$

$$A_U(\alpha) = u+y\left((1-\alpha)^{1/s_R}-\frac{s_R}{s_R+1}\right), \tag{1.24}$$

another representation of the extended parametric fuzzy number A, where

$$l=a-\sigma\frac{s_L}{s_L+1}, \quad u=b+\beta\frac{s_R}{s_R+1}, x=\sigma, \quad y=\beta. \tag{1.25}$$

We denote by $[l, u, x, y]_{s_L, s_R}$ an extended parametric fuzzy number represented as in (1.23) and (1.24). When $s_L = s_R = 1$ then we obtain the representation of an extended trapezoidal fuzzy number and the indexes s_L, s_R will be omitted. If $A = [l, u, x, y]_{s_L, s_R}$ and $B = [l', u', x', y']_{s_L, s_R}$, the Euclidean distance between A and B becomes (see [21], Proposition 2)

$$\begin{aligned} & d^2(A, B) \\ = \quad & (l - l')^2 + (u - u')^2 + \frac{s_L}{(s_L + 2)(s_L + 1)^2}(x - x')^2 + \frac{s_R}{(s_R + 2)(s_R + 1)^2}(y - y')^2. \end{aligned} \tag{1.26}$$

Finally, let us note that $A = [l, u, x, y]_{s_L, s_R}$ is a proper parametric fuzzy number of type (s_L, s_R) if and only if

$$x \geq 0, y \geq 0, x \cdot \frac{s_L}{s_L + 1} + y \cdot \frac{s_R}{s_R + 1} \leq u - l. \tag{1.27}$$

1.12 Important characteristics of a fuzzy number

This section contains original contributions from the paper [28]. In addition, starting with Theorem 1.12.2 until the end, the section contains original unpublished results and some of these results may be included in some ongoing researches such as the approximation of fuzzy numbers by F-transform (see [48]).

The expected interval of a fuzzy number A was introduced independently by Dubois and Prade ([53]) and Heilpern ([65]). It is the real interval

$$EI(A) = \left[\int_0^1 A_L(\alpha) d\alpha, \int_0^1 A_U(\alpha) d\alpha \right]. \tag{1.28}$$

The expected value of the fuzzy number A is computed with the formula

$$EV(A) = \frac{1}{2} \left(\int_0^1 A_L(\alpha) d\alpha + \int_0^1 A_U(\alpha) d\alpha \right). \tag{1.29}$$

A reducing function ([49]) is a nondecreasing function $s : [0, 1] \to [0, 1]$ with the property that $s(0) = 0$ and $s(1) = 1$. However, in this thesis we relax the requirements by considering positive nondecreasing functions. Let A be a fuzzy number. The ambiguity of A with respect to s is

$$Amb_s(A) = \int_0^1 s(\alpha)(A_U(\alpha) - A_L(\alpha)) d\alpha \tag{1.30}$$

and the value of A with respect to s is

$$Val_s(A) = \int_0^1 s(\alpha)(A_U(\alpha) + A_L(\alpha)) d\alpha. \tag{1.31}$$

When $s = 1_{[0,1]}$, for simplicity we denote $Amb_s(A) = Amb(A)$ and $Val_s(A) = Val(A)$. Hence,

$$Amb(A) = \int_0^1 (A_U(\alpha) - A_L(\alpha)) d\alpha \tag{1.32}$$

and

$$Val(A) = \int_0^1 (A_U(\alpha) + A_L(\alpha))d\alpha. \tag{1.33}$$

The characteristics introduced in this section are identically defined when instead of fuzzy numbers we consider extended fuzzy numbers with the exception of the expected interval where a slight modification is needed in the sense that if A is an extended fuzzy number such that $\int_0^1 A_U(\alpha)d\alpha < \int_0^1 A_L(\alpha)d\alpha$, then $EI(A) = \left[\int_0^1 A_U(\alpha)d\alpha, \int_0^1 A_L(\alpha)d\alpha\right]$. Otherwise, if $\int_0^1 A_U(\alpha)d\alpha \geq \int_0^1 A_L(\alpha)d\alpha$, then $EI(A)$ has the classical definition, that is $EI(A) = \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha\right]$.

In what follows we will give an interpretation for the expected interval of a fuzzy number and also we will generalize this concept. Grzegorzewski ([59]) proved that for any fuzzy number A, $EI(A)$ is the unique nearest (with respect to the Euclidean distance d) interval fuzzy number to A, that is $d(A, EI(A)) = \min_{B\in Int(\mathbb{R})} d(A,B)$. In addition, it can be easily proved that the expected value of A is the unique nearest (with respect to the Euclidean distance d) crisp fuzzy number to A, that is $d(A, EV(A)) = \min_{c\in\mathbb{R}^c} d(A,c)$. The above considerations suggests that in the case of a weighted L_2-type metric we should adjust the definition of the expected interval so that the interpretation would be the same.

Definition 1.12.1 *([28], Definition 9) Let d_λ, $\lambda = (\lambda_L, \lambda_U)$ be a weighted L_2-type metric defined on $F(\mathbb{R})$ given by formula (1.9). For a fuzzy number A we call the weighted expected interval of A the interval*

$$EI^\lambda(A) = \left[\frac{1}{a}\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha, \frac{1}{b}\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha\right],$$

where a and b are introduced in relation (1.18).

We have

$$\begin{aligned}
&\frac{1}{a}\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha \leq \frac{1}{a}\int_0^1 A_L(1)\lambda_L(\alpha)d\alpha = A_L(1) \\
\leq\ & A_U(1) = \frac{1}{b}\int_0^1 A_U(1)\lambda_U(\alpha)d\alpha \leq \frac{1}{b}\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha,
\end{aligned}$$

therefore $EI^\lambda(A)$ is well-defined. The weighted expected value of A is given by

$$EV^\lambda(A) = \frac{1}{a+b}\left(a\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha + b\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha\right).$$

It can be proved that in the case of the weighted expected interval and in the case of the weighted expected value, we have the same interpretation with respect to the weighted metric d_λ, as in the case of the usual expected interval and expected value respectively. The extension of the weighted expected interval and of the weighted expected value for the case of extended fuzzy numbers is done in the same way as in the case of the usual ones.

All that follows in this section are original unpublished results.

So far, in this section the formulas of the expected interval, ambiguity or value are depending on the parametric representation of fuzzy numbers. If the fuzzy number is continuous then we can express these characteristics in terms of membership function. More exactly we have the following.

Theorem 1.12.2 *Let u denotes a continuous fuzzy number with supp$(u) = [a, b]$ and core$(u) = [c, d]$. Moreover, suppose that $s : [0, 1] \to [0, 1]$ is a continuous reduction function. Then we have*

$$\int_0^1 s(\alpha)u_L(\alpha)d\alpha = \int_a^c xd(S(u(x)) \text{ and } \int_0^1 s(\alpha)u_U(\alpha)d\alpha = -\int_d^b xd(S(u(x)) \tag{1.34}$$

In addition we obtain

$$Amb_s(u) = -\int_a^c xd(S(u(x)) - \int_d^b xd(S(u(x)) \text{ and } Val_s(u) = \int_a^c xd(S(u(x)) - \int_d^b xd(S(u(x)), \tag{1.35}$$

where $S(x) = \int_0^x s(t)dt,\ x \in [0, 1]$.

It worth to be noticed that the above formulas are immediate when the sides of u are strictly monotone. It suffices to use the change of variable in the Riemann-Stieltjes integral. However, if the sides are not strictly monotone than the proof seems to be much more technical and it also requires some auxiliary results.

Suppose that fuzzy number v approximates fuzzy number u with respect to the Chebyshev metric D_C given in (1.5), with a certain precision. How does this influence the quality of the approximation of the important characteristics of u using the characteristics of v? When we say important characteristics we refer to the expected interval, ambiguity and value respectively. Using the conclusions of the previous theorem we obtain.

Theorem 1.12.3 *Let u and v denote two continuous fuzzy numbers such that supp$(u) = [a, b]$, core$(u) = [c, d]$ and supp$(v) = [a', b']$, core$(v) = [c', d']$. Assume that $D_C(u, v) \le M$. If $s : [0, 1] \to [0, 1]$, is a continuous reducing function and $S(x) = \int_0^x s(t)dt,\ x \in [0, 1]$ then*

$$\left| \int_a^c xd(S(u(x)) - \int_{a'}^{c'} xd(S(v(x)) \right| \le M_1(u, v)M; \left| \int_d^b xd(S(u(x)) - \int_{d'}^{b'} xd(S(v(x)) \right| \le M_2(u, v)M, \tag{1.36}$$

where $M_1(u, v) = c - a + 3|a| + 3|c| + |a'| + 2|c'|$ and $M_2(u, v) = b - d + 3|b| + 3|d| + |b'| + 2|d'|$.

In addition we obtain

$$\max\{|Amb_s(u) - Amb_s(v)|, |Val_s(u) - Val_s(v)|\} \le (M_1(u, v) + M_2(u, v))\, M. \tag{1.37}$$

When v preserves better the shape of u then we obtain even better estimations.

Corollary 1.12.4 *Consider that we are under the same hypothesis as in Theorem 1.12.3.*
(i) If supp(u) = supp(v) then

$$\left|\int_a^c xd(S(u(x)) - \int_{a'}^{c'} xd(S(v(x))\right| \leq M_3(u,v)M \text{ and } \left|\int_d^b xd(S(u(x)) - \int_{d'}^{b'} xd(S(v(x))\right| \leq M_4(u,v)M,$$

where

$$M_3(u,v) = c - a + |c| + \min\{2|c| + 2|c'|, 2|c - c'| + |c|\}$$

and

$$M_4(u,v) = b - d + |d| + \min\{2|d| + 2|d'|, 2|d - d'| + |d|\}.$$

In addition we have

$$\max\{|Amb_s(u) - Amb_s(v)|, |Val_s(u) - Val_s(v)|\} \leq (M_3(u,v) + M_4(u,v))M.$$

(ii) If supp(u) = supp(v) and core(u) ⊆ core(v) then

$$\left|\int_a^c x\ (S(u(x)) - \int_{a'}^{c'} xd(S(v(x))\right| \leq (c - a + |c|)M; \quad \left|\int_d^b xd(S(u(x)) - \int_{d'}^{b'} xd(S(v(x))\right| \leq (b - d + |d|)M.$$

In addition we have

$$\max\{|Amb_s(u) - Amb_s(v)|, |Val_s(u) - Val_s(v)|\} \leq (c - a + |c| + b - d + |d|)M.$$

(iii) If supp(u) = supp(v) and core(v) ⊆ core(u) then

$$\left|\int_a^c x\ (S(u(x)) - \int_{a'}^{c'} xd(S(v(x))\right| \leq (c' - a + |c'|)M$$

and

$$\left|\int_d^b xd(S(u(x)) - \int_{d'}^{b'} xd(S(v(x))\right| \leq (b - d' + |d'|)M.$$

In addition we have

$$\max\{|Amb_s(u) - Amb_s(v)|, |Val_s(u) - Val_s(v)|\} \leq (c' - a + |c'| + b - d' + |d'|)M. \tag{1.38}$$

Analyzing the results obtained so far in this section we conclude that when we model a fuzzy number u approximating it by a fuzzy number v then, the better the shape of the fuzzy number is preserved the better estimation holds when we compare the ambiguity and the value, in particular the expected interval.

Chapter 2

Extended approximations, convergence, convexity and ranking in space of fuzzy numbers

This chapter contains original contributions from the papers [19], [21], [25], [41].

2.1 Approximations of fuzzy numbers by extended fuzzy numbers with simpler form

This section contains original contributions from the paper [21].

In this section we will find the algorithms to compute the nearest extended parametric fuzzy number to a given fuzzy number. Then, as a consequence we will obtain the algorithms to compute the nearest extended trapezoidal fuzzy number to a given fuzzy number. In both cases the results will be obtained with respect to the Euclidean distance. More generally, we will give the algorithms to compute the weighted trapezoidal approximation of a fuzzy number. We call it weighted approximation because the approximation is taken with respect to the weighted L_2-type distances introduced by Yeh. Finally, we will provide some important distance properties with respect to these approximations. The technique used to prove of the results of this section is inspired from the paper [81]. We will often use here the notations from Section 1.11 and therefore we suppose that the reader will easily recognize them.

So, let us choose arbitrary a fuzzy number A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0,1]$. For fixed $s_L > 0$

and $s_R > 0$ we introduce the extended parametric fuzzy number $A^e_{s_L,s_R} = [l_e, u_e, x_e, y_e]_{s_L,s_R}$, where

$$l_e = \int_0^1 A_L(\alpha)d\alpha, \quad u_e = \int_0^1 A_U(\alpha)d\alpha, \tag{2.1}$$

$$x_e = -\frac{(s_L+2)(s_L+1)^2}{s_L}\int_0^1 \left((1-\alpha)^{1/s_L} - \frac{s_L}{s_L+1}\right) A_L(\alpha)d\alpha, \tag{2.2}$$

$$y_e = \frac{(s_R+2)(s_R+1)^2}{s_R}\int_0^1 \left((1-\alpha)^{1/s_R} - \frac{s_R}{s_R+1}\right) A_U(\alpha)d\alpha. \tag{2.3}$$

When $s_L = s_R = 1$ then we obtain an extended trapezoidal fuzzy number denoted with $T_e(A) = [l_e, u_e, x_e, y_e]$ given by the equations

$$l_e = \int_0^1 A_L(\alpha)d\alpha, \quad u_e = \int_0^1 A_U(\alpha)d\alpha, \tag{2.4}$$

$$x_e = 12\int_0^1 (\alpha - 1/2)A_L(\alpha)d\alpha, \quad y_e = -12\int_0^1 (\alpha - 1/2)A_U(\alpha)d\alpha. \tag{2.5}$$

By Proposition 2.1 in [81] or by some simple verifications we have

$$T_e(\alpha A + \beta B) = \alpha T_e(A) + \beta T_e(B), \tag{2.6}$$

for every $A, B \in F(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$ which means that the operator $T_e : F(\mathbb{R}) \to F^T_e(\mathbb{R}),\ A \to T_e(A)$ is linear in respect to addition and scalar multiplication of fuzzy numbers.

We propose now some auxiliary result.

Lemma 2.1.1 *([21], Proposition 3) For any $s_L > 0, s_R > 0$ and for any fuzzy number A, we have:*

(i)

$$\int_0^1 \left(A_L(\alpha) - \left(A^e_{s_L,s_R}\right)_L(\alpha)\right) d\alpha = \int_0^1 \left(A_U(\alpha) - \left(A^e_{s_L,s_R}\right)_U(\alpha)\right) d\alpha = 0;$$

(ii)

$$\int_0^1 \left((1-\alpha)^{1/s_L} - \frac{s_L}{s_L+1}\right)\left(A_L(\alpha) - \left(A^e_{s_L,s_R}\right)_L(\alpha)\right) d\alpha = 0,$$

$$\int_0^1 \left((1-\alpha)^{1/s_R} - \frac{s_R}{s_R+1}\right)\left(A_U(\alpha) - \left(A^e_{s_L,s_R}\right)_U(\alpha)\right) d\alpha = 0.$$

Proposition 2.1.2 *([21], Proposition 4) For any $s_L > 0, s_R > 0$, one has*

$$d^2(A,B) = d^2(A, A^e_{s_L,s_R}) + d^2(A^e_{s_L,s_R}, B), (\forall)\ A \in F(\mathbb{R}), (\forall)\ B \in F^{s_L,s_R}_e(\mathbb{R}). \tag{2.7}$$

Corollary 2.1.3 *([81], Proposition 4.2.) One has*

$$d^2(A,B) = d^2(A,T_e(A)) + d^2(T_e(A),B), (\forall)\ A \in F(\mathbb{R}), (\forall)\ B \in F_e^{s_L,s_R}(\mathbb{R}). \tag{2.8}$$

Now, we easily obtain the following.

Theorem 2.1.4 *(see also Theorem 3 in [21]) If $A \in F(\mathbb{R})$ is arbitrarily chosen, then $A^e_{s_L,s_R}$ is the nearest extended (s_L, s_R) parametric fuzzy number to A with respect to the Euclidean distance d and it is unique with this property.*

As a corollary we easily deduce the following result already known from other papers (see e.g. [82]).

Corollary 2.1.5 *If $A \in F(\mathbb{R})$ is arbitrarily chosen, then $T_e(A)$ is the nearest extended trapezoidal fuzzy number to A with respect to the Euclidean distance d and it is unique with this property.*

In the case of weighted L_2-type distances, having in mind the notations presented in Section 1.11, we present some similar results as follows.

Theorem 2.1.6 *([85], Proposition 3.2) Let d_λ be a weighted L_2-type distance like in formula (1.9), with $\lambda = (\lambda_L, \lambda_U)$ and let A denotes a fuzzy number. Making use of relations (1.18)-(1.20) we introduce the following numerical values:*

$$l_e = \frac{1}{a}\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha, \quad u_e = \frac{1}{b}\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha, \tag{2.9}$$

$$x_e = \frac{1}{c}\int_0^1 A_L(\alpha)(\alpha - \omega_L)\lambda_L(\alpha)d\alpha, \quad y_e = \frac{1}{d}\int_0^1 A_U(\alpha)(\alpha - \omega_U)\lambda_U(\alpha)d\alpha. \tag{2.10}$$

Then, taking $T_{e,\lambda}(A) = [l_e, u_e, x_e, y_e]_\lambda$ (see again representation (1.21)), one has

$$d_\lambda^2(A,B) = d_\lambda^2(A,T_{e,\lambda}(A)) + d_\lambda^2(T_{e,\lambda}(A),B), (\forall)\ A \in F(\mathbb{R}), (\forall)\ B \in F_e^T(\mathbb{R}). \tag{2.11}$$

By similar reasonings as in the case of the Euclidean metric we obtain the following corollary.

Corollary 2.1.7 *([85], Proposition 3.3) If $A \in F(\mathbb{R})$ is arbitrarily chosen, then $T_{e,\lambda}(A)$ is the nearest extended trapezoidal fuzzy number to A with respect to the weighted distance d_λ and it is unique with this property.*

Let A denotes an arbitrary fuzzy number. Then let s_L, s_R be arbitrary strictly positive reals and d_λ, $\lambda = (\lambda_L, \lambda_R)$ be some weighted distance. From now one, $A^e_{s_L,s_R}$ is called the extended (s_L, s_R) parametric approximation of A, $T_e(A)$ is called the extended trapezoidal approximation of A and $T_{e,\lambda}(A)$ is called the weighted extended trapezoidal approximation of A.

We present now some properties of Lipschitz continuity.

Lemma 2.1.8 *(see also Theorem 8 in [21]) Let φ denotes the extended trapezoidal, extended weighted trapezoidal or extended parametric approximation operator and let D denotes the corresponding Euclidean or weighted L_2-type distance. Then, one has $D(\varphi(A), \varphi(B)) \leq D(A,B)$, for every $A, B \in F(\mathbb{R})$.*

We notice here that the above lemma generalizes Proposition 4.4 in [81] and its proof is similar with the proof of the before mentioned proposition.

Finally, we present an interesting property of the extended parametric approximation operator, in particular extended trapezoidal approximation operator, to preserve some important characteristics associated to fuzzy numbers.

Proposition 2.1.9 *([21], Proposition 7) The extended (s_L, s_R) parametric approximation of a fuzzy number A and the fuzzy number A have the same expected interval, that is $EI\left(A^e_{s_L,s_R}\right) = EI(A)$. If the reducing function $S : [0,1] \to [0,1]$ is defined by $S(\alpha) = 1 - (1-\alpha)^{1/s}$, $s > 0$, then the extended (s,s) parametric approximation of a fuzzy number A and the fuzzy number A have the same value and ambiguity, that is $Val_S\left(A^e_{s,s}\right) = Val_S(A)$ and $Amb_S\left(A^e_{s,s}\right) = Amb_S(A)$.*

2.2 Some convergence properties in the space of extended fuzzy numbers

All the results of this section can be found in the paper [19] for the particular case of a weighted distance d_λ, $\lambda = (\lambda_L, \lambda_U)$, $\lambda_L = \lambda_U$ and for the more restrictive case of trapezoidal fuzzy numbers. The main result of this section, Lemma 2.2.2, will be used later in the study of the continuity of the weighted trapezoidal approximation operator preserving the core.

Lemma 2.2.1 *([19], Lemma 2) If $(T_n)_{n\in\mathbb{N}}$, $T_n = (t_1(n), t_2(n), t_3(n), t_4(n))$, is a sequence of extended trapezoidal fuzzy numbers such that $\lim_{n\to\infty} t_i(n) = t_i < \infty$, $i = \overline{1,4}$, then $\lim_{n\to\infty} T_n = T$ with respect to any weighted metric d_λ, $\lambda = (\lambda_L, \lambda_U)$, where T is the extended trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$.*

Lemma 2.2.2 *([19], Lemma 3) If $(T_n)_{n\in\mathbb{N}}$, $T_n = (t_1(n), t_2(n), t_3(n), t_4(n))$, is a convergent sequence of extended trapezoidal fuzzy numbers with respect to a weighted metric d_λ, then its limit is an extended trapezoidal fuzzy number $T = (t_1, t_2, t_3, t_4)$ and in addition we have $\lim_{n\to\infty} t_i(n) = t_i$, $i = \overline{1,4}$.*

Using equations (1.13)-(1.14) and the previous two lemmas, we easily obtain the following.

Corollary 2.2.3 *If $(T_n)_{n\in\mathbb{N}}$, $T_n = [l(n), u(n), x(n), y(n)]$, is a convergent sequence of extended trapezoidal fuzzy numbers with respect to a weighted metric d_λ, then its limit is an extended trapezoidal fuzzy number $T = [l, u, x, y]$ and in addition we have $\lim_{n\to\infty} l(n) = l$, $\lim_{n\to\infty} u(n) = u$, $\lim_{n\to\infty} x(n) = x$ and $\lim_{n\to\infty} y(n) = y$. Conversely, if $T_n = [l(n), u(n), x(n), y(n)]$, is a sequence of extended trapezoidal fuzzy numbers such that $\lim_{n\to\infty} l(n) = l$, $\lim_{n\to\infty} u(n) = u$, $\lim_{n\to\infty} x(n) = x$ and $\lim_{n\to\infty} y(n) = y$, then $\lim_{n\to\infty} T_n = [l, u, x, y]$, with respect to any weighted metric d_λ.*

Remark 2.2.4 *Similar results can be obtained for extended parametric fuzzy numbers.*

2.3 Convexity in the space of fuzzy numbers

This section contains original contributions from the paper [41].

Usually, the concept of convex set is given in relation with a vector space structure. We already know that the addition and scalar multiplication of fuzzy numbers do not form a vector space. However, since these operations are closed in $F(\mathbb{R})$ and mostly because it will be of great help later in the obtaining of some important results of this thesis, we need the notion of convex set in the space of fuzzy numbers.

Definition 2.3.1 *([41], Definition 1) A nonempty set $\Omega \subseteq F(\mathbb{R})$ is called a convex subset of $F(\mathbb{R})$ if for all $A, B \in \Omega$ and $\gamma \in [0,1]$, we have $((1-\gamma)A + \gamma B) \in \Omega$.*

In the paper [41] some useful results are proved for some convex sets. For example, if $A, B \in F(\mathbb{R})$, $0 \leq \gamma_1 \leq \gamma_2 \leq ... \leq \gamma_n \leq 1$ and $C_i = (1-\gamma_n)A + \gamma_n B$ for any $i \in \{1, 2, ..., n\}$, then $[A, B] = [A, C_1] \cup [C_1, C_2] \cup ... \cup [C_{n-1}, C_n] \cup [C_n, B]$. Then for any $C \in [A, B]$ we have $d(A, B) = d(A, C) + d(C, B)$. Finally, if $A, B \in F(\mathbb{R})$, $A \neq B$ then $cl((A, B)) = cl((A, B]) = cl([A, B)) = [A, B]$. It has to be mentioned that since the space of fuzzy numbers is not a vector space the proofs are more technical comparing with the approaches which could be used in normed spaces.

2.4 A characterization of Lipschitz fuzzy number-valued functions

This section contains original contributions from the paper [41]. The following lemma can be proved by using the properties from the previous section.

Lemma 2.4.1 *([41], Lemma 8) Let $A, B \in F(\mathbb{R})$, $A \neq B$. Furthermore, we consider the family of closed convex subsets of $F(\mathbb{R})$, $\mathcal{F} = \{\Omega_i : i \in \{1, 2, ..., n\}\}$, such that $[A, B] \subseteq \bigcup_{i=1}^{n} \Omega_i$. Then, there exist $k \in \{1, 2, ..., n\}$, $\{C_j : j \in \{0, 1, ..., k\}\} \subseteq [A, B]$, with $C_0 = A$ and $C_k = B$ respectively, and $\{\Omega_{l_j} : j \in \{1, 2, ..., k\}\} \subseteq \mathcal{F}$, such that: i) $[A, B] = \bigcup_{j=1}^{k} [C_{j-1}, C_j]$; ii) $d(A, B) = \sum_{j=1}^{k} d(C_{j-1}, C_j)$; iii) $[C_{j-1}, C_j] \subseteq \Omega_{l_j}$ for all $j \in \{1, 2, ..., k\}$.*

From the previous result we obtain the following characterization of Lipschitz fuzzy number-valued functions .

Theorem 2.4.2 *([41], Theorem 9) Let $\mathcal{F} = \{\Omega_i : i \in \{1, 2, ..., n\}\}$ be a family of closed convex subsets of $F(\mathbb{R})$ such that that there exist the positive real constants, c_i, $i \in \{1, 2, ..., n\}$, such that for all $i \in \{1, 2, ..., n\}$ and $A, B \in \Omega_i$, we have $d(f(A), f(B)) \leq c_i d(A, B)$. Then $d(f(A), f(B)) \leq c d(A, B)$, $(\forall) A, B \in F(\mathbb{R})$, where $c = \max\{c_i : i \in \{1, 2, ..., n\}\}$.*

2.5 Reasonable properties for ranking fuzzy numbers

This section contains original contributions from the paper [25].

In the last decades hundreds of papers were devoted to study the ranking of fuzzy numbers. The ranking of fuzzy numbers is considered a very important necessity in fuzzy numbers theory but unfortunately it seems that an efficient method to rank fuzzy numbers is still an open question. The longer version of the thesis presents a very detailed study on this mater and moreover is presents in detail the state of the art in this topic as well as numerous approaches to rank fuzzy numbers. In this summary we will insist only on the main results obtained in this topic.

Suppose that $\mathcal{S}$ is a subset of $F(\mathbb{R})$. Then let us consider a defuzifier $P : \mathcal{S} \rightarrow \mathbb{R}$ which induces on $\mathcal{S}$ an order in the following way:

(i) $P(A) > P(B)$ if and only if $A \succ B$,

(ii) $P(A) < P(B)$ if and only if $A \prec B$,

(iii) $P(A) = P(B)$ if and only if $A \sim B$,
(iv) $P(A) \geq P(B)$ if and only if $A \succ B$ or $A \sim B$,
(v) $P(A) \leq P(B)$ if and only if $A \prec B$ or $A \sim B$.

When $A \succ B$ or $A \sim B$ we often denote $A \succeq B$ and, when $A \prec B$ or $A \sim B$ we may denote $A \preceq B$.

In order to find efficient methods for ordering fuzzy numbers we consider the following basic requirements for the order $\succeq$ on the set $\mathcal{S}$.

A_1) $A \succeq A$ for any $A \in \mathcal{S}$.

A_2) For any $(A, B) \in \mathcal{S}^2$, from $A \succeq B$ and $B \succeq A$ results $A \sim B$.

A_3) For any $(A, B, C) \in \mathcal{S}^3$, from $A \succeq B$ and $B \succeq C$ results $A \succeq C$.

A_4) For any $(A, B) \in \mathcal{S}^2$, from *inf supp*$(A) \geq$*sup supp*(B) results $A \succeq B$.

A'_4) For any $(A, B) \in \mathcal{S}^2$, from *inf supp*$(A) >$*sup supp*(B) results $A \succ B$.

A_5) Let $A, B, A + C$ and $B + C$ be elements of $\mathcal{S}$. If $A \succeq B$, then $A + C \succeq B + C$.

A'_5) Let $A, B, A + C$ and $B + C$ be elements of $\mathcal{S}$. If $A \succ B$, then $A + C \succ B + C$.

A_6) For any $(A, B) \in \mathcal{S}^2$ and $\lambda \in \mathbb{R}$ such that $\lambda A, \lambda B \in \mathcal{S}$, from $A \succeq B$ results $\lambda A \succeq \lambda B$ if $\lambda \geq 0$ and $\lambda A \preceq \lambda B$ if $\lambda \leq 0$.

Let us note that if A_3) holds then from $A \sim B$ and $B \sim C$ we get $A \sim C$.

Requirements $A_1) - A'_5)$ can be found in a more general setting (i.e. the ordering approach is not necessarily induced by a precise defuzifier) in the paper of Wang and Kerre ([78]). For this reason, in their paper the formulation of the basic properties is more abstract than above. Then it is easily seen that if the order $\succeq$ is generated by a defuzifier, properties $A_1) - A_3)$ hold. Property A_6) replaces property A_7) from the same paper of Wang and Kerre. They proposed somehow a stronger requirement by replacing λ in A_6) with positive fuzzy numbers (i.e. fuzzy numbers with the support included in $[0, \infty)$). However, since the multiplying of fuzzy numbers has not a commonly accepted formula we prefer requirement A_6) as it is considered in this thesis. Another reason why we consider A_6) in this form is that if $\mathcal{S}$ coincides with the set of trapezoidal fuzzy numbers then if A, B are trapezoidal fuzzy numbers then $A \cdot B$ may fail to be a trapezoidal fuzzy number. In particular, if A_6) holds then from $A \preceq B$ it results that $-A \succeq -B$, a property which is considered very important in many papers (see e.g. [5], [11], [54]). We also have to notice that at first impression requirement A_4) in [78] seems a little different with A_4) from this thesis. In [78] the requirement says that from *inf supp*$(A) >$*sup supp*(B) should result that $A \succeq B$. But it is very easy to prove that if $\mathbb{R} \subseteq \mathcal{S}$ and $\succeq$ is generated by a defuzifier P such that the restriction of P on $\mathbb{R}$, $P\mid_{\mathbb{R}}: \mathbb{R} \to \mathbb{R}$ is continuous, then A_4) in [78] and A_4) from the present thesis are equivalent. A very simple prove of this fact can be found in the paper [25]. In some recent papers (see e.g. [5], [12]) the following requirement is considered as an important reasonable property for a defuzifier $P : \mathcal{S} \to \mathbb{R}$.

$A"_4$) For any $A \in \mathcal{S}$, $P(A)$ must belong to its support.

In some sense we can say that it suffices to study only defuzifiers for which $A"_4$) holds. This is certified by the following theorem.

Theorem 2.5.1 *([25], Theorem 1) Suppose that $P : \mathcal{S} \to \mathbb{R}$ is a defuzifier which induces on $\mathcal{S}$ an order $\succeq$ which satisfies requirements $A_4) - A'_4)$ on S. If $\mathbb{R} \subseteq \mathcal{S}$ and the restriction of P on $\mathbb{R}$, $P\mid_{\mathbb{R}}: \mathbb{R} \to \mathbb{R}$ is continuous then there exists a defuzifier $P_1 : \mathcal{S} \to \mathbb{R}$ which satisfies requirement $A"_4$) and which generates on $\mathcal{S}$ an order $\succeq^1$ which is equivalent with $\succeq$, that is, for any $A, B \in \mathcal{S}$, from $A \succeq B$ it results $A \succeq^1 B$ and from $A \succ B$ it results $A \succ^1 B$.*

In the statement of the above theorem we have assumed that $P\mid_{\mathbb{R}}: \mathbb{R} \to \mathbb{R}$ is continuous but this requirement is not at all restrictive since one can easily observe that most defuzifiers that generate orders between fuzzy numbers are continuous. Actually, we could not find an approach that uses a defuzifier having discontinuities. This would also look quite unnatural. Therefore, we will consider only defuzifiers which satisfy the continuity assumption in Theorem 2.5.1.

In what follows, we will see that depending on the reasonable properties fulfilled by an order obtained from a defuzifier, we can provide some important properties of the defuzifier.

Theorem 2.5.2 *([25], Theorem 2) Suppose that $\mathcal{S}$ is a subset of $F(\mathbb{R})$ such that $\mathbb{R} \subseteq \mathcal{S}$ and $\mathcal{S}+\mathcal{S} \subseteq \mathcal{S}$. Then suppose that $R : \mathcal{S} \to \mathbb{R}$ is a defuzifier which satisfies requirement $A"_4$) and such that the order $\succeq$ generated by R on $\mathcal{S}$ satisfies requirement A_5) on $\mathcal{S}$. Then R is additive on $\mathcal{S}$ and in addition $\succeq$ satisfies requirement A'_5) on $\mathcal{S}$.*

From the above theorem we easily obtain the following corollaries.

Corollary 2.5.3 *([25], Corollary 3) Suppose that $\mathcal{S}$ is a subset of $F(\mathbb{R})$ such that $\mathbb{R} \subseteq \mathcal{S}$ and $\mathcal{S}+\mathcal{S} \subseteq \mathcal{S}$. Then suppose that $R : \mathcal{S} \to \mathbb{R}$ is a defuzifier which satisfies requirement $A"_4$). Then the order $\succeq$ generated by R on $\mathcal{S}$ satisfies requirement A_5) on $\mathcal{S}$ if and only if it satisfies requirement A'_5) on $\mathcal{S}$.*

Corollary 2.5.4 *([25], Corollary 4) Suppose that $\mathcal{S}$ is a subset of $F(\mathbb{R})$ such that $\mathbb{R} \subseteq \mathcal{S}$ and $\mathcal{S}+\mathcal{S} \subseteq \mathcal{S}$. If $R : \mathcal{S} \to \mathbb{R}$ is a defuzifier which generates an order $\succeq$ so that requirements A_4), A'_4) and A_5) respectively are satisfied by $\succeq$ on $\mathcal{S}$, then A'_5) is satisfied too by $\succeq$ on $\mathcal{S}$. In addition there exists an additive defuzifier $R_1 : \mathcal{S} \to \mathbb{R}$ which satisfies $A"_4$) on $\mathcal{S}$ and which generates on $\mathcal{S}$ an equivalent order with $\succeq$.*

Theorem 2.5.5 *([25], Theorem 5) Suppose that $\mathcal{S}$ is a subset of $F(\mathbb{R})$ such that $\mathbb{R} \subseteq \mathcal{S}$ and $\lambda \cdot \mathcal{S} \subseteq \mathcal{S}$, for all $\lambda \in \mathbb{R}$. Then suppose that $R : \mathcal{S} \to \mathbb{R}$ is a defuzifier which satisfies requirement $A"_4$) and such that the order $\succeq$ generated by R on $\mathcal{S}$ satisfies requirement A_6) on $\mathcal{S}$. Then R is scale invariant.*

Corollary 2.5.6 *([25], Corollary 6) Suppose that $\mathcal{S}$ is a subset of $F(\mathbb{R})$ such that $\mathbb{R} \subseteq \mathcal{S}$ and $\lambda \cdot \mathcal{S} \subseteq \mathcal{S}$, for all $\lambda \in \mathbb{R}$. If $R : \mathcal{S} \to \mathbb{R}$ is a defuzifier which generates an order $\succeq$ so that requirements A_4), A'_4) and A_6) respectively are satisfied by $\succeq$ on $\mathcal{S}$, then there exists a scale invariant defuzifier $R_1 : \mathcal{S} \to \mathbb{R}$ which satisfies $A"_4$) on $\mathcal{S}$ and which generates on $\mathcal{S}$ an equivalent order with $\succeq$.*

Obviously, an efficient ranking between fuzzy numbers from a set $\mathcal{S}$ should satisfy requirements $A_1) - A_6)$ from above. For this reason let us denote with $M(\mathcal{S})$ the set of all continuous defuzifiers defined on $\mathcal{S}$ and generating on $\mathcal{S}$ an order such that $A_1) - A_6)$ are satisfied. Then, inspired by Theorem 2.5.1, we consider the set $M_1(\mathcal{S})$ of all continuous defuzifiers where $P \in M_1(\mathcal{S})$ if and only if $A"_4$) holds for P and such that the order generated by P on $\mathcal{S}$ satisfies requirements $A_1) - A_3)$ and $A_5) - A_6)$. From the above considerations it results that in general $M_1(\mathcal{S})$ is strictly included in $M(\mathcal{S})$. But from Theorem 2.5.1 it also results that if some order $\succeq$ on $\mathcal{S}$ is generated by a defuzifier from $M(\mathcal{S})$ then there exists a defuzifier from $M_1(\mathcal{S})$ which generates an order $\succeq^1$over $\mathcal{S}$ which is equivalent with $\succeq$. Therefore, in order to find efficient orders over $\mathcal{S}$ it suffices to study only defuzifiers from $M_1(\mathcal{S})$ and this might just simplify the procedure of finding efficient orders on $\mathcal{S}$ since requirement $A"_4$) should simplify on the calculations part. This clearly is the case when $\mathcal{S} = F^T(\mathbb{R})$ as it will be seen in the next section. But before that, we conclude this section with some useful results in which we can characterize the elements from the classes $M(\mathcal{S})$ and $M_1(\mathcal{S})$ respectively.

Theorem 2.5.7 *([25], Theorem 7) Suppose that $\mathcal{S}$ is a subset of $F(\mathbb{R})$ such that $\mathbb{R} \subseteq \mathcal{S}$, $\mathcal{S} + \mathcal{S} \subseteq \mathcal{S}$ and $\lambda \cdot \mathcal{S} \subseteq \mathcal{S}$ for all $\lambda \in \mathbb{R}$. Moreover, let us consider some defuzifier $R : \mathcal{S} \to \mathbb{R}$. Then we have:*

(i) $R \in M_1(\mathcal{S})$ if and only if R satisfies $A"_4$) on $\mathcal{S}$ and R is linear on $\mathcal{S}$;

(ii) $R \in M(\mathcal{S})$ if and only if there exists $R_1 \in M_1(\mathcal{S})$ such that R and R_1 generate equivalent orders on $\mathcal{S}$.

2.6 Finding the class $M_1(F^T(\mathbb{R}))$

This section contains original contributions from the paper [25].

As stated in the title, in this section we will determine (Theorem 2.6.2) all the continuous defuzifiers of the class $M_1(F^T(\mathbb{R}))$ and hence, making abstraction of equivalent orders over $F^T(\mathbb{R})$ we will determine all of them which generate orders satisfying the reasonable properties $A_1) - A_6)$.

In this section and in the following two sections, sometimes we will use a new notation for trapezoidal fuzzy numbers which seems to be more suitable in obtaining the main results on the ranking of fuzzy numbers. The α-cut of a trapezoidal fuzzy number T can be written in the form (see [5])

$$T_\alpha = (x_0 - \sigma + \sigma\alpha, y_0 + \beta - \beta\alpha), \tag{2.12}$$

where $x_0, y_0, \sigma, \beta \in \mathbb{R}$, $\sigma \geq 0, \beta \geq 0$ and $x_0 \leq y_0$. We denote $T = (x_0, y_0, \sigma, \beta)$.

For a trapezoidal fuzzy number $T = (x_0, y_0, \sigma, \beta)$, let us consider the quantity

$$R(T) = ax_0 + by_0 + c\sigma + d\beta, \tag{2.13}$$

where $a, b, c, d \in \mathbb{R}$ are fixed. It is immediate that the function $R : F^T(\mathbb{R}) \to \mathbb{R}$ is additive and positively homogenous. Let us introduce the set $\Omega = \{R : R : F^T(\mathbb{R}) \to \mathbb{R}$ and R is of the form (2.13)$\}$. In the long version of the thesis it is proved proved that $M_1(F^T(\mathbb{R})) \subseteq \Omega$. This is why in the thesis at first defuzifiers from the set Ω are investigated. The most important result is the following corollary which unites some useful results which characterize different types of defuzifiers from Ω.

Corollary 2.6.1 *([25], Corollary 12) Let us consider a defuzifier $R \in \Omega$, $R(T) = ax_0 + by_0 + c\sigma + d\beta$ for $T = (x_0, y_0, \sigma, \beta)$. Then $R \in M_1(F^T(\mathbb{R}))$ if and only if $a = b = \frac{1}{2}$, $c \in [-1, 0]$, $c + d = 0$.*

We have mentioned above that $M_1(F^T(\mathbb{R})) \subseteq \Omega$. More exactly we have the following.

Theorem 2.6.2 *([25], Theorem 13) Let us consider a defuzifier $R : F^T(\mathbb{R}) \to \mathbb{R}$. Then $R \in M_1(F^T(\mathbb{R}))$ if and only if there exist $c \in [-1, 0]$ such that for some $T \in F^T(\mathbb{R})$, $T = (x_0, y_0, \sigma, \beta)$, we have*

$$R(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 + c\sigma - c\beta. \tag{2.14}$$

In addition it is easy to check (see also Theorem 2.5.7) that R is linear on $F^T(\mathbb{R})$.

In what follows, we will characterize classes of defuzifiers over $F^T(\mathbb{R})$ which generate orders satisfying entirely or just a part of the basic requirements $A_1) - A_6)$ on $F^T(\mathbb{R})$. Moreover, by Theorem 2.5.1 too, we can simplify the searching of such orders using equivalent orders that satisfy requirement $A_4^")$ on $F^T(\mathbb{R})$.

Corollary 2.6.3 *([25], Corollary 14) (i) If $R \in M(F^T(\mathbb{R}))$ then there exists $R_1 \in M_1(F^T(\mathbb{R}))$ such that the orders $\succeq$ and $\succeq^1$ over $F^T(\mathbb{R})$ generated by R and R_1 respectively are equivalent. In addition (by Theorem 2.6.2) there exist $c \in [-1, 0]$ such that for some $T \in F^T(\mathbb{R})$, $T = (x_0, y_0, \sigma, \beta)$, we have $R_1(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 + cx - cy$.*

(ii) If $R : F^T(\mathbb{R}) \to \mathbb{R}$ is a defuzifier such that the order $\succeq$ generated by R on $F^T(\mathbb{R})$ satisfies basic requirements $A_4), A_4')$ and $A_5)$, then $\succeq$ satisfies $A_5')$ too and moreover, there exists an additive defuzifier $R_1 : F^T(\mathbb{R}) \to \mathbb{R}$, which satisfies $A_4^")$ on $F^T(\mathbb{R})$ and which generates on $F^T(\mathbb{R})$ an equivalent order with $\succeq$.

(iii) If $R : F^T(\mathbb{R}) \to \mathbb{R}$ is a defuzifier such that the order $\succeq$ generated by R on $F^T(\mathbb{R})$ satisfies basic requirements $A_4), A_4')$ and $A_6)$, then there exists a scalar invariant defuzifier $R_2 : F^T(\mathbb{R}) \to \mathbb{R}$, which satisfies $A_4^")$ on $F^T(\mathbb{R})$ and which generates on $F^T(\mathbb{R})$ an equivalent order with $\succeq$.

2.7 Examples of ranking approaches

This section contains original contributions from the paper [25].

Example 2.7.1 *([25], Example 16) Let us consider the function* $EV : F^T(\mathbb{R}) \to \mathbb{R}$*, which for any trapezoidal fuzzy number* $T = (x_0, y_0, \sigma, \beta)$*, associates its expected value that is (with the present notations)* $EV(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 - \frac{1}{4}\sigma + \frac{1}{4}\beta$*. It is immediate that* $EV \in M_1(F^T(\mathbb{R}))$*. It seems that Yagger ([79]) has been the first who proposed to rank fuzzy numbers through their expected values. This procedure was considered also more recently in the paper [12].*

Example 2.7.2 *([25], Example 17) In the paper [5], the authors considered the magnitude of a trapezoidal fuzzy number, namely the function* $Mag : F^T(\mathbb{R}) \to \mathbb{R}$*,*

$$Mag(T) = \frac{1}{2}\left(\int_0^1 (T_L(\alpha) + T_U(\alpha) + x_0 + y_0) f(\alpha) d\alpha\right),$$

where $T = (x_0, y_0, \sigma, \beta)$ *is an arbitrary trapezoidal fuzzy number and* f *is a nonnegative and nondecreasing function on* $[0,1]$ *with* $f(0) = 0$*,* $f(1) = 1$ *and* $\int_0^1 f(\alpha)d\alpha = 1/2$*. Since by simple calculations we get* $Mag(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 + \frac{\sigma}{2}\int_0^1 f(\alpha)(\alpha - 1)d\alpha + \frac{\beta}{2}\int_0^1 f(\alpha)(1-\alpha)d\alpha$*, one can easily prove that* $Mag \in M_1(F^T(\mathbb{R}))$*. In the paper [5], the authors dealt with the particular case* $f(\alpha) = \alpha$*, when* $Mag(T) = \frac{1}{2}x_0 + \frac{1}{2}y_0 - \frac{1}{12}\sigma + \frac{1}{12}\beta$*.*

Example 2.7.3 *([25], Example 20) In the paper [3] the authors propose to rank fuzzy numbers by using* L_p*-type distances. Then in the paper [7] the same approach is proposed with a small modification. In what follows we will describe this approach. Let us fix a real* $p \geq 1$*and consider an arbitrary fuzzy number* A*. If* $\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha > 0$*, we consider the quantity* $\delta_p(A) = \left(\int_0^1 |A_L(\alpha)|^p d\alpha + \int_0^1 |A_U(\alpha)|^p d\alpha\right)^{1/p}$*. If* $\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha < 0$ *then we take* $\delta_p(A) = -\left(\int_0^1 |A_L(\alpha)|^p d\alpha + \int_0^1 |A_U(\alpha)|^p d\alpha\right)^{1/p}$*. Finally, if* $\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha = 0$ *then we take* $\delta_p(A) = 0$*. Note that in paper [3] this last case is included in the first one. Now, we can introduce the defuzifier* $\delta_p : F^T(\mathbb{R}) \to \mathbb{R}$ *and let* $\succeq$ *denotes the order induced by* δ_p *on* $F^T(\mathbb{R})$*. Let us check now the reasonable properties satisfied by* $\succeq$ *on* $F^T(\mathbb{R})$*. Obviously, requirements* $A_1) - A_3)$ *hold. Unfortunately, neither* $A_4)$ *nor* $A_4')$ *holds. Let us construct an counterexample. For some* $\varepsilon > 0$ *we consider the trapezoidal fuzzy number (we use standard notations)* $T_\varepsilon = (-11, 1, 5+\varepsilon, 5+\varepsilon)$*. We observe that* $\int_0^1 (T_\varepsilon)_L(\alpha)d\alpha + \int_0^1 (T_\varepsilon)_U(\alpha)d\alpha > 0$*, therefore* $\delta_p(T_\varepsilon) = \left(\int_0^1 |(T_\varepsilon)_L(\alpha)|^p d\alpha + \int_0^1 |(T_\varepsilon)_U(\alpha)|^p d\alpha\right)^{1/p}$*. By Hölder's inequality we obtain* $\int_0^1 |(T_\varepsilon)_L(\alpha)|^p d\alpha \geq \left(\int_0^1 |(T_\varepsilon)_L(\alpha)| d\alpha\right)^p$ *and since* $\int_0^1 |(T_\varepsilon)_L(\alpha)| d\alpha =$

5.083 *and* $\int_0^1 |(T_\varepsilon)_U(\alpha)|^p \, d\alpha = (5+\varepsilon)^p$, *we get* $\delta_p(T_\varepsilon) \geq ((5.083)^p + (5+\varepsilon)^p)^{1/p}$. *On the other hand, considering the crisp number* $5+\varepsilon$ *we easily observe that* $\delta_p(5+\varepsilon) = 2^{1/p}(5+\varepsilon)$. *Obviously, for sufficiently small* ε *we have* $\delta_p(T_\varepsilon) > \delta_p(5+\varepsilon)$, *which implies that* $T_\varepsilon \succ 5+\varepsilon$. *Since* $\sup(supp(T_\varepsilon)) = 5+\varepsilon$, *it easily results now that* $A_4)$ *does not hold in general on* $F^T(\mathbb{R})$. *Now considering the crisp number* $5+2\varepsilon$, *again it is very easy to prove that for sufficiently small* ε *we have* $\delta_p(T_\varepsilon) > \delta_p(5+2\varepsilon)$. *Clearly this implies that neither* $A_4')$ *holds. At first, let us correct the above shortcoming so that both basic requirements* $A_4)$ *and* $A_4')$ *would hold. Therefore, we will modify the defuzifier* δ_p *to a defuzifier* $\overline{\delta}_p$ *as follows. If* T *is either a positive trapezoidal fuzzy number (i.e.* $T_L(0) \geq 0$*) or a negative trapezoidal fuzzy number (i.e.* $T_U(0) \leq 0$*) then we take* $\overline{\delta}_p(T) = \delta_p(T)$. *If* T *is neither positive nor negative (i.e.* $T_L(0) < 0 < T_U(0)$*) then we distinguish three cases. If* $\int_0^1 T_L(\alpha)d\alpha + \int_0^1 T_U(\alpha)d\alpha = 0$ *then we take* $\overline{\delta}_p(T) = \delta_p(T) = 0$. *If* $\int_0^1 T_L(\alpha)d\alpha + \int_0^1 T_U(\alpha)d\alpha > 0$ *then we take* $\overline{\delta}_p(T) = \min(\delta_p(T), 2^{1/p}T_U(0))$. *Finally, if* $\int_0^1 T_L(\alpha)d\alpha + \int_0^1 T_U(\alpha)d\alpha < 0$ *then we take* $\overline{\delta}_p(T) = \max(\delta_p(T), 2^{1/p}T_L(0))$. *It is easy to prove that the order* $\succeq^1$ *generated by* $\overline{\delta}_p$ *on* $F^T(\mathbb{R})$ *satisfies both* $A_4)$ *and* $A_4')$. *It remains to study whether* $\succeq^1$ *satisfies basic requirements* $A_5), A_5')$ *or* $A_6)$ *respectively. Let us define the defuzifier* $\widetilde{\delta}_p : F^T(\mathbb{R}) \to \mathbb{R}$, $\widetilde{\delta}_p = 2^{-1/p}\,\overline{\delta}_p$ *and let* $\succeq^2$ *denotes the order induced by* $\widetilde{\delta}_p$ *on* $F^T(\mathbb{R})$. *Evidently* $\succeq^2$ *and* $\succeq^1$ *are equivalent. Then by simple calculations one can easily prove that* $\widetilde{\delta}_p$ *satisfies* $A_4'')$ *on* $F^T(\mathbb{R})$. *On the other hand we notice that* $\widetilde{\delta}_p$ *is not additive nor scale invariant on* $F^T(\mathbb{R})$ *and hence by Theorems 2.5.2 and 2.5.5 respectively, it follows that neither* $A_5)$ *nor* $A_6)$ *is satisfied in general by* $\succeq^2$ *on* $F^T(\mathbb{R})$. *Now, by Corollary 2.5.3 it results that* $A_5')$ *does not hold in general for the order* $\succeq^2$ *on* $F^T(\mathbb{R})$. *By the equivalence of the orders* $\succeq^1$ *and* $\succeq^2$ *respectively, it results finally that non of properties* $A_5), A_5')$ *and* $A_6)$ *respectively, are satisfied in general by* $\succeq^1$ *on* $F^T(\mathbb{R})$.

2.8 Ranking fuzzy numbers through trapezoidal fuzzy numbers

This section contains original contributions from the paper [25].

So far, we found the class $M_1(F^T(\mathbb{R}))$ consisting of all defuzifiers that generate orders over the space of fuzzy numbers satisfying requirements $A_1) - A_3)$, $A_4'')$ and $A_5) - A_6)$. In what follows, we will prove that for any $R \in M_1(F^T(\mathbb{R}))$ there exists $\overline{R} \in M_1(F(\mathbb{R}))$ such that $\overline{R}(T) = R(T)$ for any trapezoidal fuzzy number T . This means that $\overline{R}$ is an extension of R on $F(\mathbb{R})$ so that all the desirable properties hold.

Theorem 2.8.1 *([25], Theorem 21) Let us consider the trapezoidal valued operator* $T : F(\mathbb{R}) \to F^T(\mathbb{R})$ *and let us consider the defuzzification operators* $\overline{R} : F(\mathbb{R}) \to \mathbb{R}$ *and* $R \in M_1(F^T(\mathbb{R}))$. *Suppose that the following requirements hold:*

(i) T *is linear;*

(ii) supp$(T(A)) \subseteq$*supp*(A), *for all* $A \in F(\mathbb{R})$;

(iii) $\overline{R}(A) = R(T(A))$, *for all* $A \in F(\mathbb{R})$.
Then $\overline{R}$ *is linear on* $F(\mathbb{R})$ *and in addition* $\overline{R} \in M_1(F(\mathbb{R}))$.

Now, we easily obtain the following corollary.

Corollary 2.8.2 *([25], Corollary 22) Let us consider the defuzzification operator* $EV : F(\mathbb{R}) \rightarrow \mathbb{R}$ *which associates to a fuzzy number its expected value. Then* $EV \in M_1(F(\mathbb{R}))$.

The approach proposed in the above corollary has its shortcomings because sometimes we obtain some integrals which are computed with difficulty. We present now a results which is more convenient from computational point of view.

Corollary 2.8.3 *([25], Corollary 23) Let us consider the defuzifier* $\overline{R} : F(\mathbb{R}) \rightarrow \mathbb{R}$,

$$\overline{R}(A) = \left(A_L(0) + A_L(1) + A_U(0) + A_U(1)\right)/4.$$

Then $\overline{R} \in M_1(F(\mathbb{R}))$.

Using the same reasoning as we did to obtain the conclusion of the above corollary we can extend each order given by an operator $R \in M_1(F^T(\mathbb{R}))$ to an order over $F(\mathbb{R})$, which is very easy to handle from computational point of view and in addition satisfies the basic requirements considered in this thesis.

Theorem 2.8.4 *([25], Theorem 24) For* $R \in M_1(F^T(\mathbb{R}))$ *there exists* $\overline{R} : F(\mathbb{R}) \rightarrow \mathbb{R}$ *such that* $\overline{R} \in M_1(F(\mathbb{R}))$ *and in addition* $\overline{R}(T) = R(T)$, *for all* $T \in F^T(\mathbb{R})$.

Chapter 3

Approximations of fuzzy numbers by fuzzy numbers with simpler form

In this chapter we propose the basic methods to find algorithms to determine different kinds of parametric or trapezoidal approximations.

This chapter contains original contributions from the papers [17], [21]-[22], [27]-[28], [40]. In addition, this chapter contains original unpublished results which will be mentioned at the right moment.

3.1 Approximations of fuzzy numbers; a look back

This section presents the state of the art in the topic of the approximation of fuzzy number. We also mention that the content of this section can be found in paper [40] too.

In the last few years many papers investigated on the approximations of fuzzy numbers with respect to L_2-type metrics. Mostly, two kinds of problems are considered: approximations without any other restrictions and approximations with additional conditions. We recall here some important contributions with respect to the problem of the approximation of fuzzy numbers. Firstly, we discuss about approximations without additional conditions. Chanas ([38]) and Grzegorzewski ([59]) independently proposed the interval approximation of a fuzzy number. Grzegorzewski proved that the nearest interval approximation of a fuzzy number with respect to the Euclidean metric is actually its expected interval. Abbasbandy and Asady ([4]) proposed the trapezoidal approximation of a fuzzy number with respect to the same Euclidean distance. Yeh ([82]) proposed new algorithms for computing trapezoidal and triangular approximations of fuzzy numbers with respect to the Euclidean distance. Zeng and Li ([88]) proposed the triangular approximation of a fuzzy number with respect to a weighted L_2-type metric. Unfortunately, the algorithm proposed by them does not always produce proper triangular fuzzy numbers as it was pointed out in the papers [16] and [82]. The correct algorithm is given by Ban in the paper [16]. The most general result concerning approximations with trapezoidal or triangular fuzzy numbers is given by Yeh in the paper [85] where algorithms for computing trapezoidal or triangular approximations of fuzzy numbers with respect to general weighted L_2-type metrics are given. A more general approach has been proposed by Nasibov and Peker in [68] where they introduced the parametric approximation of a fuzzy number with respect to the Euclidean metric, result improved by Ban in [14]. Then Yeh ([84]) generalized these results by considering general weighted

L_2-type metrics. In the very recent paper [43], the authors propose algorithms to compute so called piecewise linear approximations of fuzzy numbers (with respect to the Euclidean metric) which are fuzzy numbers depending on 6 parameters. This is a first step towards approximations by fuzzy numbers which depend on n parameters. More details about piecewise linear approximations are given in the conclusions section. Now, we discuss about approximations of fuzzy numbers under additional conditions. Grzegorzewski and Mrówka ([62], [63]) proposed the trapezoidal approximation of a fuzzy number preserving the expected interval with respect to the Euclidean metric. Then Ban ([13]) and Yeh ([83]) independently improved their result. Algorithms for computing the trapezoidal approximation of a fuzzy number preserving the expected interval can be found in the papers [60] and [61]. Then Abbasbandy and Hajjari ([6]) proposed approximations of fuzzy numbers preserving the core with respect to weighted L_2-type metrics. Recently, Ban et al. ([17]) proposed the trapezoidal approximation preserving the value and ambiguity with respect to the Euclidean metric. More recently, Ban and Coroianu ([22]) proposed interval, triangular and trapezoidal approximations preserving the ambiguity with respect to the Euclidean metric. More generally, Ban and Coroianu ([21]) proposed simpler methods to compute the parametric approximation of a fuzzy number with respect to the Euclidean metric preserving important characteristics such as the expected interval or the ambiguity and value. Other types of non-linear approximations can be found in the paper of Grzegorzewski and Stefanini ([77]) where they proposed classes of fuzzy numbers depending on 4 or 5 parameters which allow approximations with conservation of multiple characteristics of a fuzzy number such as the support and core or the ambiguity and value.

3.2 Existence results and applications for important classes of fuzzy numbers

This section contains entirely original contributions. Some of the results can be found in the paper [27] too but in most cases the proofs are omitted. We also mention that Theorems 3.2.5 and 3.2.7 are completely new.

To obtain the main results of this section we need an old result of Rådström and also some well-known results in the problem of best approximation which will not be mentioned in this summary.

Theorem 3.2.1 *(see [72], Theorem 1) Let us consider a triplet* $(M,+,\cdot)$ *which forms a semilinear structure. Then there exist a vector space* $(\widetilde{M},\oplus,\odot)$ *and an injective application (inclusion)* $i:M\rightarrow\widetilde{M}$ *and, regarding* M *as a subset of* $\widetilde{M}$ *(that is adopting the convention* $i(x)=x$ *for all* $x\in M$*), we have* $a\oplus b=a+b$, $\lambda\odot a=\lambda\cdot a$, *for all* $a,b\in M$ *and* $\lambda\in[0,\infty)$. *If, in addition there exits a metric* d *defined on* M *satisfying:*

(i) $d(a+c,b+c)=d(a,b)$, *for all* $a,b,c\in M$;

(ii) $d(\lambda a,\lambda b)=\lambda d(a,b)$, *for all* $\lambda\in[0,\infty)$ *and* $a,b\in M$;

(iii) $+:M\times M\rightarrow M$ *and* $\cdot:[0,\infty)\times M\rightarrow M$, *are continuous on the topology generated by* d *on* M,

then there exists a norm $\|\cdot\|:\widetilde{M}\rightarrow[0,\infty)$ *such that the metric* $\widetilde{d}$ *on* $\widetilde{M}$ *generated by* $\|\cdot\|$ *satisfies* $d(a,b)=\widetilde{d}(a,b)$, *for all* $a,b\in M$.

Before we give existence results for parametric or trapezoidal approximations we will present some theoretical results which will help us to obtain the before mentioned existence results.

Theorem 3.2.2 *(see also [27], Theorem 9) Let* d *be a metric defined on the space of fuzzy numbers* $F(\mathbb{R})$ *which satisfies requirements (i)-(iii) of Theorem 3.2.1 and let* $\left(\widetilde{F(\mathbb{R})},\widetilde{d},\oplus,\odot\right)$ *be the normed* *space which realizes the embedding of* $(F(\mathbb{R}), d, +, \cdot)$ *according to Theorem 3.2.1 (recall that we already know that the space of fuzzy numbers is a semilinear space, see the end of Section 1.7) . Then, let us consider a subset* $\mathcal{A} \subseteq F(\mathbb{R})$*, for which there exist* $\{v_2, v_3, ..., v_m\} \subseteq \mathcal{A}$*, such that:*

(i) the system $\{1, v_2, ..., v_m\}$ *is linearly independent in the vector space* $\left(\widetilde{F(\mathbb{R})}, \oplus, \odot\right)$;

(ii) $\mathcal{A} = \left\{ \lambda_1 \cdot 1 + \sum_{k=2}^{m} \lambda_i v_i : \lambda_1 \in \mathbb{R} \text{ and } \lambda_i \in [0, \infty), i \in \{2, ..., m\} \right\}.$

Then, $\mathcal{A}$ *is a closed subset of* $F(\mathbb{R})$ *in the topology generated by the metric* d.

Another very useful result is the following.

Theorem 3.2.3 *(see also [27], Theorem 12) Let* Ω *be one of the following subsets of* $F(\mathbb{R})$*:* $F^{s_L, s_R}(\mathbb{R})$ *(for some fixed* s_L *and* s_R*),* $F^T(\mathbb{R})$*,* $F^\Delta(\mathbb{R})$*,* $Int(\mathbb{R})$*,* $\mathbb{R}^c$*. If* d *is a metric on* $F(\mathbb{R})$ *which satisfies requirements (i)-(iii) of Theorem 3.2.1 then* Ω *is a closed subset of* $F(\mathbb{R})$ *in the topology generated by* d *and, if* $\left(\widetilde{F(\mathbb{R})}, \widetilde{d}, \oplus, \odot\right)$ *is a normed space such that* $(F(\mathbb{R}), d, +, \cdot)$ *can be embedded in* $\left(\widetilde{F(\mathbb{R})}, \widetilde{d}, \oplus, \odot\right)$*, then* Ω *is a closed convex subset of* $\widetilde{F(\mathbb{R})}$ *in the topology generated by* $\widetilde{d}$*.*

From the above two theorems and together with several auxiliary results we obtain the following main result of this section.

Theorem 3.2.4 *(see also [27], Theorem 13) Let* Ω *be one of the following subsets of* $F(\mathbb{R})$*:* $F^{s_L, s_R}(\mathbb{R})$ *(for some fixed* s_L *and* s_R*),* $F^T(\mathbb{R})$*,* $F^\Delta(\mathbb{R})$*,* $Int(\mathbb{R})$*,* $\mathbb{R}^c$*. If* d *is a metric on* $F(\mathbb{R})$ *satisfying requirements (i)-(iii) of Theorem 3.2.1, then for any* $A \in F(\mathbb{R})$ *there exits* $A^* \in \Omega$ *such that* $d(A, A^*) = \inf_{B \in \Omega} d(A, B)$.

The above existence result is quite general since almost all the metrics defined on the space of fuzzy numbers satisfy the hypothesis in the above theorem. Under some stronger requirements we can prove uniqueness results as follows.

Theorem 3.2.5 *Let* Ω *be one of the following subsets of* $F(\mathbb{R})$*:* $F^{s_L, s_R}(\mathbb{R})$ *(for some fixed* s_L *and* s_R*),* $F^T(\mathbb{R})$*,* $F^\Delta(\mathbb{R})$*,* $Int(\mathbb{R})$*,* $\mathbb{R}^c$*. If* d *is a metric on* $F(\mathbb{R})$ *and if there exists a Hilbert space* $\left(\widetilde{F(\mathbb{R})}, \widetilde{d}, \oplus, \odot\right)$ *(*$\widetilde{d}$ *is the metric generated by the inner product which endows* $\widetilde{F(\mathbb{R})}$ *with a Hilbert space structure) such that* $(F(\mathbb{R}), d, +, \cdot)$ *can be embedded in* $\left(\widetilde{F(\mathbb{R})}, \widetilde{d}, \oplus, \odot\right)$*, then for any* $A \in F(\mathbb{R})$ *there exits* $A^* \in \Omega$ *such that* $d(A, A^*) = \inf_{B \in \Omega} d(A, B)$*, and in addition* A^* *is unique with this property.*

The next corollary is very important since in this thesis we will study approximation operators with respect to L_2-type metrics.

Corollary 3.2.6 *(see also [84] Proposition 4.1. for the case when* $\Omega = F^{s_L, s_R}(\mathbb{R})$*) Let* Ω *be one of the following subsets of* $F(\mathbb{R})$*:* $F^{s_L, s_R}(\mathbb{R})$ *(for some fixed* s_L *and* s_R*),* $F^T(\mathbb{R})$*,* $F^\Delta(\mathbb{R})$*,* $Int(\mathbb{R})$*,* $\mathbb{R}^c$*. If* d_λ *is any weighted* L_2 *metric on* $F(\mathbb{R})$ *(see (1.9)), then for any* $A \in F(\mathbb{R})$ *there exists a unique element* $A^* \in \Omega$ *such that* $d_\lambda(A, A^*) = \inf_{B \in \Omega} d_\lambda(A, B)$*.*

Finally, we present an existence and uniqueness result in the approximation by the metrics $\delta_{p,\lambda}$ given in (1.10). The proof uses the remarkable inequalities of Clarkson ([39]).

Theorem 3.2.7 *Let* Ω *be one of the following subsets of* $F(\mathbb{R})$*:* $F^{s_L, s_R}(\mathbb{R})$ *(for some fixed* s_L *and* s_R*),* $F^T(\mathbb{R})$*,* $F^\Delta(\mathbb{R})$*,* $Int(\mathbb{R})$*,* $\mathbb{R}^c$*. If* $\delta_{p,\lambda}$*,* $\lambda = (\lambda_L, \lambda_U)$*, is any weighted* L_p*-type metric on* $F(\mathbb{R})$ *(see (1.10)) such that* $p > 1$*, then for any* $A \in F(\mathbb{R})$ *there exists a unique element* $A^* \in \Omega$ *such that* $\delta_{p,\lambda}(A, A^*) = \inf_{B \in \Omega} \delta_{p,\lambda}(A, B)$*.*

3.3 Parametric and trapezoidal approximations without other restrictions

In this section we will prove (using an original approach) the existence and uniqueness of parametric or trapezoidal approximations without other restrictions. The key element in the obtaining of these results is Corollary 3.2.6 from the previous section.

Let us consider on the space of fuzzy numbers an arbitrary weighted metric d_λ, given by

$$d_\lambda(A,B) = \left[\int_0^1 (A_L(\alpha)-B_L(\alpha))^2\,\lambda_L(\alpha)d\alpha + \int_0^1 (A_U(\alpha)-B_U(\alpha))^2\,\lambda_U(\alpha)d\alpha\right]^{1/2}, A,B\in F(\mathbb{R}),$$

where $\lambda=(\lambda_L,\lambda_U)$, $\lambda_L,\lambda_U:[0,1]\to\mathbb{R}$, are strictly positive almost everywhere on $[0,1]$ and integrable weight functions. Then, let us fix $s_L>0$ and $s_R>0$.

Now, let us choose arbitrarily a fuzzy number A. By Corollary 3.2.6 we already know that there exits a unique parametric fuzzy number of type (s_L,s_R) denoted from now one with $A^*_{s_L,s_R}$ such that $d_\lambda(A,A^*_{s_L,s_R})=\inf_{B\in F^{s_L,s_R}(\mathbb{R})} d_\lambda(A,B)$. This means that we can define an operator $\Psi_{d_\lambda,s_L,s_R}: F(\mathbb{R})\to F^{s_L,s_R}(\mathbb{R})$, $\Psi_{d_\lambda,s_L,s_R}(A)=A^*_{s_L,s_R}$.

The operator Ψ_{d_λ,s_L,s_R} will be called weighted parametric approximation operator of type (s_L,s_R) or simply weighted parametric approximation operator when there is no risk of confusion. In the particular case when $\lambda_L=\lambda_U=1$, that is when d_λ equals the Euclidean distance d, we will denote $\Psi_{d_\lambda,s_L,s_R}=\Psi_{s_L,s_R}$. Then, in the particular case when $s_L=s_R=1$, we use the notation $\Psi_{d_\lambda,1,1}=T_{d_\lambda}$ and the operator T_{d_λ} will be called from now one the weighted trapezoidal approximation operator. The most particular case of trapezoidal approximations is when $\lambda_L=\lambda_U=1$ and $s_L=s_R=1$ and we obtain the so called trapezoidal approximation operator denoted from now one with T_d.

Now naturally the following question arises. Can we find algorithms to determine Ψ_{d_λ,s_L,s_R} for any $A\in F(\mathbb{R})$? A very technical solution of this problem is given by Yeh in the paper [84] where he obtains the algorithm to compute the weighted parametric approximations by using linear algebra and many characterizations of the best approximation in Hilbert spaces.

3.4 Parametric approximations under additional conditions

The content of this section is based on the original contributions from paper [21]. However, Theorems 3.4.1-3.4.2 and the reasonings used to obtain these theorems are original unpublished results.

Through out this section and all that follows in this chapter, we will compute the ambiguity, the value and the expected interval of a fuzzy number by using formulas (1.28), (1.30)-(1.33). In particular, if $B=[l,u,x,y]_{s,s}$ is an extended (s,s) (for some $s>0$) parametric fuzzy number with $l\le u$, and if we consider the reduction function S is given by $S(\alpha)=1-(1-\alpha)^{1/s}, \alpha\in[0,1]$, then by simple calculations (see also the proofs of Lemma 2.1.1 (i), (iii) and Proposition 2.1.9) we obtain

$$\begin{aligned} EI(B) &= [l,u], \\ Val_S(B) &= \frac{1}{1+s}(u+l)+\frac{s}{(1+s)^2(2+s)}(x-y), \\ Amb_S(B) &= \frac{1}{1+s}(u-l)-\frac{s}{(1+s)^2(2+s)}(x+y). \end{aligned}$$

Therefore, if A denotes a fuzzy number and $A^e_{s,s} = [l_e, u_e, x_e, y_e]_{s,s}$ denotes the extended (s, s) parametric approximation of A then

$$\begin{aligned} EI(A^e_{s,s}) &= [l_e, u_e], \\ Val_S(A^e_{s,s}) &= \frac{1}{1+s}(u_e + l_e) + \frac{s}{(1+s)^2(2+s)}(x_e - y_e), \\ Amb_S(A^e_{s,s}) &= \frac{1}{1+s}(u_e - l_e) - \frac{s}{(1+s)^2(2+s)}(x_e + y_e). \end{aligned}$$

As we know from the previous section, when $s = 1$ then $A^e_{s,s}$ coincides with the extended trapezoidal approximation of A denoted with $T_e(A)$ and hence we obtain

$$\begin{aligned} EI(T_e(A)) &= [l_e, u_e], \\ Val_S(T_e(A)) &= \frac{1}{2}(u_e + l_e) + \frac{1}{12}(x_e - y_e), \\ Amb_S(A^e_{s,s}) &= \frac{1}{2}(u_e - l_e) - \frac{1}{12}(x_e + y_e). \end{aligned}$$

In many papers the finding of the nearest parametric or trapezoidal fuzzy number to a fuzzy number such that some parameters are preserved is based on the Karush-Kuhn-Tucker theorem (see [13], [14], [62]), proposed to be used in this topic by Grzegorzewski and Mrówka ([62]). The method is sophisticated, with complicated calculus, because a system with a great number of equations and inequalities must be solved. The properties in Propositions 2.1.2 and 2.1.9 suggest us a possibility to simplify the calculus as follows.

Let us consider the problem to find the nearest parametric fuzzy number (with respect to the Euclidean metric d) to a given fuzzy number A such that the parameters $P_k, k \in \{1, ..., n\}$ are preserved, that is

$$\begin{gathered} \min_{B \in F^{s_L, s_R}(\mathbb{R})} d(A, B), \\ P_k(A) = P_k(B), (\forall) k \in \{1, ..., n\}. \end{gathered} \tag{3.1}$$

Here any parameter P_K is described as a function (defuzzification operator) $P_K : F_e(\mathbb{R}) \to \mathbb{R}$. If the extended (s_L, s_R) parametric approximation $A^e_{s_L,s_R}$ (see Section 2.1) of A preserves the parameters $P_k, k \in \{1, ..., n\}$, that is

$$P_k\left(A^e_{s_L,s_R}\right) = P_k(A), (\forall) k \in \{1, ..., n\},$$

then, because of (2.7), the problem (3.1) is equivalent to

$$\begin{gathered} \min_{B \in F^{s_L, s_R}(\mathbb{R})} d\left(A^e_{s_L,s_R}, B\right), \\ P_k\left(A^e_{s_L,s_R}\right) = P_k(B), (\forall) k \in \{1, ..., n\}. \end{gathered} \tag{3.2}$$

A) Nearest parametric fuzzy number preserving the expected interval

For fixed $s_L > 0$, $s_R > 0$ we consider the problem

$$\begin{gathered} \min_{B \in F^{s_L, s_R}(\mathbb{R})} d(A, B), \\ EI(A) = EI(B), \end{gathered} \tag{3.3}$$

where $A \in F(\mathbb{R})$ is fixed, $A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1]$. We could try to solve this problem by using the Karush-Kuhn-Tucker theorem but this method seems to be very complicated for this particular problem. Therefore, we propose a different approach which will imply the existence and uniqueness of the solution. Taking into account the previous discussion and Proposition 2.1.9, (3.3) is equivalent to

$$\min_{B \in F^{s_L, s_R}(\mathbb{R})} d\left(A^e_{s_L,s_R}, B\right), \tag{3.4}$$
$$EI\left(A^e_{s_L,s_R}\right) = EI(B),$$

where $A^e_{s_L,s_R} = [l_e, u_e, x_e, y_e]_{s_L,s_R}$ is the extended (s_L, s_R) parametric approximation of A. After simple calculations (see again relations (1.27) which characterize the set of parametric fuzzy numbers of type (s_L, s_R)) (3.4) becomes equivalent with (we used (1.26) to express $d\left(A^e_{s_L,s_R}, B\right)$)

$$\min_{l,u,x,y \in \mathbb{R}} \left((l_e - l)^2 + (u_e - u)^2 + \tfrac{s_L}{(s_L+2)(s_L+1)^2}(x_e - x)^2 + \tfrac{s_R}{(s_R+2)(s_R+1)^2}(y_e - y)^2\right), \tag{3.5}$$
$$l = l_e, u = u_e, x \geq 0, y \geq 0, l + \tfrac{s_L}{s_L+1}x \leq u - \tfrac{s_R}{s_R+1}y. \tag{3.6}$$

Therefore, if (l_0, u_0, x_0, y_0) is a solution of the above problem then necessarily $l_0 = l_e$, $u_0 = u_e$ and (x_0, y_0) is a solution of the problem

$$\min_{x,y \in \mathbb{R}} \left(\tfrac{s_L}{(s_L+2)(s_L+1)^2}(x_e - x)^2 + \tfrac{s_R}{(s_R+2)(s_R+1)^2}(y_e - y)^2\right)$$
$$x \geq 0, y \geq 0, \tfrac{s_L}{s_L+1}x + \tfrac{s_R}{s_R+1}y \leq u_e - l_e,$$

which is a minimization problem in $\mathbb{R}^2$. In what follows we prove that this later problem has (always) a unique solution. Indeed, the problem can be written in the equivalent form

$$\min_{(x,y) \in M_A} \left(\tfrac{s_L}{(s_L+2)(s_L+1)^2}(x_e - x)^2 + \tfrac{s_R}{(s_R+2)(s_R+1)^2}(y_e - y)^2\right)$$

where $M_A = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, \frac{s_L}{s_L+1}x + \frac{s_R}{s_R+1}y \leq u_e - l_e\}$. But since $u_e - l_e \geq 0$ it is easily seen that M_A is non-empty. Moreover, it is easy to check that M_A is a closed convex subset of $\mathbb{R}^2$. Then, let us define on $\mathbb{R}^2$ an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$,

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \tfrac{s_L}{(s_L+2)(s_L+1)^2}x_1x_2 + \tfrac{s_R}{(s_R+2)(s_R+1)^2}y_1y_2.$$

This inner product generates a metric $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$D^2((x_1, y_1), (x_2, y_2)) = \tfrac{s_L}{(s_L+2)(s_L+1)^2}(x_1 - x_2)^2 + \tfrac{s_R}{(s_R+2)(s_R+1)^2}(y_1 - y_2)^2.$$

Since the metric is of Euclidean type (is very easy to prove that D is equivalent with the Euclidean metric on $\mathbb{R}^2$), we conclude that $\left(\mathbb{R}^2, D\right)$ is a Hilbert space. Summarizing, the minimization problem is equivalent now with

$$\min_{(x,y) \in M_A} D((x,y),(x_e,y_e)),$$

and noting again that M_A is a non-empty closed convex subset of $\mathbb{R}^2$, it follows by elementary convex analysis that the problem has a unique solution (x_0, y_0) which is the orthogonal projection of (x_e, y_e) with respect to the metric D onto the set M_A. In conclusion we have the following theorem of existence and uniqueness of the parametric approximation operator preserving the expected interval.

Theorem 3.4.1 *Let us consider arbitrarily $s_L > 0$ and $s_R > 0$. Then for any fuzzy number A there exists a unique parametric (s_L, s_R) fuzzy number $\Psi_{EI}^{s_L,s_R}(A)$, so that $EI(A) = EI(\Psi_{EI}^{s_L,s_R}(A))$ and which satisfies the property that for any parametric (s_L, s_R) fuzzy number B satisfying $EI(A) = EI(B)$, we have $d(A, \Psi_{EI}^{s_L,s_R}(A)) \leq d(A, B)$. That is, $\Psi_{EI}^{s_L,s_R}(A)$ is the nearest parametric (s_L, s_R) fuzzy number to A with respect to the metric d which preserves the expected interval of A.*

The operator $\Psi_{EI}^{s_L,s_R} : F(\mathbb{R}) \to F^{s_L,s_R}(\mathbb{R})$ will be called parametric (of type (s_L, s_R)) approximation operator preserving the expected interval.

B) Nearest parametric fuzzy number preserving the value and the ambiguity

We consider the problem

$$\min_{B\in F^{s,s}(\mathbb{R})} d(A,B),$$
$$Val_S(A) = Val_S(B), Amb_S(A) = Amb_S(B), \tag{3.7}$$

where $A \in F(\mathbb{R})$ is fixed, $A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1]$ and the reducing function S is given by $S(\alpha) = 1 - (1-\alpha)^{1/s}, s > 0$. Taking into account the discussion from the beginning of this section and Proposition 2.1.9, the minimization problem is equivalent to

$$\min_{B\in F^{s,s}(\mathbb{R})} d\left(A^e_{s,s}, B\right),$$
$$Val_S\left(A^e_{s,s}\right) = Val_S(B), Amb_S\left(A^e_{s,s}\right) = Amb_S(B), \tag{3.8}$$

where $A^e_{s,s} = [l_e, u_e, x_e, y_e]_{s,s}$ is the extended (s,s) parametric approximation of A. After simple calculations, (3.8) becomes a minimization problem in $\mathbb{R}^4$,

$$\min_{l,u,x,y\in\mathbb{R}} \left((l_e - l)^2 + (u_e - u)^2 + \tfrac{s}{(s+2)(s+1)^2}\left((x_e - x)^2 + (y_e - y)^2\right)\right), \tag{3.9}$$

$$u + l + \frac{s}{(1+s)(2+s)}(x - y) = u_e + l_e + \frac{s}{(1+s)(2+s)}(x_e - y_e), \tag{3.10}$$

$$u - l - \frac{s}{(1+s)(2+s)}(x + y) = u_e - l_e - \frac{s}{(1+s)(2+s)}(x_e + y_e), \tag{3.11}$$

$$x \geq 0, y \geq 0, l + \frac{s}{s+1}x \leq u - \frac{s}{s+1}y. \tag{3.12}$$

Conditions (3.10) and (3.11) imply $u - u_e = \frac{s}{(1+s)(2+s)}(y - y_e)$ and $l - l_e = \frac{s}{(1+s)(2+s)}(x_e - x)$, therefore if (l_0, u_0, x_0, y_0) is a solution of (3.9)-(3.12) then necessarily (x_0, y_0) is a solution of

$$\min_{x,y\in\mathbb{R}} \left((x_e - x)^2 + (y_e - y)^2\right),$$
$$x \geq 0, y \geq 0, x + y \leq \frac{2+s}{s}(u_e - l_e) - \frac{1}{1+s}(x_e + y_e),$$

which is a minimization problem in $\mathbb{R}^2$ with respect to the Euclidean distance from $\mathbb{R}^2$. It is easy to prove that the problem has a unique solution. Firstly, by some elementary calculus (see also equations (2.1)-(2.3)) we get that

$$\frac{2+s}{s}(u_e - l_e) - \frac{1}{1+s}(x_e + y_e)$$
$$= \frac{(2+s)(1+s)}{s}\int_0^1 (A_U(\alpha) - A_L(\alpha))\left(1 - (1-\alpha)^{1/s}\right)d\alpha \geq 0 \tag{3.13}$$

and from here it easily follows that we actually have a Euclidean distance minimization problem with respect to a non-empty closed convex set which means that the solution is the orthogonal projection of (x_e, y_e) onto the non-empty closed convex set given by the inequalities from the minimization problem.

In the case $s = 1$ the solution of the above minimization problem supplies the nearest trapezoidal approximation of the initial fuzzy number, which preserves its value and ambiguity. Summarizing, we obtain the following existence and uniqueness result.

Theorem 3.4.2 *Let us consider the reduction function S, $S(\alpha) = 1 - (1-\alpha)^{1/s}$, $s > 0$. Then for any $A \in F(\mathbb{R})$ there exists a unique (s,s) parametric fuzzy number denoted with $\Psi^s_{AV}(A)$, such that $Amb_S(A) = Amb_S(\Psi^s_{AV}(A))$, $Val_S(A) = Val_S(\Psi^s_{AV}(A))$ and for any (s,s) parametric fuzzy number B satisfying $Amb_S(A) = Amb_S(B)$, $Val_S(A) = Val_S(B)$, we have $d(A, \Psi^s_{AV}(A)) \leq d(A,B)$. That is, $\Psi^s_{AV}(A)$ is the nearest parametric (s,s) fuzzy number to A with respect to the metric d which preserves the ambiguity and the value of A with respect to the reducing function S.*

The operator $\Psi^s_{AV} : F(\mathbb{R}) \to F^{s_L,s_R}(\mathbb{R})$ will be called parametric (of type (s,s)) approximation operator preserving the value and ambiguity with respect to the reducing function S.

3.5 Trapezoidal approximations preserving the expected interval

From the more general case of the parametric approximations we will deduce the minimization problem which gives the trapezoidal approximation preserving the expected interval (see also the paper [83]).

In this section, for a fuzzy number A we are interested in finding a trapezoidal fuzzy number denoted with $T_{EI}(A)$, with the property that if T is a trapezoidal fuzzy number satisfying $EI(A) = EI(T)$, then $d(A, T_{EI}(A) \leq d(A,T)$ where as usual, d denotes the Euclidean metric between fuzzy numbers. By Theorem 3.4.1 taking there $s_L = s_R = 1$, we already know that the trapezoidal approximation preserving the expected interval always exists and in addition it is unique. Hence we can define the operator $T_{EI} : F(\mathbb{R}) \to F^T(\mathbb{R})$, which will be called the trapezoidal approximation operator preserving the expected interval. Firstly, let us notice that $[l,u,x,y]_{1,1} = [l,u,x,y]$ whenever the quadruple (l,u,x,y) represents a trapezoidal fuzzy number since it is easy to check that formulas (1.23)-(1.24) give the representation (1.12) in the particular case when $s_L = s_R = 1$. Similarly, we observe that in this particular case the coordinates of $A^e_{1,1}$ coincide with the coordinates of $T_e(A)$ for any fuzzy number A. Recall, $T_e(A)$ is the extended trapezoidal approximation of A which can be computed by using equations (2.4)-(2.5). Therefore, from the minimization problem which gives the parametric approximation preserving the expected interval we easily obtain the minimization problem which gives the trapezoidal approximation preserving the expected interval. This implies that following the steps of the minimization problem which gives the parametric approximation preserving the expected interval, we will obtain the minimization problem which gives the trapezoidal approximation preserving the expected interval. Therefore, let us choose arbitrarily a fuzzy number A and let us denote with $T_{EI}(A) = [l_0, u_0, x_0, y_0]$ its trapezoidal approximation preserving the expected interval. By relations (3.5)-(3.6) (remember $s_L = s_R = 1$) it results that (l_0, u_0, x_0, y_0) solves the minimization problem

$$\min_{l,u,x,y \in \mathbb{R}} \left((l_e - l)^2 + (u_e - u)^2 + \tfrac{1}{12} (x_e - x)^2 + \tfrac{1}{12} (y_e - y)^2 \right),$$
$$l = l_e, u = u_e, x \geq 0, y \geq 0, l + \tfrac{1}{2}x \leq u - \tfrac{1}{2}y.$$

It is immediate that we have two out of the four components of $T_{EI}(A)$, $l_0 = l_e$ and $u_0 = u_e$. The remaining two components are obtained as solutions of the minimization problem

$$\min_{l,u,x,y\in\mathbb{R}} (x_e - x)^2 + (y_e - y)^2,$$
$$x \geq 0, y \geq 0, x + y \leq 2u_e - 2l_e.$$

It easily results that the pair (x_0, y_0) is the orthogonal projection (x_e, y_e) onto the non-empty set M_A with respect to the Euclidean metric on $\mathbb{R}^2$ denoted with d_E (see Fig. 3.1), where

$$M_A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 2u_e - 2l_e\}. \tag{3.14}$$

Therefore, we can write

$$(x_0, y_0) = P_{M_A}(x_e, y_e). \tag{3.15}$$

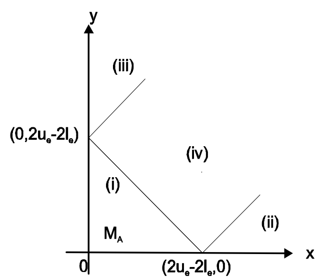

Fig. 3.1

We obtain an algorithm (corresponding to (i), (ii), (iii) and (iv) in Fig. 3.1) in four steps as follows.

Theorem 3.5.1 *([83], Theorem 4.2.) Let A, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0, 1]$, be a fuzzy number, $T_e(A) = [l_e, u_e, x_e, y_e]$, be the extended trapezoidal approximation of A and $T_{EI}(A) = [l_0, u_0, x_0, y_0]$, be the nearest (with respect to the metric d) trapezoidal fuzzy number to fuzzy number A preserving the expected interval of A.*

(i) If $x_e + y_e - 2u_e + 2l_e \leq 0$, then

$$l_0 = l_e, u_0 = u_e, x_0 = x_e, y_0 = y_e.$$

(ii) If $x_e - y_e - 2u_e + 2l_e \geq 0$, then

$$l_0 = l_e, u_0 = u_e, x_0 = 2u_e - 2l_e, y_0 = 0.$$

(iii) If $x_e - y_e + 2u_e - 2l_e \leq 0$, then

$$l_0 = l_e, u_0 = u_e, x_0 = 0, y_0 = 2u_e - 2l_e.$$

(iv) If

$$x_e + y_e - 2u_e + 2l_e \geq 0, \tag{3.16}$$
$$x_e - y_e - 2u_e + 2l_e \leq 0, \tag{3.17}$$
$$x_e - y_e + 2u_e - 2l_e \geq 0, \tag{3.18}$$

then

$$l_0 = l_e, u_0 = u_e, \tag{3.19}$$

$$x_0 = -l_e + u_e + \frac{1}{2}x_e - \frac{1}{2}y_e, \tag{3.20}$$

$$y_0 = -l_e + u_e - \frac{1}{2}x_e + \frac{1}{2}y_e. \tag{3.21}$$

In the above theorem the cases are not disjoint as in the paper of Yeh because this small modification by taking everywhere non-strict inequalities will help us later when we will find the best Lipschitz constant of the operator T_{EI}.

Now, using equations (1.15)-(1.16) and (2.4)-(2.5), we can obtain an equivalent theorem using the classical representation of trapezoidal fuzzy numbers which can be found in Theorem 7 the paper [13].

3.6 Trapezoidal approximations preserving the ambiguity and value

This section contains original contributions from the paper [17].

In this section, for a fuzzy number A we are interested in finding a trapezoidal fuzzy number denoted with $T_{AV}(A)$ with the property that if T is a trapezoidal fuzzy number satisfying $Val(A) = Val(T)$ and $Amb(A) = Amb(T)$, then $d(A, T_{AV}(A) \leq d(A,T)$. By Theorem 3.4.2 taking there $s = 1$, we already know that the trapezoidal approximation preserving the ambiguity and value always exists and in addition it is unique. Hence we can define the operator $T_{AV} : F(\mathbb{R}) \to F^T(\mathbb{R})$ which will be called the trapezoidal approximation operator preserving the ambiguity and value. Let us choose arbitrarily a fuzzy number A and let us denote with $T_{AV}(A) = [l_0, u_0, x_0, y_0]$ its trapezoidal approximation preserving the ambiguity and value. By relations (3.9)-(3.12) (remember $s = 1$) it results that (l_0, u_0, x_0, y_0) solves the minimization problem

$$\min_{l,u,x,y\in\mathbb{R}} \left((l_e - l)^2 + (u_e - u)^2 + \tfrac{1}{12}\left((x_e - x)^2 + (y_e - y)^2 \right) \right),$$

$$u + l + \frac{1}{6}(x - y) = u_e + l_e + \frac{1}{6}(x_e - y_e),$$

$$u - l - \frac{1}{6}(x + y) = u_e - l_e - \frac{1}{6}(x_e + y_e),$$

$$x \geq 0, y \geq 0, l + \frac{1}{2}x \leq u - \frac{1}{2}y.$$

As in the case of the approximations preserving the expected interval, we easily substitute two components of the trapezoid,

$$l_0 = -\frac{1}{6}(x_0 - x_e) + l_e \tag{3.22}$$

and

$$u_0 = \frac{1}{6}(y_0 - y_e) + u_e. \tag{3.23}$$

Therefore, it suffices to provide a minimization problem which gives (x_0, y_0). Substituting l and u in the previous minimization problem, after some simple calculations we obtain that (x_0, y_0) solves the minimization problem

$$\min \left((x - x_e)^2 + (y - y_e)^2 \right), \tag{3.24}$$

under the conditions

$$x \geqslant 0, y \geqslant 0, x + y \leqslant 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e.$$

We already know from (3.13) by taking $s = 1$ there, that the set $M_A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leqslant 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e\}$, is a non-empty closed convex subset of $\mathbb{R}^2$ and hence (x_0, y_0) is the orthogonal projection of (x_e, y_e) onto the set M_A.

Because $x_e \geq 0$ and $y_e \geq 0$, the following cases (corresponding to $(i), (ii), (iii)$ and (iv) in Fig. 3.1, where the only modification is that the base of the triangle intersects the coordinates axes in $(3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e, 0)$ and $(0, 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e)$ respectively) are possible to find (x_0, y_0), the orthogonal projection of (x_e, y_e) on M_A.

(i) $(x_e, y_e) \in M_A$, that is $x_e + y_e \leqslant 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e$.

The inequality is equivalent with $x_e + y_e \leqslant 2(u_e - l_e)$ and we get $P_{M_A}(x_e, y_e) = (x_e, y_e)$, that is

$$x_0 = x_e, y_0 = y_e.$$

(ii) $\frac{3}{2}x_e - \frac{1}{2}y_e - 3u_e + 3l_e > 0$.

Then $P_{M_A}(x_e, y_e) = \left(3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e, 0\right)$, that is

$$x_0 = 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e, y_0 = 0.$$

(iii) $\frac{1}{2}x_e - \frac{3}{2}y_e + 3u_e - 3l_e < 0$.

Then $P_{M_A}(x_e, y_e) = \left(0, 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e\right)$, that is

$$x_0 = 0, y_0 = 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e.$$

(iv) (x_e, y_e) is not in the cases $(i) - (iii)$, that is

$$\begin{aligned} x_e + y_e &> 2(u_e - l_e), \\ \frac{3}{2}x_e - \frac{1}{2}y_e - 3u_e + 3l_e &\leq 0, \\ \frac{1}{2}x_e - \frac{3}{2}y_e + 3u_e - 3l_e &\geq 0. \end{aligned}$$

Then (x_0, y_0) is the orthogonal projection of (x_e, y_e) on the line $x + y = 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e$, that is

$$\begin{aligned} x_0 &= \frac{3}{2}u_e - \frac{3}{2}l_e + \frac{1}{4}x_e - \frac{3}{4}y_e, \\ y_0 &= \frac{3}{2}u_e - \frac{3}{2}l_e - \frac{3}{4}x_e + \frac{1}{4}y_e. \end{aligned}$$

It is easy to verify that in the above cases we can use non-strict inequalities (it is immediate by the geometrical interpretation in Fig. 3.1). Now, by relations (3.22)-(3.23) and by equations (1.15)-(1.16) and (2.4)-(2.5), we easily get (see [17], Corollary 8) the algorithms by using classical representations for trapezoidal fuzzy numbers.

3.7 Trapezoidal approximations preserving the ambiguity

This section contains original contributions from the paper [22].

In this section we prove that for any fuzzy number A there exists a unique trapezoidal fuzzy number $T_{Amb}(A)$ such that $Amb(A) = Amb(T_{Amb}(A))$ and which is the nearest to A with respect to the metric d from all the trapezoidal fuzzy numbers with the same ambiguity as A. This means that it make sense to introduce the operator $T_{Amb} : F(\mathbb{R}) \to F^T(\mathbb{R})$ which will be called from now one trapezoidal approximation operator preserving the ambiguity. By Corollary 2.1.3 and Proposition 2.1.9 (for the particular case when $s = 1$) it follows that the problem to find the trapezoidal approximation preserving the ambiguity of a fuzzy number A is equivalent with the problem to find a trapezoidal fuzzy number $T_{Amb}(A)$ such that $Amb(T_{Amb}(A)) = Amb(T_e(A))$ and $d(T_{Amb}(A), T_e(A)) \leq d(T, T_e(A))$ for all $T \in F^T(\mathbb{R})$ satisfying $Amb(T) = Amb(T_e(A))$. Therefore, $T_{Amb}(A) = [l_0, u_0, x_0, y_0]$, is a solution of the discussed problem if and only if the quadruple $(l_0, u_0, x_0, y_0) \in \mathbb{R}^4$ is a solution of the minimization problem

$$\min\left((l-l_e)^2 + (u-u_e)^2 + \frac{1}{12}(x-x_e)^2 + \frac{1}{12}(y-y_e)^2\right), \tag{3.25}$$

under the conditions

$$x \geq 0, y \geq 0, x + y \leq 2(u-l), \tag{3.26}$$

$$6u - 6l - x - y = 6u_e - 6l_e - x_e - y_e. \tag{3.27}$$

Condition (3.27) implies

$$u - l = u_e - l_e + \frac{1}{6}(x+y) - \frac{1}{6}(x_e + y_e)$$

and

$$l - l_e = u - u_e - \frac{1}{6}(x - x_e) - \frac{1}{6}(y - y_e), \tag{3.28}$$

therefore problem (3.25)-(3.27) becomes

$$\min F(l, u, x, y),$$

where

$$\begin{aligned} F(l,u,x,y) &= 2(u-u_e)^2 + \frac{1}{9}(x-x_e)^2 + \frac{1}{9}(y-y_e)^2 - \frac{1}{3}(u-u_e)(x-x_e) \\ &\quad - \frac{1}{3}(u-u_e)(y-y_e) + \frac{1}{18}(x-x_e)(y-y_e) \end{aligned}$$

under the conditions

$$x \geq 0, y \geq 0, 2x + 2y \leq 6u_e - 6l_e - x_e - y_e. \tag{3.29}$$

After elementary calculus we get

$$F(l,u,x,y) = 2\left(u - u_e - \frac{1}{12}(x - x_e + y - y_e)\right)^2 + \frac{7}{72}\left[(x-x_e)^2 + (y-y_e)^2\right] + \frac{1}{36}(x-x_e)(y-y_e).$$

Because conditions (3.29) are independent of u and taking into account (3.28), $T_{Amb}(A) = [l_0, u_0, x_0, y_0]$ is the trapezoidal approximation of A preserving the ambiguity if and only if

$$u_0 = u_e + \frac{1}{12}(x_0 - x_e + y_0 - y_e), \tag{3.30}$$

$$l_0 = l_e - \frac{1}{12}(x_0 - x_e + y_0 - y_e) \tag{3.31}$$

and (x_0, y_0) is the solution of the minimization problem

$$\min \left((7(x-x_e)^2 + 7(y-y_e)^2 + 2(x-x_e)(y-y_e)\right), \tag{3.32}$$

$$x \geq 0, y \geq 0, 2x + 2y \leq 6u_e - 6l_e - x_e - y_e. \tag{3.33}$$

Let us denote

$$M_A = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, 2x + 2y \leq 6u_e - 6l_e - x_e - y_e\} \tag{3.34}$$

and d_E the Euclidean metric on $\mathbb{R}^2$.

The key element in the proof of the next coming theorem which solves the problem from above is the set M_A which is the surface of the triangle given by the points $(0,0)$, $(3u_e - 3l_e - 1/2x_e - 1/2y_e, 0)$ and $(0, 3u_e - 3l_e - 1/2x_e - 1/2y_e)$ (see again Fig. 3.1 where the only modification is that the base of the triangle intersects the coordinates axes in $(3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e, 0)$ and $(0, 3u_e - 3l_e - \frac{1}{2}x_e - \frac{1}{2}y_e)$ respectively). The key result in the obtaining of the algorithms to compute the trapezoidal approximation preserving the ambiguity is the following.

Theorem 3.7.1 *([22], Theorem 9) The problem (3.32)-(3.33) has a unique solution (x_0, y_0), where $(x_0, y_0) = P_{M_A}(x_e, y_e)$ and $P_M(a,b)$ denotes the orthogonal projection of $(a,b) \in \mathbb{R}^2$ onto the non-empty set $M \subset \mathbb{R}^2$, with respect to the Euclidean metric d_E on $\mathbb{R}^2$.*

Theorem 3.7.1, together (3.30) and (3.31), suggest us the following method to compute $T_{Amb}(A) = [l_0, u_0, x_0, y_0]$, the nearest trapezoidal fuzzy number of a fuzzy number A preserving the ambiguity (see Fig. 3.1 taking into account the previously mentioned modification).

(i) $(x_e, y_e) \in M_A$, that is $-2l_e + 2u_e - x_e - y_e \geq 0$. Then

$$x_0 = x_e, y_0 = y_e, l_0 = l_e, u_0 = u_e.$$

If $(x_e, y_e) \notin M_A$ then the following cases are possible.

(ii) If $6l_e - 6u_e + 3x_e - y_e \geq 0$, then

$$x_0 = -3l_e + 3u_e - \frac{1}{2}x_e - \frac{1}{2}y_e, y_0 = 0, \tag{3.35}$$

$$u_0 = -\frac{1}{4}l_e + \frac{5}{4}u_e - \frac{1}{8}x_e - \frac{1}{8}y_e, \tag{3.36}$$

$$l_0 = \frac{5}{4}l_e - \frac{1}{4}u_e + \frac{1}{8}x_e + \frac{1}{8}y_e. \tag{3.37}$$

(iii) If $-6l_e + 6u_e + x_e - 3y_e \leq 0$, then

$$\begin{aligned} x_0 &= 0, y_0 = -3l_e + 3u_e - \frac{1}{2}x_e - \frac{1}{2}y_e, \\ u_0 &= -\frac{1}{4}l_e + \frac{5}{4}u_e - \frac{1}{8}x_e - \frac{1}{8}y_e, \\ l_0 &= \frac{5}{4}l_e - \frac{1}{4}u_e + \frac{1}{8}x_e + \frac{1}{8}y_e. \end{aligned}$$

(iv) If

$$\begin{aligned} -2l_e + 2u_e - x_e - y_e &\leq 0, \\ 6l_e - 6u_e + 3x_e - y_e &\leq 0, \\ -6l_e + 6u_e + x_e - 3y_e &\geq 0, \end{aligned}$$

then

$$\begin{aligned}
x_0 &= -\frac{3}{2}l_e + \frac{3}{2}u_e + \frac{1}{4}x_e - \frac{3}{4}y_e, \\
y_0 &= -\frac{3}{2}l_e + \frac{3}{2}u_e - \frac{3}{4}x_e + \frac{1}{4}y_e, \\
u_0 &= -\frac{1}{4}l_e + \frac{5}{4}u_e - \frac{1}{8}x_e - \frac{1}{8}y_e, \\
l_0 &= \frac{5}{4}l_e - \frac{1}{4}u_e + \frac{1}{8}x_e + \frac{1}{8}y_e.
\end{aligned}$$

Taking into account (1.15)-(1.16) and (2.4)-(2.5), as in the case of the previous approximation operators, we can easily compute $T_{Amb}(A) = (t_1, t_2, t_3, t_4)$ using standard notations.

3.8 Trapezoidal approximations preserving the weighted expected interval

This section contains original contributions from the paper [28].

In the last sections we discussed different kinds of trapezoidal approximations with respect to the Euclidean metric d. In this section we will prove that for a given fuzzy number A there exists a unique trapezoidal fuzzy number which is the nearest to A with respect to a weighted metric $d_\lambda, \lambda = (\lambda_L, \lambda_U)$, from those that preserve the weighted expected interval of A, denoted with $T_{EI,\lambda}(A)$. Recall, the weighted expected interval of a fuzzy number A is given by (see Definition 1.12.1)

$$EI^\lambda(A) = \left[\frac{1}{a}\int_0^1 A_L(\alpha)\lambda_L(\alpha)d\alpha, \frac{1}{b}\int_0^1 A_U(\alpha)\lambda_U(\alpha)d\alpha\right],$$

where $a = \int_0^1 \lambda_L(\alpha)d\alpha, b = \int_0^1 \lambda_U(\alpha)d\alpha.$

If $T_{e,\lambda}(A) = [l_e, u_e, x_e, y_e]_\lambda$ (we use the representation given in (1.21)) is the weighted extended trapezoidal approximation of A where l_e, u_e, x_e, y_e are given by (2.9)-(2.10) (see also formulas (1.18)-(1.20)) then after some simple calculations we get $EI^w(A) = EI^w(T_{e,\lambda}(A)) = [l_e, u_e]$.

Now taking into account Theorem 2.1.6 and the above equality, the problem of finding the nearest (with respect to metric d_λ) trapezoidal fuzzy number $T_{EI,\lambda}(A) = [l_0, u_0, x_0, y_0]$ to a given fuzzy number A such that $EI^w(A) = EI^w(T_{EI,\lambda}(A))$ is equivalent to the minimization problem

$$\min d_\lambda\left(T_{e,\lambda}(A), [l, u, x, y]\right),$$

under conditions

$$\begin{aligned}
&EI^w(T_e(A)) = EI^w([l, u, x, y]), \\
&x \geq 0, y \geq 0, x + y \leq 2(u - l).
\end{aligned}$$

After some simple calculations the above problem becomes

$$a\left(l_e - l - x\left(\omega_L - \frac{1}{2}\right)\right)^2 + b\left(u_e - u + y\left(\omega_U - \frac{1}{2}\right)\right)^2 + c(x_e - x)^2 + d(y_e - y)^2 \to \min,$$

under conditions

$$l_e = l + x\left(\omega_L - \frac{1}{2}\right), u_e = u - y\left(\omega_U - \frac{1}{2}\right),$$
$$x \geq 0, y \geq 0, x + y \leq 2(u - l).$$

This problem is equivalent to

$$c(x_e - x)^2 + d(y_e - y)^2 \to \min,$$
$$x \geq 0, y \geq 0, x(1-\omega_L) + y(1-\omega_U) \leq u_e - l_e \tag{3.38}$$

and

$$l = l_e - x\left(\omega_L - \frac{1}{2}\right), u = u_e + y\left(\omega_U - \frac{1}{2}\right). \tag{3.39}$$

Problem (3.38) has a unique solution. Indeed, let us define an inner product (it is important here that $c > 0$ and $d > 0$) $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^2$, by $\langle(x_1, y_1), (x_2, y_2)\rangle = cx_1x_2 + dy_1y_2$. If $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ then

$$D^2((x_1, y_1), (x_2, y_2)) = \langle(x_1 - x_2, y_1 - y_2), (x_1 - x_2, y_1 - y_2)\rangle = c(x_1 - x_2)^2 + d(y_1 - y_2)^2,$$

introduces a distance in $\mathbb{R}^2$, therefore $(\mathbb{R}^2, \langle\cdot,\cdot\rangle)$ is a Hilbert space. Let us consider now the set

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x(1-\omega_L) + y(1-\omega_U) \leq u_e - l_e\right\}.$$

Since Ω is a non-empty closed convex subset of $\mathbb{R}^2$, it follows that (3.38) has a unique solution, which is the projection of $(x_e, y_e) \in \mathbb{R}^2$ onto Ω under D. We immediately obtain the following main result of this section which can also be found in paper [28], although not as a theorem.

Theorem 3.8.1 *For any fuzzy number A there exists a unique trapezoidal fuzzy number $T_{EI,\lambda}(A)$, which is the nearest to A with respect to metric d_λ, from those that preserve the weighted expected interval of A.*

3.9 Weighted trapezoidal approximations preserving-cores of fuzzy numbers

The core of a fuzzy number is perhaps the most important characteristic of a fuzzy number. All the basic algebraic operations between fuzzy numbers are usual arithmetic operations when they are performed on the core. Therefore, it is a good idea to find simpler representations for fuzzy numbers with the additional condition of the core preservation. The following algorithm to find the weighted trapezoidal approximation of a fuzzy number with the additional requirement of core preservation was proposed in the paper [6]. In that paper the authors considered weight functions like $f : [0,1] \to \mathbb{R}$ where f is positive, nondecreasing and in addition $\int_0^1 f(\alpha)d\alpha = 1/2$. However, the proofs of the main results in [6] do not use these supplementary assumptions and therefore, in this section we consider arbitrary weight functions that need only to be strictly positive on $(0,1)$ and integrable on $[0,1]$.

Theorem 3.9.1 *([6], formulas (3.15)) Let* $A, A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1],$ *be a fuzzy number and*

$$T_{c,d_{f,f}}(A) = (t_1(A), t_2(A), t_3(A), t_4(A)) = (t_1, t_2, t_3, t_4),$$

be the nearest (with respect to metric $d_{f,f}$ *where* $d_{f,f}$ *means the weighted* L_2*-metric* d_λ*, with* $\lambda = (f,f)$*) trapezoidal fuzzy number to fuzzy number* A *which preserves the core. Then*

$$\begin{aligned} t_1 &= \frac{-\int_0^1 (\alpha-1)A_L(\alpha)f(\alpha)d\alpha + A_L(1)\int_0^1 \alpha(\alpha-1)f(\alpha)d\alpha}{\int_0^1 (\alpha-1)^2 f(\alpha)d\alpha}, \\ t_2 &= A_L(1), t_3 = A_U(1), \\ t_4 &= \frac{-\int_0^1 (\alpha-1)A_U(\alpha)f(\alpha)d\alpha + A_U(1)\int_0^1 \alpha(\alpha-1)f(\alpha)d\alpha}{\int_0^1 (\alpha-1)^2 f(\alpha)d\alpha}. \end{aligned} \tag{3.40}$$

When $f(\alpha) = 1$ for all $\alpha \in [0,1]$, then we get the algorithm to compute the trapezoidal approximation operator (with respect to the Euclidean metric d) preserving the core.

Corollary 3.9.2 *([6], formulas (3.17)) Let* $A, A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1],$ *be a fuzzy number and*

$$T_{c,d}(A) = (t_1(A), t_2(A), t_3(A), t_4(A)) = (t_1, t_2, t_3, t_4),$$

be the nearest (with respect to the Euclidean metric d*) trapezoidal fuzzy number to fuzzy number* A *which preserves the core. Then*

$$\begin{aligned} t_1 &= -3\int_0^1 (\alpha-1)A_L(\alpha)d\alpha - \frac{1}{2}A_L(1), \\ t_2 &= A_L(1), t_3 = A_U(1), \\ t_4 &= -3\int_0^1 (\alpha-1)A_U(\alpha)d\alpha - \frac{1}{2}A_U(1). \end{aligned} \tag{3.41}$$

We observe that comparing with the operators from the previous sections, the algorithm consists in a precise formula. In addition, by simple calculations one can prove that $T_{c,d_{f,f}}$ is linear with respect to the addition and scalar multiplication of fuzzy numbers. That is, for any weight function f we have $T_{c,d_{f,f}}(\lambda_1 A + \lambda_2 B) = \lambda_1 T_{c,d_{f,f}}(A) + \lambda_2 T_{c,d_{f,f}}(B)$, for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $A, B \in F(\mathbb{R})$. On the other hand (see Section 4.7 in the next chapter), these operators are discontinuous in any fuzzy number which is not unimodal. Clearly, this is an unnatural behavior since one would expect that an approximation operator is continuous. If fuzzy numbers A and B are close one to another then their approximations should be also close one to another.

3.10 Calculus of trapezoidal approximations

Having in mind the algorithms from the previous sections we can easily compute different kinds of trapezoidal approximations. In what follows we present some simple examples. We start with the operator T_{EI} mentioning that all the examples are taken from [13].

Example 3.10.1 *Let us consider fuzzy number A given by $A_L(\alpha) = 2\alpha - 2$, $A_U(\alpha) = 1 - \sqrt{\alpha}$, $\alpha \in [0,1]$. We observe that case (iv) in Theorem 3.5.1 is suitable to be applied to A. Applying the algorithm and then passing to standard notations we get $T_{EI}(A) = \left(-\frac{59}{30}, -\frac{1}{30}, -\frac{1}{30}, \frac{7}{10}\right)$. Now let us consider the Bodjanova type fuzzy numbers $B = (1,2,3,30)_2$ and $C = (1,28,29,30)_2$ (see (1.4) to recall the representation of Bodjanova type fuzzy numbers). By simple verifications we observe that case (iii) in Theorem 3.5.1 is suitable to be applied to B and case (ii) in Theorem 3.5.1 is suitable to be applied to C. Therefore, applying the algorithm we get $T_{EI}(B) = \left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{67}{3}\right)$ and $T_{EI}(C) = \left(\frac{26}{3}, \frac{88}{3}, \frac{88}{3}, \frac{88}{3}\right)$.*

We continue with the operator T_{AV} with some examples taken from [17].

Example 3.10.2 *Let us consider fuzzy numbers A and B given by $A_L(\alpha) = 2\alpha - 2$, $A_U(\alpha) = 1 - \sqrt{\alpha}$, $\alpha \in [0,1]$, and $B_L(\alpha) = 2\alpha - 20$, $B_U(\alpha) = 1 - \sqrt{\alpha}$, $\alpha \in [0,1]$. Case (iv) of the algorithm to compute $T_{AV}(A)$ is applicable for A and case (i) of the algorithm to compute $T_{AV}(B)$ is applicable for B. Thus, applying the algorithm we get $T_{AV}(A) = \left(-\frac{29}{30}, -\frac{1}{30}, -\frac{1}{30}, \frac{2}{3}\right)$ and $T_{AV}(B) = \left(-20, -18, -\frac{1}{15}, \frac{11}{15}\right)$.*

Next, we propose some examples for the operator T_{Amb} which are taken from [22].

Example 3.10.3 *We consider fuzzy numbers A and B, $A = (1,2,3,4)_2$, $B = (1,2,4,35)_2$. Case (i) of the algorithm to compute $T_{Amb}(A)$ is applicable for A and we obtain $T_{Amb}(A) = \left(\frac{19}{15}, \frac{31}{15}, \frac{44}{15}, \frac{56}{15}\right)$. Then, case (iv) of the algorithm to compute $T_{Amb}(B)$ is applicable for B and we obtain $T_{Amb}(B) = \left(\frac{7}{5}, 2, 2, \frac{133}{5}\right)$.*

Chapter 4

Properties of fuzzy approximation operators

The quality of a trapezoidal, triangular or parametric approximation operator is important nevertheless. For this reason, Grzegorzewski and Mrówka ([62]) proposed a list of criteria that a trapezoidal approximation operator should posses. Most of these approximation operators own important properties such as: translation invariance, scale invariance, or identity criterion. Another important property that an approximation operator should posses is the continuity. One would expect that if fuzzy number A is close to fuzzy number B then theirs approximations are also close one to another. Yeh ([82], [84], [85]) proved that the approximation operators without additional conditions are nonexpansives. Ban and Coroianu ([18]) proved that the trapezoidal approximation operator preserving the expected interval satisfies the Lipschitz condition. Then, Coroianu ([40]) found the best Lipschitz constant of the discussed operator. Recently, in the paper ([17]) it was proved that the trapezoidal approximation operator preserving the value and ambiguity satisfies the Lipschitz condition but the best Lipschitz constant was not provided because the method presented in paper [40] was not feasible for the case of the trapezoidal approximation operator preserving the value and ambiguity. For this reason, Coroianu proposed in the paper [41] a characterization of fuzzy number-valued functions which can be used to prove that such functions satisfy the Lipschitz condition and as an application the best Lipschitz constant of the trapezoidal approximation operator preserving the value and ambiguity was finally determined.

In the first sections of this chapter we will provide some quantitative results on the translation and scale invariance of fuzzy approximation operators. As a consequence, we will obtain that most of the operators from the previous chapter are both scale and translation invariant. Then we will discuss about the continuity of these operators. As we have already said, we will determine the best Lipschitz constant in the case of the trapezoidal approximation operator preserving the expected interval and in the case of the trapezoidal approximation preserving the value and ambiguity. As a negative result, we will prove that any trapezoidal fuzzy number-valued operator preserving the core is discontinuous at any fuzzy number which is not unimodal with respect to any weighted L_2- type distance. As a direct consequence, we obtain that the trapezoidal approximation operator preserving the core is discontinuous at any fuzzy number which is not unimodal. Interestingly, when we restrict ourselves to the set of unimodal fuzzy numbers then the trapezoidal approximation operator preserving the core is continuous on this subset. Thus, we get a complete characterization of the continuity of the trapezoidal approximation operator preserving the core. In the last but one section, since most of the approximation operators (with the exception of the trapezoidal approximation operator preserving the core) are not additive (we will also provide a general results implying the non-additivity), we will find estimations for the defect of additivity (according to the definition in [29]) of the trapezoidal approximation operator preserving the expected interval and for the trapezoidal approximation operator preserving the value and ambiguity. In the case of the first operator, we even find the best possible estimation. In the last section of this chapter we discuss about trapezoidal approximation in relation with aggregation another important topic in the present days.

This chapter contains original contributions from the papers [17]-[19], [21]-[24], [26], [28], [40]-[41].

4.1 General results on scale and translation invariance

This section contains original contributions from the paper [26]. However, Theorem 4.1.1 is more general comparing to the result from [26].

An operator $P : F(\mathbb{R}) \to F(\mathbb{R})$ is called:

(i) invariant to translations if $P(A+z) = P(A)+z$ for any $A \in F(\mathbb{R})$ and $z \in \mathbb{R}$;

(ii) scale invariant if $P(\lambda A) = \lambda P(A)$ for any $A \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$.

A distance D on $F(\mathbb{R})$ is translation invariant if $D(A+z, B+z) = D(A,B)$, for every $A, B \in F(\mathbb{R})$, $z \in \mathbb{R}$ and scale invariant if $D(\lambda \cdot A, \lambda \cdot B) = |\lambda| D(A,B)$, for every $A, B \in F(\mathbb{R})$, $\lambda \in \mathbb{R} \setminus \{0\}$.

One can easily check that any weighted L_2-type metric $d_\lambda = (\lambda_L, \lambda_U)$, defined on $F(\mathbb{R})$ is translation invariant and in the case when $\lambda_L = \lambda_U$, we get that d_λ is also scale invariant.

The following two theorems will permit us to find large classes of translation or scale invariant operators. The first one is a generalization of Theorem 1 in [26].

Theorem 4.1.1 *(see also Theorem1 in [26]) Let D be a translation invariant distance on $F(\mathbb{R})$ and $P_k, k = \overline{1,n}$, be parameters associated with fuzzy numbers such that $P_k(A+z) = P_k(A) + f_k(z)$, for every $A \in F(\mathbb{R})$, $k = \overline{1,n}$ and $z \in \mathbb{R}$, where, f_k, $k = \overline{1,n}$, are real functions of real variable. If $\Omega \subset F(\mathbb{R})$ satisfies $z + \Omega = \Omega, \forall z \in \mathbb{R}$ and $\omega(A) \in \Omega$ is the nearest fuzzy number to a given $A \in F(\mathbb{R})$ (with respect to D) which preserves $P_k, k \in \{1,...,n\}$, that is $P_k(\omega(A)) = P_k(A), \forall k \in \{1,...,n\}$, then $\omega(A) + z \in \Omega$ is the nearest fuzzy number to $A + z$ (with respect to D) which preserves $P_k, k = \overline{1,n}$, that is $P_k(\omega(A)+z) = P_k(A+z), \forall k \in \{1,...,n\}$.*

Note that in Theorem 1 in [26] only the particular case when each f_k is either the null function or the identity function respectively was considered.

Theorem 4.1.2 *([26], Theorem 4) Let D be a scale invariant distance on $F(\mathbb{R})$ and $P_k, k = \overline{1,n}$ real parameters or intervals associated to fuzzy numbers such that $P_k(\lambda \cdot A) = \lambda P_k(A)$, for every $A \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$ or $P_k(\lambda \cdot A) = |\lambda| P_k(A)$, for every $A \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$. If $\Omega \subset F(\mathbb{R}), \lambda \cdot \Omega \subset \Omega, \forall \lambda \in \mathbb{R}$ and $\omega(A) \in \Omega$ is the nearest fuzzy number to a given $A \in F(\mathbb{R})$ (with respect to D) which preserves $P_k, k = \overline{1,n}$, that is $P_k(\omega(A)) = P_k(A), \forall k \in \{1,...,n\}$, then $\lambda \cdot \omega(A) \in \Omega$ is the nearest fuzzy number to $\lambda \cdot A$ (with respect to D) which preserves $P_k, k \in \{1,...,n\}$, that is $P_k(\omega(\lambda \cdot A)) = P_k(\lambda \cdot A), \forall k \in \{1,...,n\}$.*

Remark 4.1.3 *Note that the assumption $\lambda \cdot \Omega \subset \Omega, \forall \lambda \in \mathbb{R}$ in Theorem 4.1.2 as well as the assumption $z + \Omega = \Omega, \forall z \in \mathbb{R}$ in Theorem 4.1.1 are important. Indeed, if $s_L, s_R > 0, s_L \neq s_R$ and $\Omega = F^{s_L, s_R}(\mathbb{R})$, then $\lambda \cdot \Omega \subsetneq \Omega$ for $\lambda < 0$. The operator $\omega : F(\mathbb{R}) \to \Omega$, where $\omega(A)$ is the best approximation of fuzzy number A relatively to the set Ω with respect to the Euclidean distance d, is not scale invariant (see [14], Theorem 12, (iii)) even if all the other hypotheses in Theorem 4.1.2 are satisfied.*

4.2 Classes of scale/translation invariant operators

This section contains original contributions from the paper [26].

The two theorems from the previous section are quite general since they allow us to find many examples of scale or translation invariant operators. These examples include almost all the operators from Chapter 3 and in addition we can find examples of other operators too, which were not highlighted in this thesis.

The following corollaries are immediate consequences of Theorems 4.1.1-4.1.2.

Corollary 4.2.1 *([26], Corollary 8)* (*i*) *The operator* $O_{\mathbb{R}^c} : F(\mathbb{R}) \to \mathbb{R}^c$, *where* $O_{\mathbb{R}^c}(A)$ *is the nearest crisp value to* A *with respect to Euclidean distance* d, *is scale and translation invariant.*

(*ii*) *The operator* $O_{\mathbb{I}} : F(\mathbb{R}) \to Int(\mathbb{R})$, *where* $O_{\mathbb{I}}(A)$ *is the nearest interval to* A *with respect to distance* d, *is scale and translation invariant.*

(*iii*) *The operator* $O_{\triangle} : F(\mathbb{R}) \to F^{\Delta}(\mathbb{R})$, *where* $O_{\triangle}(A)$ *is the nearest triangular fuzzy number to* A *with respect to distance* d, *is scale and translation invariant.*

(*iv*) *The operator* $T_d : F(\mathbb{R}) \to F^T(\mathbb{R})$, *where* $T_d(A)$ *is the nearest trapezoidal fuzzy number to* A *with respect to distance* d, *is scale and translation invariant.*

(*v*) *The operator* $\Psi_{s,s} : F(\mathbb{R}) \to F^{s,s}(\mathbb{R})$, *where* $\Psi_{s,s}(A)$ *is the nearest parametric* (s,s)*-fuzzy number to* A *(for some* $s > 0$*) with respect to distance* d, *is scale and translation invariant.*

(*vi*) *The operator* $O_{\triangle^s} : F(\mathbb{R}) \to F^{\Delta s}(\mathbb{R})$, *where* $O_{\triangle^s}(A)$ *is the nearest symmetric triangular fuzzy number to* A *with respect to distance* d, *is scale and translation invariant.*

Similarly we can find many example of fuzzy approximation operators which are translation and scale invariant (see also Corollary 9 in [26]). In addition it is easy to prove that the operators T_{EI}, T_{AV} and T_{Amb} from Chapter 3, they all satisfy the hypothesis of Theorems 4.1.1-4.1.2 and therefore they are translation and scale invariant.

4.3 Lipschitz continuity of parametric approximation operators without additional conditions

This section contains an original approach which leads in a more simple way to some conclusions regarding the continuity of parametric or trapezoidal approximation operators which can be found in the papers [21], [82], [84]. Before we start our reasoning we need the following auxiliary result which is known from Hilbert space theory. Recall, if A is a closed convex subset of a Hilbert space $(X, \langle \cdot, \cdot \rangle)$ then for some $x \in X$ we denote with $P_A(x)$ the unique element in A which satisfies $D(x, P_A(x)) = \inf_{y \in A} D(x, y)$ and D is the metric generated by $\langle \cdot, \cdot \rangle$ on X.

Theorem 4.3.1 *(see e. g. [82] Fact 6.4) If* $(X, \langle \cdot, \cdot \rangle)$ *is a Hilbert space and* A *is a non-empty closed convex subset of* X, *then* $d(P_A(x), P_A(y)) \leq d(x, y)$, $(\forall)\ x, y \in X$. *Here* d *denotes the metric generated by the inner product* $\langle \cdot, \cdot \rangle$.

Let us consider the weighted metric d_λ, $\lambda = (\lambda_L, \lambda_U)$ and for fixed $s_L > 0$ and $s_R > 0$ let us consider the weighted parametric approximation operator (see Section 3.3) $\Psi_{d_\lambda, s_L, s_R} : F(\mathbb{R}) \to F^{s_L, s_R}(\mathbb{R})$, where $\Psi_{d_\lambda, s_L, s_R}(A) = A^*_{s_L, s_R}$ is the nearest parametric fuzzy number of type (s_L, s_R) to A with respect to the metric d_λ. By the proof of Corollary 3.2.6, we know that $(F(\mathbb{R}), d_\lambda, +, \cdot)$ can be embedded in a Hilbert space $\left(\widetilde{F(\mathbb{R})}, \widetilde{d_\lambda}, \oplus, \odot\right)$, where $\widetilde{F(\mathbb{R})} = L_2^{\lambda_L}[0,1] \times L_2^{\lambda_U}[0,1]$. Then, by Theorem 3.2.3 it results that $F^{s_L,s_R}(\mathbb{R})$ is a closed convex subset of $\widetilde{F(\mathbb{R})}$ in the topology generated by $\widetilde{d_\lambda}$. Now, if A and B denote two fuzzy numbers, as they can be viewed as elements of $\widetilde{F(\mathbb{R})}$, then by Theorem 4.3.1 we get that $\widetilde{d_\lambda}\left(P_{F^{s_L,s_R}(\mathbb{R})}(A), P_{F^{s_L,s_R}(\mathbb{R})}(B)\right) \leq \widetilde{d_\lambda}(A,B)$. Since the restriction of $\widetilde{d_\lambda}$ to the space $F(\mathbb{R})$ coincides with d_λ and since by the definition of the operator Ψ_{d_λ,s_L,s_R}, we actually have $\Psi_{d_\lambda,s_L,s_R}(A) = P_{F^{s_L,s_R}(\mathbb{R})}(A)$ and $\Psi_{d_\lambda,s_L,s_R}(B) = P_{F^{s_L,s_R}(\mathbb{R})}(B)$, it easily follows from the above relation that $d_\lambda(\Psi_{d_\lambda,s_L,s_R}(A), \Psi_{d_\lambda,s_L,s_R}(B)) \leq d_\lambda(A,B)$. Consequently, we obtain the following result which can be also found in the paper [84].

Theorem 4.3.2 *The parametric approximation operator $\Psi_{d_\lambda,s_L,s_R} : F(\mathbb{R}) \to F^{s_L,s_R}(\mathbb{R})$ is nonexpansive, that is $d_\lambda(\Psi_{d_\lambda,s_L,s_R}(A), \Psi_{d_\lambda,s_L,s_R}(B)) \leq d_\lambda(A,B)$, for all $A, B \in F(\mathbb{R})$.*

The above theorem says that the operator Ψ_{d_λ,s_L,s_R} is Lipschitz continuous with Lipschitz constant 1. It is immediate that this is the best possible Lipschitz constant since for any $A \in F^{s_L,s_R}(\mathbb{R})$ we have $\Psi_{d_\lambda,s_L,s_R}(A) = A$.

From Theorem 4.3.2 we can easily obtain similar results for all the other approximation operators without additional conditions. By similar reasonings we obtain the following even stronger results.

Theorem 4.3.3 *For the parametric approximation operator $\Psi_{s_L,s_R} : F(\mathbb{R}) \to F^{s_L,s_R}(\mathbb{R})$, it holds that $d(\Psi_{s_L,s_R}(A), \Psi_{s_L,s_R}(B)) \leq d(A^e_{s_L,s_R}, B^e_{s_L,s_R}) \leq d(A,B)$, for all fuzzy numbers A, B. Here $A^e_{s_L,s_R}$ denotes the extended (s_L, s_R) parametric approximation of A (see Section 2.1).*

In the particular case when $s_L = s_R = 1$, we obtain the following corollary which can also be collected from the paper [82].

Corollary 4.3.4 *Let us consider the trapezoidal approximation operator $T_d : F(\mathbb{R}) \to F^T(\mathbb{R})$, where recall, $T_d(A)$ denotes the nearest trapezoidal fuzzy number to A with respect to the Euclidean metric d. Then for any arbitrary fuzzy numbers A and B it holds that $d(T_d(A), T_d(B)) \leq d(T_e(A), T_e(B)) \leq d(A,B)$. Here (see again Section 2.1) $T_e(A)$ denotes the extended trapezoidal approximations of A.*

It is easy to prove that in Theorem 4.3.3 and in particular in Corollary 4.3.4, we have the best possible Lipschitz constants, since for any $s_L > 0$ and $s_R > 0$ and for any $A \in F^{s_L,s_R}(\mathbb{R})$ we have $\Psi_{s_L,s_R}(A) = A$.

4.4 Best Lipschitz constant of the trapezoidal approximation operator preserving the expected interval

This section contains original contributions from the paper [40].

In the case of the trapezoidal approximation operator (with respect to the Euclidean metric d) preserving the expected interval, $T_{EI} : F(\mathbb{R}) \to F^T(\mathbb{R})$, we will prove in this section that this operator is Lipschitz continuous too, but comparing with the approximation operators without additional conditions from the previous section, we will see that T_{EI} is not nonexpansive and this also holds for most of the approximation operators with additional requirements as for example the operator T_{AV} from the next coming section. However, we will find the best possible Lipschitz constant of the operator T_{EI} (and of the operator T_{AV} in the next coming section). The fact that the operator T_{EI} is not nonexpansive has a quite simple explanation. Let us take in discussion the parametric approximation operator (in the most general case) Ψ_{d_λ,s_L,s_R}. In the previous section we proved that actually this operator is a projector onto a closed convex subset of the Hilbert space $\left(\widetilde{F(\mathbb{R})}, \widetilde{d_\lambda}, \oplus, \odot\right)$

and it is well known that such projectors are actually nonexpansive functions. But in the case of the operator T_{EI}, it is immediate by its definition that $T_{EI}(A)$ is the projection of A onto a "moving" closed convex subset $\Omega(A) \subseteq F^T(\mathbb{R})$, where $\Omega(A) = \{T \in F^T(\mathbb{R}) : EI(A) = EI(T)\}$. It is very easy to check that we can always find two fuzzy numbers A and B so that $\Omega(A) \neq \Omega(B)$. This is the main reason why the operator T_{EI} is not nonexpansive. The Lipschitz continuity of the operator T_{EI} was proved for the first time in the paper [18]. The best Lipschitz constant of the operator T_{EI} is given in the following main result of this section.

Theorem 4.4.1 *([40], Theorem 7) The nearest trapezoidal approximation operator preserving the expected interval $T_{EI} : F(\mathbb{R}) \rightarrow F^T(\mathbb{R})$, satisfies the inequality $d(T_{EI}(A), T_{EI}(B)) \leq \sqrt{\frac{5}{3}} d(A, B)$, for all $A, B \in F(\mathbb{R})$.*

To prove that the result from Theorem 4.4.1 cannot be improved, we need to find $A, B \in F(\mathbb{R})$ such that $d(A, B) > 0$ and $d(T_{EI}(A), T_{EI}(B)) = \sqrt{\frac{5}{3}} d(A, B)$. In the following example we prove that there are fuzzy numbers such that this equality holds.

Example 4.4.2 *([40], Example 1) Let us consider the fuzzy numbers A and B given by $A_L(\alpha) = 90\sqrt{\alpha} + 1$, $A_U(\alpha) = 93$ and $B_L(\alpha) = 90\sqrt{\alpha}$, $B_U(\alpha) = 94$, $\alpha \in [0, 1]$. Since the case (ii) in Theorem 3.5.1 is suitable to be applied to fuzzy numbers A and B, we obtain (with standard notations according to the algorithm in Theorem 3.5.1) $T_{EI}(A) = (29, 93, 93, 93), T_{EI}(B) = (26, 94, 94, 94)$. Then, $d^2(T_{EI}(A), T_{EI}(B)) = \frac{10}{3}$. Because $d^2(A, B) = 2$, it follows that $d(T_{EI}(A), T_{EI}(B)) = \sqrt{\frac{5}{3}} d(A, B)$.*

Example 4.4.3 *([40], Example 3) Let us consider fuzzy number A, $A_L(\alpha) = e^{\alpha^2}$, $A_U(\alpha) = 4 - \alpha$, $\alpha \in [0, 1]$. We will determine $T_{EI}(A)$ with an error less than 10^{-2} with respect to the Euclidean distance d. For this purpose, let us consider the sequence of fuzzy numbers $(A_n)_{n \geq 1}$, $A_{n_L}(\alpha) = 1 + \alpha^2 + \frac{\alpha^4}{2!} + \frac{\alpha^6}{3!} + \ldots + \frac{\alpha^{2n}}{n!}$, $A_{n_U}(\alpha) = 4 - \alpha$. From the Taylor formula we have $e^\alpha = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \ldots + \frac{\alpha^n}{n!} + \int_0^\alpha \frac{(\alpha - t)^n}{n!} e^t dt$, for any $\alpha \in [0, 1]$. Then*

$$d^2(A_n, A) = \int_0^1 \left(e^{\alpha^2} - \left(1 + \alpha^2 + \frac{\alpha^4}{2!} + \frac{\alpha^6}{3!} + \ldots + \frac{\alpha^{2n}}{n!} \right) \right)^2 d\alpha = \int_0^1 \left(\int_0^{\alpha^2} \frac{(\alpha^2 - t)^n}{n!} e^t dt \right)^2 d\alpha.$$

By the mean value theorem there exists $c_n \in (0, 1)$ such that

$$\begin{aligned} d^2(A_n, A) &= e^{2c_n} \int_0^1 \left(\int_0^{\alpha^2} \frac{(\alpha^2 - t)^n}{n!} dt \right)^2 d\alpha = \frac{e^{2c_n}}{((n+1)!)^2} \int_0^1 \alpha^{4n+4} d\alpha \\ &= \frac{e^{2c_n}}{((n+1)!)^2 (4n+5)} \leq \frac{e^2}{((n+1)!)^2 (4n+5)}. \end{aligned}$$

By Theorem 4.4.1 it follows that $d^2(T_{EI}(A_n), T_{EI}(A)) \leq \frac{5e^2}{3((n+1)!)^2(4n+5)}$. It is obvious that for $n \geq 4$ we have $d(T_{EI}(A_n), T_{EI}(A)) \leq 10^{-2}$. For $n = 4$, the case (i) in Theorem 3.5.1 is applicable to compute the nearest trapezoidal fuzzy number to fuzzy number A_4 preserving the expected interval, therefore $T_{EI}(A_4) = \left(\frac{527}{756}, \frac{2104}{945}, 3, 4 \right)$. Hence we obtained an approximation of $T_{EI}(A)$ with an error less then 10^{-2}.

4.5 Best Lipschitz constant of the trapezoidal approximation operator preserving the value and ambiguity

This section contains original contributions from the paper [41].

In this section we will determine the best Lipschitz constant of the operator T_{AV} introduced in Section 3.6.

Theorem 4.5.1 *([41], Theorem 11) If* $T_{AV} : F(\mathbb{R}) \rightarrow F^T(\mathbb{R})$ *is the trapezoidal approximation operator which preserves the value and the ambiguity then* $d(T_{AV}(A), T_{AV}(B)) \leq \sqrt{\frac{10+2\sqrt{10}}{3}} d(A,B)$*, for all* $A, B \in F(\mathbb{R})$ *and the value* $\sqrt{\frac{10+2\sqrt{10}}{3}}$ *is the best possible Lipschitz constant of the operator* T_{AV}*.*

The proof of the above theorem is based on the fact that the space of fuzzy numbers is covered by a finite family of closed convex sets (we mean convexity as according to Definition 2.3.1) Then it is proved that T_{AV} is Lipschitz continuous on each set of the covering, which by Theorem 2.4.2 implies that T_{AV} is Lipschitz continuous on the whole domain. Finally, by using a concrete example it is proved that the Lipschitz constant obtained in the proof is actually the best possible one.

In the final part of this section we will apply the estimation obtained in Theorem 4.5.1 to calculate the trapezoidal approximation of a fuzzy number preserving the value and ambiguity within a reasonable error in the case when the direct algorithm is not applicable. We will consider the same example as in the case of the trapezoidal approximation operator preserving the expected interval (see Example 4.4.3).

Example 4.5.2 *([41], Example 14) Let us consider the fuzzy number* A, $A_L(\alpha) = e^{\alpha^2}$, $A_U(\alpha) = 4-\alpha$, $\alpha \in [0,1]$. *We will determine* $T_{AV}(A)$ *with an error less than* 10^{-2} *with respect to the Euclidean metric* d. *For this purpose let us consider the same sequence of fuzzy numbers as in Example 4.4.3,* $(A_n)_{n\geq 1}$, $(A_n)_L(\alpha) = 1 + \alpha^2 + \alpha^4/2! + \alpha^6/3! + ... + \alpha^{2n}/n!$, $A_U(\alpha) = 4 - \alpha$, $\alpha \in [0,1]$. *Following the same root as in the case of the trapezoidal approximation operator preserving the expected interval (see Example 4.4.3) we get that* $d^2(A_n, A) \leq \frac{e^2}{((n+1)!)^2(4n+5)}, n \geq 1$. *Applying Theorem 4.5.1 we obtain* $d^2(T_{AV}(A_n), T_{AV}(A)) \leq \frac{10+2\sqrt{10}}{3} \cdot \frac{e^2}{((n+1)!)^2(4n+5)}$. *Obviously, for* $n \geq 5$ *we have* $d(T_{AV}(A_n), T_{AV}(A)) \leq 10^{-2}$. *For* $n = 5$, *case* (i) *of the algorithm to compute* $T_{AV}(A_5)$ *is applicable, therefore* $T_{AV}(A_5) = (0.69595, 2.2291, 3, 4)$ *and we thus obtained an approximation of* $T_{AV}(A)$ *with an error less than* 10^{-2}.

4.6 Lipschitz continuity of the trapezoidal approximation operator preserving the ambiguity

This section contains original contributions from the paper [22]. We mention that in Theorem 4.6.1 we obtain a better constant comparing with that from paper [22]. The approach is the same as in the case of the operator T_{AV}. However, the finding of the best possible Lipschitz constant remains an open question.

Theorem 4.6.1 *The nearest trapezoidal approximation operator preserving the ambiguity* T_{Amb} : $F(\mathbb{R}) \rightarrow F^T(\mathbb{R})$ *satisfies the inequality* $d(T_{Amb}(A), T_{Amb}(B)) \leq \sqrt{6} d(A,B)$*, for all* $A, B \in F(\mathbb{R})$.

4.7 On the continuity of trapezoidal fuzzy number-valued operators preserving the core

This section contains original contributions from the papers [19] and [24].

Perhaps one of the most important characteristics of a fuzzy number is the 1-cut level set (the core). Therefore, when it comes to simplify on the representation of a fuzzy number by using for example trapezoidal fuzzy numbers, the trapezoidal approximation preserving the core would count as an important one. However, we will see in this section that this approximation process has important points of discontinuity and hence, it is questionable if such operators are effective in any practical application. For the main results of this section we will use Lemmas 2.2.1-2.2.2. The following results are taken from the paper [19] with the only difference that here the weights λ_L and λ_U of the weighted metric $d_\lambda = (\lambda_L, \lambda_U)$ are considered bounded and not necessarily nondecreasing as in the before mentioned paper.

Theorem 4.7.1 *([19], Theorem 4) Let $T : F(\mathbb{R}) \to F^T(\mathbb{R})$ be a trapezoidal fuzzy number-valued operator preserving core, that is core $(A) =$ core $(T(A))$, for every $A \in F(\mathbb{R})$. If $A \in F(\mathbb{R}), A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1]$, satisfies $A_L(1) < A_U(1)$ (that is A is not unimodal), then T is discontinuous in A with respect to any weighted metric d_λ, $\lambda = (\lambda_L, \lambda_U)$.*

Since the weighted trapezoidal approximation operator preserving the core $T_{c,d_{f,f}}$ (for some bounded weight function f) given in Theorem 3.9.1 is included in the class of operators considered in the previous theorem, we easily get the following corollary.

Theorem 4.7.2 *([19], Theorem 6) $T_{c,d_{f,f}} : F(\mathbb{R}) \to F^T(\mathbb{R})$ is discontinuous in any fuzzy number A satisfying $A_L(1) < A_U(1)$, with respect to any weighted metric d_λ.*

Interestingly, for a sequence of fuzzy numbers with uniformly convergent sides, we get the following convergence result.

Theorem 4.7.3 *([19], Theorem 8) If $A, A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1]$, is a fuzzy number and $(A_n)_{n\in\mathbb{N}}$ is a sequence of fuzzy numbers such that $((A_n)_L)_{n\in\mathbb{N}}$ and $((A_n)_U)_{n\in\mathbb{N}}$ are uniformly convergent sequences to A_L and A_U, respectively, then $\lim_{n\to\infty} T_{c,d_{f,f}}(A_n) = T_{c,d_{f,f}}(A)$, with respect to any weighted metric d_λ, $\lambda = (\lambda_L, \lambda_U)$.*

Using the above theorem we can compute trapezoidal approximations preserving the core in the case when the direct algorithm is not applicable (see Example 9 in [19]).

All that follows in this section is taken from paper [24] where a more general problem is discussed, more exactly the authors study the properties of trapezoidal fuzzy number-valued operators preserving a given $\alpha-$cut, $\alpha \in [0,1]$. However, in this section we discuss only the case when $\alpha = 1$ because this case corresponds to our interest on investigating trapezoidal fuzzy number-valued operators preserving the core.

So far, for a weighted trapezoidal approximation operator preserving the core $T_{c,d_{f,f}}$, we know that the set of discontinuities contains the set of all fuzzy numbers which are not unimodal (also known as fuzzy intervals). Interestingly, in the case when the weight function f is nondecreasing, we will prove that the remaining points that is unimodal fuzzy numbers (see Section 1.6), all of them are continuity points for the operator $T_{c,d_{f,f}}$. In this way, the continuity points of the operator $T_{c,d_{f,f}}$ are completely determined. To prove the continuity of $T_{c,d_{f,f}}$ on the set of unimodal fuzzy numbers the following lemma is essential.

Lemma 4.7.4 *([24]) Let* $(g_n)_{n\geq 0}$, $g_n : [0,1] \to \mathbb{R}$, *be a sequence of functions and consider the function* $g : [0,1] \to \mathbb{R}$. *Also, let us consider a weight function* $f : [0,1] \to \mathbb{R}$. *Suppose that the following requirements hold:*

(i) g_n, $n \in \mathbb{N}$ *and* g, *are all nondecreasing and left continuous functions;*

(ii) $g_n(1) \leq g(1)$, *for all* $n \in \mathbb{N}$;

(iii) $\lim_{n\to\infty} \int_0^1 f(\alpha)\left(g_n(\alpha) - g(\alpha)\right)^2 d\alpha = 0.$

Then $\lim_{n\to\infty} g_n(1) = g(1)$.

By performing similar reasonings we get a similar result as above for the case when g_n, $n \in \mathbb{N}$ and g are all nonincreasing and left continuous functions.

By using Lemma 4.7.4 and its corresponding result for the case of noincreasing and left continuous functions, we can prove the following important result concerning the continuity of trapezoidal approximation operators preserving the core.

Theorem 4.7.5 *([24]) Let* $T_{c,d_{f,f}} : F(\mathbb{R}) \to F^T(\mathbb{R})$ *denotes the weighted trapezoidal approximation operator preserving core with respect to a weighted metric* $d_{f,f}$, *given by equations (3.40). Then* $T_{c,d_{f,f}}$ *is continuous on the space of unimodal fuzzy numbers* $UF(\mathbb{R})$.

From Theorems 4.7.2 and 4.7.5, it is immediate that we can determine exactly the points of continuity and discontinuity respectively for the operator $T_{c,d_{f,f}}$.

4.8 On the defect of additivity of fuzzy approximation operators

This section contains original contributions from the papers [17]-[18], [23] and [41] .

With the exception of the weighted trapezoidal approximation operator preserving the core, the operators proposed in this thesis are given on cases. Let us take the trapezoidal approximation operator preserving the expected interval T_{EI} for example. Denote with Ω_i, $i \in \{1,2,3,4\}$, the subfamilies of fuzzy numbers corresponding to the cases (i), (ii), (iii) and (iv) in Theorem 3.5.1. It is very easy to check that for any $i \in \{1,2,3,4\}$ and for any $A, B \in \Omega_i$, it holds that $T_{EI}(A+B) = T_{EI}(A) + T_{EI}(B)$. But in general it does not hold that $T_{EI}(A+B) = T_{EI}(A) + T_{EI}(B)$ (see Example 2 in [18]) .

Actually, we can prove a strong result on the non-additivity which will imply the non-additivity of the most of the approximation operators proposed in this thesis. We have the following.

Lemma 4.8.1 *([23], Lemma 8) Let us consider the trapezoidal valued operator* $T : F(\mathbb{R}) \to F^T(\mathbb{R})$ *which satisfies the condition that if* $A \in F(\mathbb{R})$ *satisfies* $T_e(A) \in F^T(\mathbb{R})$, *then* $T(A) = T_e(A)$. *Then* T *is non-additive.*

It is immediate that the approximation operators T_d, T_{EI}, T_{AV} and T_{Amb}, they all satisfy the hypothesis of the above lemma and therefore it follows that they are non-additive. In addition, as in the case of the operator T_{EI}, it can be proved that the operators T_d, T_{AV} and T_{Amb} are only piecewise additive.

Similarly, it can be proved that for some fixed $s_L > 0$ and $s_R > 0$ and for a weighted metric d_λ, the operator Ψ_{d_λ,s_l,s_R} is only piecewise additive but it is not additive on the whole domain.

Following the ideas in [29], the notion of defect of additivity of a trapezoidal approximation operator was introduced in the paper [17].

Definition 4.8.2 *([17], Definition 25) Let $A \in F(\mathbb{R})$ and $t : F(\mathbb{R}) \to F^T(\mathbb{R})$ be a trapezoidal approximation operator with respect to a distance denoted with D. The defect of additivity of the operator t with respect to fuzzy number A and distance D is given by $\delta_{t,D}(A) = \sup_{B \in F(\mathbb{R})} D(t(A) + t(B), t(A+B))$.*

The following result will help us to find an estimation of the defect of additivity for the fuzzy approximation operators.

Lemma 4.8.3 *([41], Lemma 16) Let $t : F(\mathbb{R}) \to F^T(\mathbb{R})$ be a trapezoidal approximation operator with respect to a distance denoted with D. Suppose that the following requirements hold:*
(i) $t(O) = O$ where O is the trapezoidal fuzzy number $(0,0,0,0)$;
(ii) there exists a positive real constant c such that $D(t(A), t(B)) \le cD(A,B)$, $A, B \in F(\mathbb{R})$;
(iii) we have $D(A+C, B+C) = D(A,B)$, for all $A, B, C \in F(\mathbb{R})$.
Then $\delta_{t,D}(A) \le 2cD(O,A)$, for all $A \in F(\mathbb{R})$.

The following corollary is immediate.

Corollary 4.8.4 *([17], Corollary 17) Let A be a fuzzy number and let T_{AV} denotes the trapezoidal approximation operator preserving the value and the ambiguity. Then*

$$\delta_{T_{AV},d}(A) \le 2\sqrt{\frac{10+2\sqrt{10}}{3}} \left(\int_0^1 \left(A_L^2(\alpha) + A_U^2(\alpha) \right) d\alpha \right)^{1/2}.$$

By similar reasonings, having in mind the Lipschitz constants of the operators T_{Amb} and $\Psi_{d_\lambda, s_L, s_R}$, we may obtain estimations for the defect of additivity of these operators.

In the case of the operator T_{EI}, by using some geometrical reasonings we can obtain the best possible estimation for the defect of additivity.

Theorem 4.8.5 *([23], Theorem 16) We have*

$$d(T_{EI}(A+B), T_{EI}(A) + T_{EI}(B)) \le \frac{2\sqrt{3}}{3} \min\{length(EI(A)), length(EI(B))\}, \tag{4.1}$$

for all $A, B \in F(\mathbb{R})$.

In what follows, by some reasonings which can be found also in [23], we will prove that the value $\frac{2\sqrt{3}}{3}$ from the conclusion of Theorem 4.8.5 is the "lowest" possible applied for the entire set of fuzzy numbers. For any $\varepsilon \in (0,1)$ we consider the fuzzy number $A_\varepsilon = (A_\varepsilon)_\alpha = [(A_\varepsilon)_L(\alpha), (A_\varepsilon)_U(\alpha)]$, $\alpha \in [0,1]$, where

$$(A_\varepsilon)_L(\alpha) = \begin{cases} \frac{\varepsilon\alpha}{6} - \frac{\alpha^2}{2}; & \alpha \in \left[0, \frac{\varepsilon}{6}\right], \\ \frac{\varepsilon^2}{72}; & \alpha \in \left[\frac{\varepsilon}{6}, 1\right] \end{cases}$$

and $(A_\varepsilon)_U(\alpha) = (A_\varepsilon)_L(1) = \frac{\varepsilon^2}{72}$, for all $\alpha \in [0,1]$. Let $T_e(A_\varepsilon) = [l_e(A_\varepsilon), u_e(A_\varepsilon), x_e(A_\varepsilon), y_e(A_\varepsilon)]$ denotes the extended trapezoidal approximation of A_ε. Noting that $(A_\varepsilon)_L$ is differentiable and using integration by parts, we obtain (see formulas (2.4)-(2.5))

$$\begin{aligned} & x_e(A_\varepsilon) + (6-\varepsilon) l_e(A_\varepsilon) \\ = & \int_0^1 (A_\varepsilon)_L(\alpha)(12\alpha - \varepsilon) d\alpha = (6-\varepsilon)(A_\varepsilon)_L(1) - \int_0^1 (A_\varepsilon)'_L(\alpha)(6\alpha^2 - \varepsilon\alpha) d\alpha \\ = & (6-\varepsilon) u_e(A_\varepsilon) - \int_0^1 (A_\varepsilon)'_L(\alpha)(6\alpha^2 - \varepsilon\alpha) d\alpha. \end{aligned}$$

Since

$$(A_\varepsilon)'_L(\alpha) = \begin{cases} \frac{\varepsilon}{6} - \alpha; & \alpha \in \left[0, \frac{\varepsilon}{6}\right], \\ 0; & \alpha \in \left[\frac{\varepsilon}{6}, 1\right], \end{cases}$$

it is immediate that $\int_0^1 (A_\varepsilon)'_L(\alpha)(6\alpha^2 - \varepsilon\alpha) d\alpha < 0$. This implies

$$x_e(A_\varepsilon) > (6-\varepsilon) u_e(A_\varepsilon) - (6-\varepsilon) l_e(A_\varepsilon). \tag{4.2}$$

In addition, since by formula (2.5) it results $y_e(A_\varepsilon) = 0$, we obtain

$$x_e(A_\varepsilon) - y_e(A_\varepsilon) > 2u_e(A_\varepsilon) - 2l_e(A_\varepsilon).$$

From the above inequality it results that for any $\varepsilon \in (0,1)$ the case (ii) in Theorem 3.5.1 is suitable to be applied to compute $T_{EI}(A_\varepsilon)$ and consequently we obtain

$$T_{EI}(A_\varepsilon) = [l_e(A_\varepsilon), u_e(A_\varepsilon), 2u_e(A_\varepsilon) - 2l_e(A_\varepsilon), 0], \tag{4.3}$$

for all $\varepsilon \in (0,1)$.

Now, let us consider the trapezoidal fuzzy number $B = [0,1,0,0]$. Since B is a trapezoidal fuzzy number and since T_{EI} satisfies the identity criterion we get $T_{EI}(B) = B$ which together with relation (4.3) gives

$$T_{EI}(A_\varepsilon) + T_{EI}(B) = [l_e(A_\varepsilon), u_e(A_\varepsilon) + 1, 2u_e(A_\varepsilon) - 2l_e(A_\varepsilon), 0]. \tag{4.4}$$

On the other hand, after some simple calculations (or using the linearity of the extended trapezoidal approximation T_e) we obtain $T_e(A_\varepsilon + B) = [l_e(A_\varepsilon), u_e(A_\varepsilon) + 1, x_e(A_\varepsilon), y_e(A_\varepsilon)]$. Noting that $0 \leq l_e(A_\varepsilon) \leq u_e(A_\varepsilon)$, and that by very simple calculations we have $x_e(A_\varepsilon) \leq 6u_e(A_\varepsilon) - 6l_e(A_\varepsilon) \leq 6u_e(A_\varepsilon)$ (recall that $y_e(A_\varepsilon) = 0$), it follows that

$$2(u_e(A_\varepsilon) + 1 - l_e(A_\varepsilon)) - x_e(A_\varepsilon) - y_e(A_\varepsilon) \geq 2 - x_e(A_\varepsilon) \geq 2 - 6u_e(A_\varepsilon) = 2 - \frac{\varepsilon^2}{12} > 0,$$

for all $\varepsilon \in (0,1)$. This implies that $T_e(A_\varepsilon + B)$ is a trapezoidal fuzzy number and thus since case (i) is applicable in Theorem 3.5.1 we obtain that $T_{EI}(A_\varepsilon + B) = T_e(A_\varepsilon + B)$, that is

$$T_{EI}(A_\varepsilon + B) = [l_e(A_\varepsilon), u_e(A_\varepsilon) + 1, x_e(A_\varepsilon), 0]. \tag{4.5}$$

Relations (1.17), (4.4) and (4.5) imply

$$d(T_{EI}(A_\varepsilon) + T_{EI}(B), T_{EI}(A_\varepsilon + B)) = \frac{1}{\sqrt{12}} (x_e(A_\varepsilon) - 2u_e(A_\varepsilon) + 2l_e(A_\varepsilon))$$

and from relation (4.2) we obtain

$$\begin{aligned} & d(T_{EI}(A_\varepsilon)+T_{EI}(B),T_{EI}(A_\varepsilon+B)) \\ > & \frac{1}{\sqrt{12}}(4-\varepsilon)(u_e(A_\varepsilon)-l_e(A_\varepsilon))=\left(\frac{2\sqrt{3}}{3}-\frac{\varepsilon}{\sqrt{12}}\right)length(EI(A_\varepsilon)).\end{aligned}$$

The above inequality combined with the fact that $length(EI(B)) > length(EI(A_\varepsilon)) > 0$, for all $\varepsilon \in (0,1)$, proves that in general the value $\frac{2\sqrt{3}}{3}$ from the conclusion of Theorem 4.8.5 is the smallest possible.

Even if $\frac{2\sqrt{3}}{3}$ is the best possible constant in the conclusion of Theorem 4.8.5, it can be proved that we cannot find two fuzzy numbers A and B such that $length(EI(A)) > 0$, $length(EI(B)) > 0$ and such that relation (4.1) would become equality.

4.9 Trapezoidal approximation and aggregation

This section contains original contributions from paper [28].

The theory of aggregation operators has had an impressive development in the last years. In the present it is one of the most popular topics. This is so because aggregation operators have so many applications. Almost every field of research interacts with the necessity of aggregating data. A comprehensive study on aggregation operator is the work of Grabisch et al. ([57]) where the reader can find the most important results of this topic.

Suppose that I is a real bounded or unbounded interval. The most general definition of an aggregation operator is given in the following.

Definition 4.9.1 *(see e.g. [57], Definition 1.1 pp. 3) An aggregation function in I^n is a function $A: I^n \to I$, that satisfies the following requirements:*

(i) A is nondecreasing in each variable;

(ii) it holds that $\inf_{x\in I^n} A(x) = \inf I$ and $\sup_{x\in I^n} A(x) = \sup I$.

Between the most popular aggregation operator are the arithmetic mean function $AM: I^n \to I$, as well as the geometric mean function $GM: I^n \to I$ (here we necessarily have $\inf I \geq 0$), where $AM(x_1,...,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $GM(x_1,...,x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$.

Other important aggregation functions are the minimum, maximum or projection function respectively. One can easily guess their definition. Other interesting examples can be found in [57], pp. 6-9.

It seems that Definition 4.9.1 cannot easily be extended when we work with fuzzy numbers instead of real (crisp) values. Both requirements of Definition 4.9.1 can hardly be adapted to the fuzzy case since the ranking of fuzzy numbers is an issue which so far, has not a satisfactory method.

But there are some aggregation functions that can naturally be extended to the fuzzy case. For example the arithmetic mean function or the projection function. In what follows we deal with the arithmetic mean function. Therefore, if $A_1, A_2, ..., A_n$, represents a sample of fuzzy numbers then the arithmetic mean of the sample is given by $AM(A_1,...,A_n) = \frac{1}{n}\sum_{i=1}^{n} A_i$. When there is no danger of confusion we denote $\overline{A} = AM(A_1,...,A_n)$.

Suppose now that we have a sample $A_1, A_2, ..., A_n$, of fuzzy numbers that should be aggregated efficiently in the sense of the arithmetic mean. Having in mind all the discussions related to trapezoidal approximation and its benefits, let us firstly try to find such trapezoidal fuzzy number $T_{A_1,A_2,...,A_n}$, which is the nearest one to all members of the initial data set $A_1, A_2, ..., A_n$, with respect to the weighted distance d_λ, $\lambda = (\lambda_L, \lambda_U)$. In other words, we are looking for such trapezoidal fuzzy number $T_{A_1,A_2,...,A_n}$, which minimizes the distance $D_\lambda((A_1, ..., A_n), T_{A_1,A_2,...,A_n})$, where

$$D_\lambda^2((A_1, ..., A_n), T_{A_1,A_2,...,A_n}) = \sum_{i=1}^{n} d_\lambda^2(A_i, T_{A_1,A_2,...,A_n}). \tag{4.6}$$

We say that $T_{A_1,A_2,...,A_n}$ is the nearest trapezoidal fuzzy number to fuzzy numbers $A_1, A_2, ..., A_n$. Obviously, the metric D_λ is considered on the power space $(F(\mathbb{R}))^n$ and $T = T_{A_1,A_2,...,A_n}$ is uniquely identified with the n-tuple $(T, ..., T)$.

Using the theory of Hilbert spaces we obtain the following main result.

Theorem 4.9.2 *([28], Theorem 6) The trapezoidal fuzzy number nearest to fuzzy numbers $A_1, ..., A_n$ (with respect to metric D_λ), is the trapezoidal fuzzy number nearest to fuzzy number $\overline{A} = \frac{1}{n}\sum_{i=1}^{n} A_i$ (with respect to metric d_λ), which means that we have $T_{A_1,A_2,...,A_n} = T_{d_\lambda}(\overline{A})$, where T_{d_λ} is the weighted trapezoidal approximation operator with respect to the metric d_λ.*

Theorem 4.9.2 shows that the trapezoidal fuzzy number nearest to a given family of fuzzy numbers equals the trapezoidal fuzzy number nearest to their average (provided any solution exists). In other words, there is no difference whether the approximation is performed before or after aggregation when the average is chosen as aggregation operator.

We have a similar result if in addition we impose the preservation of the weighted expected interval. We denote with $EI^\lambda(A)$ (see Definition 1.12.1) the weighted expected interval of the fuzzy number A where $\lambda = (\lambda_L, \lambda_U)$ and λ_L,λ_U are weights. Then we define the weighted expected interval of the sample of fuzzy numbers $A_1, ..., A_n$, as

$$EI^\lambda(A_1, A_2, ..., A_n) = \frac{1}{n}\left(EI^\lambda(A_1) + ... + EI^\lambda(A_n)\right).$$

Theorem 4.9.3 *([28], Theorem 10) The trapezoidal fuzzy number nearest to fuzzy numbers $A_1, ..., A_n$ (with respect to metric D_λ), which preserves the weighted expected interval of $A_1, ..., A_n$, is the trapezoidal fuzzy number nearest to fuzzy number $\overline{A} = \frac{1}{n}\sum_{i=1}^{n} A_i$ (with respect to metric d_λ), which preserves the weighted expected interval of $\overline{A}$, that is $T_{A_1,A_2,...,A_n} = T_{EI,\lambda}(\overline{A})$.*

The conclusion is the same as in the case of approximation without conditions.

Example 4.9.4 *([28], Example 11) Let us consider the weighted functions $\lambda_L(\alpha) = \lambda_L(\alpha) = 1$, for every $\alpha \in [0,1]$ and fuzzy numbers A, B, C, with $A_\alpha = \left[-1+\alpha^2, 4-2\alpha^2\right]$, $B_\alpha = \left[1+\alpha^2, 3-\alpha^2\right], \alpha \in [0,1]$, and $C_\alpha = [45\sqrt{\alpha}, 46-\sqrt{\alpha}], \alpha \in [0,1]$. According with Theorem 3.5.1, (i) we get the trapezoidal fuzzy number nearest to $\frac{1}{2}\cdot(A+B)$ preserving the expected interval of $\frac{1}{2}\cdot(A+B)$, as $T_{EI}\left(\frac{1}{2}\cdot(A+B)\right) = \left(-\frac{1}{6}, \frac{5}{6}, \frac{9}{4}, \frac{15}{4}\right)$. Theorem 4.9.3 implies that $\left(-\frac{1}{6}, \frac{5}{6}, \frac{9}{4}, \frac{15}{4}\right)$ is the trapezoidal fuzzy number nearest to A and B which preserves the expected interval of the set of fuzzy numbers A, B. We obtain (Theorem 3.5.1, (iv)) the trapezoidal fuzzy number nearest to $\frac{1}{3}\cdot(A+B+C)$ preserving the expected interval of $\frac{1}{3}\cdot(A+B+C)$, as $T_{EI}\left(\frac{1}{3}\cdot(A+B+C)\right) = \left(\frac{707}{180}, \frac{991}{60}, \frac{991}{60}, \frac{3187}{180}\right)$. According with Theorem 4.9.3 we get that $\left(\frac{707}{180}, \frac{991}{60}, \frac{991}{60}, \frac{3187}{180}\right)$ is the trapezoidal fuzzy number nearest to A, B and C, which preserves the expected interval of the set of fuzzy numbers A, B, C.*

Chapter 5

Approximations of fuzzy numbers by Bernstein operators of max-product kind

There exists an extensive literature about linear approximation operators. However, there are certain situations when linear operators are not efficient. For example when we approximate a fuzzy number u which has its core as a proper interval by using the linear Bernstein operator B_n (actually we approximate the restriction of the membership function to its support), then the quality of the approximation is questionable. First of all, it is easy to check that in general $B_n(u)$ is not a fuzzy number. Of course, if we normalize $B_n(u)$, noting that the Bernstein linear operators preserve the quasi-concavity, then we obtain a fuzzy number but the normalization process is not always an easy task since there are many examples when the maximum value of a function cannot be computed exactly. Then, since the core of $B_n(u)$ is reduced to a single element, it follows that the segment cores of the sequence $(B_n(u))_{n\geq 1}$ do not converge towards $core(u)$. Therefore, we propose a different approach to the problem of the approximation of the membership function of a fuzzy number by sequences of approximation operators. We will use the so called max-product operators introduced recently. We will see that besides the convergence in the uniform norm or L_2-type metrics, they also preserve the support and they converge with respect to the core. In addition, it holds the convergence of the important characteristics such as the expected interval, ambiguity or value.

We need to discuss the notations used through out this chapter. If $I \subseteq \mathbb{R}$ is an interval and $(T_n)_{n\geq 1}$, $T_n : C(I) \to X$, is a sequence of operators then the image of $f \in C(I)$ will be denoted with $T_n(f)$ for some $n \geq 1$. If $J \subseteq \mathbb{R}$ is an interval such that $I \subseteq J$ and $f : J \to \mathbb{R}$ is a continuous function, then we can apply T_n to the restriction of f on I and in this case we will use the notation $T_n(f;I)$ for some $n \geq 1$. Sometimes, to avoid confusions we may use this notation even if $J = I$.

Then, if $I \subseteq \mathbb{R}$ is an interval and $f \in C(I)$ (here $C(I)$ denotes the space of all continuous real functions defined on I), we denote by $\|f\|$ the Chebyshev norm of f in $C(I)$, that is $\|f\| = \sup_{x\in I} |f(x)|$. However, if u denotes a fuzzy number then we prefer the notation $\|u\|_C$ for its Chebyshev norm, that is (see formula (1.6)) $\|u\|_C = \sup_{x\in\mathbb{R}} |u(x)|$. We denote by $D_c(u,v)$ the Chebyshev distance between fuzzy numbers u and v according to formula (1.5).

Mostly, in this chapter we are interested in approximating fuzzy numbers by sequences of fuzzy numbers generated by approximation operators. The estimates will be measured by using the uniform modulus of continuity. For a real interval I and a continuous function $f: I \to \mathbb{R}$, the function $\omega_1(f, \cdot) : [0, \infty) \to [0, \infty)$,

$$\begin{aligned} \omega_1(f,\delta) &= \sup\{|f(x)-f(y)| : x, y \in I, |x-y| \le \delta\} \\ &= \sup\{|f(x+h)-f(x)| : x, x+h \in I, 0 \le h \le \delta\}, \end{aligned}$$

is called the modulus of continuity of the function f. If J is an interval included in I then we often denote

$$\begin{aligned} \omega_1(f,\delta)_J &= \sup\{|f(x)-f(y)| : x, y \in J, |x-y| \le \delta\} \\ &= \sup\{|f(x+h)-f(x)| : x, x+h \in J, 0 \le h \le \delta\}. \end{aligned}$$

If I is a compact interval then $\omega_1(f, \cdot)$ is uniformly continuous and from $\delta_n \searrow 0$, it easily results that $\omega_1(f, \delta_n) \searrow 0$. For more details about the modulus of continuity we refer to [8] and [10].

This chapter contains original contributions from the papers [30], [44], [46] and [47]. In addition the chapter contains original unpublished results and also some results are improved comparing to their published versions.

5.1 A discussion on sequences of fuzzy numbers

All of this section contains original unpublished results. They are well connected to the unpublished results from Section 1.12. All these results are likely to be incorporated in one or more papers dealing with the approximation of important characteristics of fuzzy numbers. This problem has been already discussed in the paper [47] and actually Example 5.1.1 is taken from this paper. As we have already mentioned in Section 1.12, there is an ongoing research on this topic too (see [48]), therefore, some results from this section as well as some results from Section 1.12 are subject to be included in this work.

We mention that through out this chapter, for a fuzzy number u we will use the parametric representation (u^-, u^+) instead of (u_L, u_U), because we find it more appropriate as we will use many notations from approximation theory.

For a fixed fuzzy number u, we are interested in finding a sequence of fuzzy numbers $(u_n)_{n\ge1}$ such that the following requirements would hold:

(i) $(\delta) \lim\limits_{n\to\infty} u_n = u$;
(ii) $core(u_n) \to core(u)$;
(iii) $supp(u_n) \to supp(u)$;
(iv) $EI(u_n) \to EI(u)$;
(v) $Amb_s(u_n) \to Amb_s(u)$ and $Val_s(u_n) \to Val_s(u)$, for some reducing function s.

Sometimes we do not need all the requirements since some of them could be consequences of others. This will be the case of the Chebyshev type distance between fuzzy numbers D_C presented in Section 1.9. We will see later that in the case of this metric we need only requirement (i) together with a much weaker condition than $(ii)-(iii)$ from above, to get $(iv)-(v)$. Therefore, this will easily imply that if $(i)-(iii)$ hold then $(iv)-(v)$ hold too.

It is easy to prove that depending on the metric, there exist convergent sequences of fuzzy numbers for which some of the above requirements do not hold. Here is one example which involves the metric D_C but also please note that in the longer version of the thesis there exists an example in the support of the same idea, which involves the metric d_p (see formula (1.11)).

Example 5.1.1 *(this example is discussed in [47] too) Perhaps one of the most popular approximation operators that come in our mind are the linear Bernstein operators. For a continuous function* $f : [a,b] \to \mathbb{R}$, *we denote with* $B_n(f)$ *the Bernstein operator of order* n *attached to the function* f. *Let* u *denotes a continuous fuzzy number with* $supp(u) = [a,b]$, $a < b$ *and* $core(u) = [c,d]$, $c < d$. *We define the Bernstein operator of order* n *attached to* u *denoted with* $\widetilde{B}_n(u)$, *given by*

$$\widetilde{B}_n(u)(x) = 0, \text{ for } x \text{ outside } [a,b]$$

and

$$\widetilde{B}_n(u)(x) = B_n(u,[a,b]) = \sum_{k=0}^{n} p_{n,k}(x) \cdot u\left(a + (b-a)k/n\right), \ x \in [a,b],$$

where $p_{n,k}(x) = \binom{n}{k}\left(\frac{x-a}{b-a}\right)^k \cdot \left(\frac{b-x}{b-a}\right)^{n-k}$, $k \in \{0,1,...,n\}$, *are the fundamental Bernstein polynomials. It is well known that the Bernstein operators have interpolative properties at the endpoints and that they preserve the quasi-concavity (see [69]) . However, it is easily seen that since* u *is continuous and since* $\|u\|_C = 1$ *it results that* $\|B_n(u,[a,b])\| < 1$, *for sufficiently large* $n \in \mathbb{N}$. *For this reason, in order to produce proper fuzzy numbers we need to normalize* $\widetilde{B}_n(u)$. *In this way we obtain the sequence of fuzzy numbers* $\left(\frac{1}{\|B_n(u,[a,b])\|} \cdot \widetilde{B}_n(u)\right)_{n \geq 1}$. *Now, it is well known that* $\widetilde{B}_n(u)$ *converges uniformly to* u *since there exists an absolute constant* C *such that*

$$|B_n(u,[a,b]) - u(x)| \leq C\omega_1(u, 1/\sqrt{n})_{[a,b]}, (\forall)\, x \in [a,b],$$

which easily implies that $\frac{1}{\|B_n(u,[a,b])\|} \cdot \widetilde{B}_n(u)$ *converges to* u *with respect to the metric* D_C. *On the other hand, it is very easy to prove that the core of* $\frac{1}{\|B_n(u,[a,b])\|} \cdot \widetilde{B}_n(u)$ *is reduced to a single element (this is immediate since the restriction of* $\frac{1}{\|B_n(u,[a,b])\|} \cdot \widetilde{B}_n(u)$ *to the interval* $[a,b]$ *is a polynomial function of degree at least* 1 *and hence it cannot be constant on an interval as it is actually the core). This means that it does not hold the convergence of the core since we have assumed that the core of* u *is a nondegenerated interval. This is not surprising since the Bernstein operators as they are polynomials cannot approximate with accuracy the shape of a function which has points where it is not differentiable and clearly this is the case of the endpoints of the core of* u. *This lack of property will be discussed again later in Subsection 5.5.1 of this chapter where a comparison with the Bernstein max-product operators will be done. The same phenomena of divergence on the core appears when we consider other types of linear Bernstein operators since most of them are polynomials.*

In what follows we propose some minimal requirements in order to get properties $(iv)-(v)$ described at the beginning of this section.

Lemma 5.1.2 *Suppose that* $s : [0,1] \to [0,1]$, *is a continuous reducing function and consider its antiderivative* $S : [0,1] \to \mathbb{R}$, $S(x) = \int_0^x s(x)dx$. *Let* u *denotes a continuous fuzzy number and let us consider the sequence of continuous fuzzy numbers* $(u_n)_{n \geq 1}$ *satisfying the following requirements:*

(i) $(D_C) \lim_{n \to \infty} u_n = u$;

(ii) There exists a constant $\beta_0 > 0$ *which may depend only on* u *such that* $supp(u_n) \subseteq [-\beta_0, \beta_0]$, *for all* $n \geq 1$.

Then it holds that $EI(u_n) \to EI(u)$, $Amb_s(u_n) \to Amb_s(u)$, *and* $Val_s(u_n) \to Val_s(u)$.

Remark 5.1.3 *If in the previous lemma* $\sup\{\max\{|a_n|, |d_n|\} : n \in \mathbb{N}\} = \infty$*, then even if* $(D_C)\lim_{n\to\infty} u_n = u$*, we can have*

$$\max\left\{\limsup_{n\to\infty}\left|\int_a^c xd(S(u(x)) - \int_{a_n}^{c_n} xd(S(u_n(x))\right|, \limsup_{n\to\infty}\left|\int_d^b xd(S(u(x)) - \int_{d_n}^{b_n} xd(S(u_n(x))\right|\right\} = \infty.$$

From Theorem 5.1.2 we easily obtain the following corollary.

Corollary 5.1.4 *Let* u *denotes a continuous fuzzy number and let us consider the sequence of continuous fuzzy numbers* $(u_n)_{n\geq 1}$*, such that:*

(i) $(D_c)\lim_{n\to\infty} u_n = u$ *;*

(ii) $core(u_n) \to core(u)$;

(iii) $supp(u_n) \to supp(u)$;

Then we have $EI(u_n) \to EI(u)$*,* $Amb_s(u_n) \to Amb_s(u)$*, and* $Val_s(u_n) \to Val_s(u)$*, for any reduction function* $s : [0,1] \to [0,1]$.

In the case of the metric d_p we have the following.

Lemma 5.1.5 *Let* u *be a fuzzy number and let us consider the sequence of fuzzy numbers* $(u_n)_{n\geq 1}$*, such that* $(d_p)\lim_{n\to\infty} u_n = u$*, for some* $p > 1$*. If* $s : [0,1] \to [0,1]$*, is a reducing function then it holds that* $EI(u_n) \to EI(u)$*,* $Amb_s(u_n) \to Amb_s(u)$*, and* $Val_s(u_n) \to Val_s(u)$*.*

In view of the previous two lemmas, we are interested in finding sequences of fuzzy numbers such that the requirements in the above lemmas would hold since this would imply the convergence of the important characteristics too. We will see later that the Bernstein operators of max-product kind fulfil these requirements.

5.2 Examples of max-product operators

In this section we discuss about the so called max-product type operators introduced for the first time in the paper [34]. For some interval $I \subseteq \mathbb{R}$, we denote with $CB_+(I)$ the space of all positive, bounded and continuous real functions defined on I. The general form of $L_n : CB_+(I) \to CB_+(I)$, (called here a discrete max-product type approximation operator) will be

$$L_n(f)(x) = \bigvee_{i=0}^{n} K_n(x, x_i) \cdot f(x_i), x \in I, \tag{5.1}$$

or

$$L_n(f)(x) = \bigvee_{i=0}^{\infty} K_n(x, x_i) \cdot f(x_i), x \in I, \tag{5.2}$$

where $n \in \mathbb{N}$, $f \in CB_+(I)$, $K_n(\cdot, x_i) \in CB_+(I)$ and $x_i \in I$, for all i. These operators are nonlinear, positive and moreover they satisfy a pseudo-linearity condition of the form

$$L_n(\alpha \cdot f \vee \beta \cdot g) = \alpha \cdot L_n(f) \vee \beta \cdot L_n(g), \forall \alpha, \beta \in \mathbb{R}_+, f, g \in CB_+(I).$$

Let us consider the interval $I = [0, 1]$ and a function $f \in C_+(I)$ (here $C_+(I)$ is the space of all positive and continuous real functions defined on I and, obviously in our case $C_+(I)$ coincides with $CB_+(I)$). If in relation (5.1) we take $I = [0, 1]$ and

$$K_n(x, x_i) = \frac{p_{n,i}(x)}{\bigvee_{k=0}^{n} p_{n,k}(x)}, x \in [0, 1], i \in \{0, 1, ..., n\},$$

then we obtain the so called Bernstein operator of max-product kind (introduced for the first time by Gal in the book [55]) attached to the function f, given by

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(x)}, x \in [0, 1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k \in \{0, 1, ..., n\}$, $x \in [0, 1]$. Another method to obtain the Bernstein max-product operator is to write the linear Bernstein operator attached to f in the form

$$B_n(f)(x) = \frac{\sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)}{\sum_{k=0}^{n} p_{n,k}(x)}, x \in [0, 1]$$

and then replacing in the numerator and in the denominator the "sum" operator with the "max" operator we obtain again the max-product Bernstein operator.

The approximation and shape preserving properties of the Bernstein operators of max-product kind have been studied (in this order) in the papers [33], [30], [44]. Some of these properties will be mentioned in the next sections.

Reasoning as in the case of the Bernstein operator of max-product kind, for $I = [0, \infty)$ and $f \in CB_+(I)$, the Favard-Szász-Mirakjan operator of max-product kind ([33]) attached to f is given by

$$F_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty} s_{n,k}(x)}, x \in [0, 1],$$

where $s_{n,k}(x) = \frac{(nx)^k}{k!}$, $k \in \mathbb{N}$, $x \in [0, \infty)$. The approximation and shape preserving properties of the Favard-Szász-Mirakjan operator of max-product kind have been studied firstly in [33] and then in [31].

There are also other types of max-product operators which are recalled in the longer version of the thesis.

5.3 Approximation and shape preserving properties of the Bernstein operator of max-product kind

This section contains original contributions from the papers [30] and [44]. The theoretical results of this section will be used to study the approximation and shape preserving properties when we approximate fuzzy numbers by the Bernstein max-product operators.

In the paper [33] an upper estimate of the approximation error by the Bernstein operator of max-product kind of the form $C\omega_1(f, 1/\sqrt{n})$ (with $C > 0$ unexplicit absolute constant) was obtained for functions from the space $C_+([0,1])$. Then in the paper [30] this result was improved by finding an explicit constant in front of $\omega_1(f, 1/\sqrt{n})$, as follows.

Theorem 5.3.1 *([30], Theorem 4.1) If $f : [0,1] \to \mathbb{R}_+$ is continuous then we have the estimate*

$$|B_n^{(M)}(f)(x) - f(x)| \leq 12\omega_1\left(f, \frac{1}{\sqrt{n+1}}\right), n \in \mathbb{N}, x \in [0,1].$$

In the paper [44] it was proved that the above estimation is the best possible with respect to $\omega_1(f, \cdot)_{[0,1]}$, by proving that for the function $f : [0,1] \to [0, \infty)$, $f(x) = 0$ if $x \in [0, 1/2]$ and $f(x) = x - 1/2$ if $x \in [1/2, 1]$, we have ([44], Example 3.1) $\left\|B_n^{(M)}(f) - f\right\| \geq \frac{e^{-5}}{6}\omega_1(f, 1/\sqrt{n})$.

For concave functions we have a Jackson type estimation.

Theorem 5.3.2 *([30], Corollary 4.6) If $f : [0,1] \to \mathbb{R}_+$ is concave on $[0,1]$, then we have the estimate $|B_n^{(M)}(f)(x) - f(x)| \leq 2\omega_1\left(f; \frac{1}{n}\right)$, for all $n \in \mathbb{N}, x \in [0,1]$.*

Another kind of estimation is the following.

Theorem 5.3.3 *([44], Theorem 4.6) Let $f : [0,1] \to [0, \infty)$ be a continuous and strictly positive function. Then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 4\right)\omega_1(f, 1/n), x \in [0,1], n \in \mathbb{N},$$

where $m_f = min\{f(x); x \in [0,1]\}$.

From the above theorem we obtain the following estimation for Lipschitz functions.

Corollary 5.3.4 *([44], Corollary 4.7) If $f : [0,1] \to [0, \infty)$ is a strictly positive function satisfying the Lipschitz condition, then there exists a constant C independent of n and x, but depending on f, such that $\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \frac{C}{n}, x \in [0,1]$, for all $n \in \mathbb{N}$.*

It is known that there exist Lipschitz functions on $[0,1]$ (for example concave polygonal lines) for which worst estimations are obtained than the Jackson type estimate from above when we approximate them by using the linear Bernstein operator. It means that the Bernstein operator of max-product kind has better approximation properties than its linear counterpart relatively to the class of strictly positive Lipschitz functions defined on the interval $[0,1]$.

We discuss now the shape preserving properties of the Bernstein operators of max-product kind.

In the paper [30] it was proved that relatively to the space $C_+([0,1])$, the Bernstein operator of max-product kind has interpolative properties at the endpoints, preserves the monotonicity and more generally the quasi-convexity. Then, in the paper [44] the following result was proposed.

Theorem 5.3.5 *([44], Theorem 5.1) Let us consider the function $f : [0,1] \to \mathbb{R}_+$ and let us fix $n \in \mathbb{N}$, $n \geq 1$. Suppose in addition that there exists $c \in [0,1]$ such that f is nondecreasing on $[0,c]$ and nonincreasing on $[c,1]$. Then, there exists $c' \in [0,1]$ such that $B_n^{(M)}(f)$ is nondecreasing on $[0,c']$ and nonincreasing on $[c',1]$. In addition we have $|c - c'| \leq \frac{1}{n+1}$ and $\left|B_n^{(M)}(f)(c) - f(c)\right| \leq \omega_1\left(f, \frac{1}{n+1}\right)$.*

From the above theorem it follows that $B_n^{(M)}$ preserves the quasi-concavity relatively to the space $C_+([0,1])$. Moreover, any global maximum point of the function f can be approximated with a global maximum point of the function $B_n^{(M)}(f)$ with an error of order $O(1/n)$ and any global maximum value of f can be approximated with a global maximum value of $B_n^{(M)}(f)$ with an error of order $O(\omega_1(f, \frac{1}{n}))$.

5.4 Max-product Bernstein operators defined on compact intervals

This section contains original contributions from the papers [46] and [47].

For a function $f \in C_+([a,b])$, we define the corresponding max-product Bernstein operator on $[a,b]$, by ([46])

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f(a + k \cdot \frac{b-a}{n})}{\bigvee_{k=0}^{n} p_{n,k}(x)}, x \in [a,b], \tag{5.3}$$

where $p_{n,k}(x) = \binom{n}{k}\left(\frac{x-a}{b-a}\right)^k \cdot \left(\frac{b-x}{b-a}\right)^{n-k}$. Since $\sum_{k=0}^{n} p_{n,k}(x) = 1$ for all $x \in [a,b]$, it is immediate that $\bigvee_{k=0}^{n} p_{n,k}(x) > 0$ for all $x \in [a,b]$, which means that $B_n^{(M)}(f)$ is well defined. Also, we easily get $B_n^{(M)}(f)(a) = f(a)$ and $B_n^{(M)}(f)(b) = f(b)$. Then, since the maximum of a finite number of continuous functions is a continuous function, we get that for any $f \in C_+([a,b])$, $B_n^{(M)}(f) \in C_+([a,b])$ too. Actually, if $f : [a,b] \to \mathbb{R}_+$ is only bounded, then one can easily prove that $B_n^{(M)}(f) \in C_+([a,b])$.

It can be proved that $B_n^{(M)} : C_+([a,b]) \to C_+([a,b])$ has the same order of uniform approximation as the linear Bernstein operator and that it preserves the quasi-concavity too.

Theorem 5.4.1 *(i) ([46], Theorem 5) If $a, b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}_+$ is continuous, then we have the estimate $|B_n^{(M)}(f)(x) - f(x)| \leq 12([b-a]+1)\omega_1\left(f, \frac{1}{\sqrt{n+1}}\right)$, for all $n \in \mathbb{N}, x \in [a,b]$.*

(ii) ([47], Theorem 6 (ii)) If $f : [a,b] \to \mathbb{R}_+$ is concave on $[a,b]$, then we have the estimate $|B_n^{(M)}(f)(x) - f(x)| \leq 2([b-a]+1)\omega_1\left(f, \frac{1}{n}\right)$, for all $n \in \mathbb{N}, x \in [a,b]$.

Theorem 5.4.2 *([46], Theorem 6) Let us consider the function $f : [a,b] \to \mathbb{R}_+$ and let us fix $n \in \mathbb{N}$, $n \geq 1$. Suppose in addition that there exists $c \in [a,b]$ such that f is nondecreasing on $[a,c]$ and nonincreasing on $[c,b]$ respectively. Then, there exists $c' \in [a,b]$ such that $B_n^{(M)}(f)$ is nondecreasing on $[a,c']$ and nonincreasing on $[c',b]$. In addition we have $|c - c'| \leq \frac{b-a}{n+1}$ and $\left|B_n^{(M)}(f)(c) - f(c)\right| \leq ([b-a]+1)\,\omega_1\left(f, \frac{1}{n+1}\right)$.*

Remark 5.4.3 *From the above theorem and by the comment just below Theorem 5.3.5, it results that if $f : [a,b] \to \mathbb{R}_+$ is continuous and quasi-concave then $B_n^{(M)}(f)$ is quasi-concave too.*

Remark 5.4.4 *As we have mentioned in the Section 5.3, for functions in the space $C_+([0,1])$, $B_n^{(M)}$ preserves the monotonicity and the quasi-convexity. Reasoning similarly as in the proof of Theorem 5.4.2, it can be proved that these preservation properties hold in the general case of the space $C_+([a,b])$.*

5.5 Applications to the approximation of fuzzy numbers

This section contains original contributions from the papers [46] and [47].

5.5.1 Approximations with respect to the metric D_C

Suppose that u is a fuzzy number such that $supp(u) = [a, b]$ and $core(u) = [c, d]$. For $n \in \mathbb{N}$ we introduce the function $\widetilde{B}_n^{(M)}(u) : \mathbb{R} \to [0, 1]$, $\widetilde{B}_n^{(M)}(u)(x) = 0$, for all x outside $[a, b]$ and we have $\widetilde{B}_n^{(M)}(u)(x) = B_n^{(M)}(u; [a, b])(x)$, for all $x \in [a, b]$. From Theorem 5.4.1, it results that the order of uniform approximation of the fuzzy number u by $\widetilde{B}_n^{(M)}(u)$ is $O\left(\omega_1(u, 1/\sqrt{n})_{[a,b]}\right)$ in the case when u is continuous. Then, since the restriction of u to the interval $[a, b]$ is a function like those considered in Theorem 5.4.2, it results that $\widetilde{B}_n^{(M)}(u)$ is a quasi-concave function on $[a, b]$. Moreover, we have the following result which improves the main result of paper [46].

Theorem 5.5.1 *([47], Theorem 14) Let u denotes a fuzzy number with $supp(u) = [a, b]$ and $core(u) = [c, d]$ such that $a \le c < d \le b$. Then for sufficiently large n, it results that $\widetilde{B}_n^{(M)}(u)$ is a fuzzy number such that :*

(i) $supp(u) = supp\left(\widetilde{B}_n^{(M)}(u)\right)$;

(ii) If $core(\widetilde{B}_n^{(M)}(u)) = [c_n, d_n]$, then $|c - c_n| \le \frac{b-a}{n}$ and $|d - d_n| \le \frac{b-a}{n}$. Moreover, we can determine exactly $core(\widetilde{B}_n^{(M)}(u))$;

(iii) If, in addition, u is continuous on $[a, b]$, then

$$\left|\widetilde{B}_n^{(M)}(u)(x) - u(x)\right| \le 12([b-a]+1)\omega_1\left(u, \frac{1}{\sqrt{n+1}}\right)_{[a,b]},$$

for all $x \in \mathbb{R}$.

The proof of the following corollary which improves Corollary 15 from [47], is immediate from the previous theorem and Corollary 1.12.4, (i).

Corollary 5.5.2 *(see also [47], Corollary 15) Let u denotes a fuzzy number with $supp(u) = [a, b]$ and $core(u) = [c, d]$, such that $a \le c < d \le b$. In addition, let us consider the continuous reduction function $s : [0, 1 \to [0, 1]$ and its antiderivative $S(x) = \int_0^1 s(x)dx$, $x \in [0, 1]$. Then we have:*

(i) $core(\widetilde{B}_n^{(M)}(u)) \to core(u)$;

(ii) If u is continuous then $\lim_{n\to\infty} D_C(\widetilde{B}_n^{(M)}(u), u) = 0$ and denoting $\delta_n = 12([b-a] + 1)\omega_1\left(u, \frac{1}{\sqrt{n+1}}\right)_{[a,b]}$, for sufficiently large n we obtain $\left|Amb_s(\widetilde{B}_n^{(M)}(u)) - Amb_s(u)\right| \le k_n(u)\delta_n$ and $\left|Val_s(\widetilde{B}_n^{(M)}(u)) - Val_s(u)\right| \le k_n(u)\delta_n$, where $k_n(u) \to c - a + 2|c| + b - d + 2|d|$. In particular we obtain $EI(\widetilde{B}_n^{(M)}(u)) \to EI(u)$.

From Theorem 5.5.1 and Corollary 5.5.2 it follows that the sequence of Bernstein max-product operators attached to a continuous fuzzy number fulfil the approximation and shape preserving properties mentioned in Section 5.1 and hence they are a good example of an efficient convergent sequence of fuzzy numbers.

Remarks (these remarks can be found in [47] too) (i) If the fuzzy number u is unimodal, that is $c = d$, then $\widetilde{B}_n^{(M)}(u)$ is not necessarily a fuzzy number. But normalizing $B_n^{(M)}(u;[a,b])$, we obtain the fuzzy number $\frac{1}{\left\|B_n^{(M)}(u;[a,b])\right\|}\widetilde{B}_n^{(M)}(u)$ (recall that $\|\cdot\|$ denotes the uniform norm). Since $B_n^{(M)}(u;[a,b]) \to u$ uniformly on $[a,b]$, we easily get that $\frac{1}{\left\|B_n^{(M)}(u;[a,b])\right\|}\widetilde{B}_n^{(M)}(u) \to u$, uniformly on $\mathbb{R}$ and thus we get that $\lim_{n\to\infty} D_C\left(\frac{1}{\left\|B_n^{(M)}(u;[a,b])\right\|}\widetilde{B}_n^{(M)}(u), u\right) = 0$. As in the case of Theorem 5.5.1 (ii), we can determine precisely the core of $\frac{1}{\left\|B_n^{(M)}(u;[a,b])\right\|}\widetilde{B}_n^{(M)}(u)$.

(ii) Comparing the conclusions of Theorem 5.5.1 with the comments from the end of Example 5.1.1, it follows that the max-product Bernstein operator $B_n^{(M)}$, is more convenient for approximating fuzzy numbers than the classical linear Bernstein operator, B_n. While the order of uniform approximation is the same, the max-product Bernstein operator preserves better the shape of the approximated fuzzy number.

(iii) It can be easily proved that if u is a unimodal continuous fuzzy number then the sequences considered in the present Remark (i) satisfy all the conclusions of Corollary 5.5.2.

Example 5.5.3 *([47], Example 16) We approximate the fuzzy number*

$$u(x) = \begin{cases} 4x^2 & \text{if } 0 \le x < 1/2, \\ 1 & 1/2 \le x \le 3/4, \\ 4-4x & 3/4 < x \le 1, \\ 0 & \text{otherwise}, \end{cases} .$$

using both the classical and the nonlinear max-product Bernstein operators. In Fig. 5.1.a and Fig. 5.1.b (firstly for $n = 30$ and then for $n = 80$) we can compare the classical and nonlinear max-product operators in approximating the above fuzzy number. We can easily see that the classical linear operator marked with dotted line is outperformed by the max-product operator marked with dashed line, this being almost coincident with the target fuzzy number at its core. The theoretical conclusions of this section are well illustrated by this particular example.

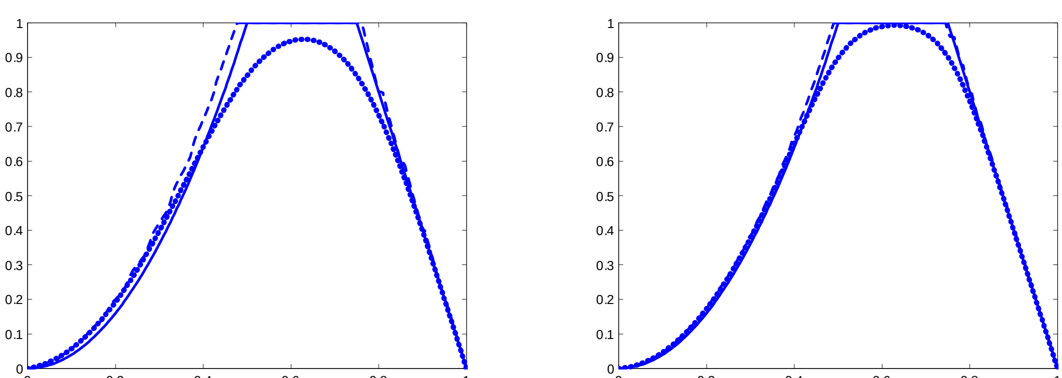

Fig. 5.1.a Fig. 5.1.b

5.5.2 Approximations with respect to the metrics d_p

Recall that the parametric form of an arbitrary fuzzy number u is $u = (u^-, u^+)$, where u^- is left-continuous and nondecreasing, u^+ is left-continuous and nonincreasing and in addition we have $u^-(1) \leq u^+(1)$. All these properties are essential in the obtaining of the main results of this subsection.

Firstly, we deal with the case when the fuzzy number u is positive, i.e. $u^-(0) \geq 0$. This means that both functions u^- and u^+ are positive and hence we can apply all the approximation results from Section 5.3. We can attach the Bernstein max-product operators $B_n^{(M)}(u^-)$ and $B_n^{(M)}(u^+)$ and then it is immediate that the ordered pair $\overline{B}_n^{(M)}(u) = \left(B_n^{(M)}(u^-), B_n^{(M)}(u^+)\right)$ is a proper fuzzy number. Now we are in position to present the main result of this subsection.

Theorem 5.5.4 *(see also [47], Theorem 19) If $u = (u^-, u^+)$ is a positive fuzzy number such that u^- and u^+ are continuous, then we have*

(i)

$$d_p(u, \overline{B}_n^{(M)}(u)) \leq 12\sqrt{2} \max \left\{ \omega_1 \left(u^-; \frac{1}{\sqrt{n+1}} \right), \omega_1 \left(u^+; \frac{1}{\sqrt{n+1}} \right) \right\}, \text{ for all } n \in \mathbb{N};$$

(ii) (in [47] a more particular case is discussed) $EI(u_n) \to EI(u)$, $Amb_s(u_n) \to Amb_s(u)$ and $Val_s(u_n) \to Val_s(u)$ for any reduction function $s : [0,1] \to [0,1]$.

Remark (this remark can be found in [47] too) Suppose now that the fuzzy number u is not positive, i.e. $u^-(0) < 0$. Let us introduce the functions $u_1^-, u_1^+ : [0,1] \to \mathbb{R}$, $u_1^-(\alpha) = u^-(\alpha) - u^-(0)$ and $u_1^+(\alpha) = u^+(\alpha) - u^-(0)$. From the properties of u^- and u^+, it results that u_1^- is nondecreasing and positive and u_1^+ is nonincreasing and positive respectively. For some $n \geq 1$, we attache the Bernstein max-product operators $B_n^{(M)}(u_1^-)$ and $B_n^{(M)}(u_1^+)$. Since $B_n^{(M)}$ preserves the monotonicity, it is immediate that $B_n^{(M)}(u_1^-)$ is nondecreasing and $B_n^{(M)}(u_1^+)$ is nonincreasing. In addition we have $B_n^{(M)}(u_1^-)(0) = u_1^-(0)$, $B_n^{(M)}(u_1^-)(1) = u_1^-(1)$, $B_n^{(M)}(u_1^+)(0) = u_1^+(0)$ and $B_n^{(M)}(u_1^+)(1) = u_1^+(1)$. In conclusion we obtain that $\overline{P}_n^{(M)}(u) = \left(B_n^{(M)}(u_1^-) + u^-(0), B_n^{(M)}(u_1^+) + u^-(0)\right)$, is a proper fuzzy number which in addition preserves the core and the support of u. Moreover, it can be proved that we obtain the same kind of estimates by replacing in Theorem 5.5.4 $\overline{B}_n^{(M)}(u)$ with $\overline{P}_n^{(M)}(u)$.

Conclusions

This thesis contains my contributions in the topic of the approximation of fuzzy numbers. At first, considering L_2- type metrics in the space of fuzzy numbers, numerous kinds of parametric or trapezoidal approximations are investigated. Since in the cases under study, the parametric or trapezoidal approximation exists and it is unique, we can define so called parametric or trapezoidal approximation operators. In the thesis we presented algorithms to compute the proper approximations and apart of that, important properties such as scale invariance, translations invariance, additivity, continuity or the relation with the aggregation of the data are investigated. For some operators the best possible Lipschitz constant is obtained. These results are important when we want to simplify the representation of fuzzy numbers. In some applications it suffices to work only with trapezoidal fuzzy numbers or other classes of fuzzy numbers with simpler form. Such an example can be found in Section 2.8 where fuzzy numbers are ranked by using trapezoidal fuzzy numbers. However, sometimes we are interested to preserve as much as possible from the information carried out by a fuzzy number. This topic should be regarded as a topic in the completeness of the previously discussed one, where approximations with simpler form are investigated. In this thesis we have proved that the so called Bernstein operators of max-product kind could be useful tools when it comes to approximate fuzzy numbers such that most of the informations to be preserved. Besides their capability to approximate fuzzy numbers with the same accuracy as the linear counterparts, they have important shape preserving properties such as the preservation of the support, the convergence towards the core, and the convergence with respect to the important characteristics of fuzzy numbers. In order to obtain a part of the main results of the thesis, I have proved some facts which are of independent importance with respect to this topic. Such results are those from Section 1.12 where the ambiguity and value are represented in terms of membership function and then approximations of these characteristics are investigated. As expected, the better the shape of the fuzzy number is preserved, the better estimation is obtained. Then, it should be stressed out that the thesis contains contributions in the sensible topic of the ranking of fuzzy numbers. In the case of trapezoidal fuzzy numbers we can say that defuzifiers which generate orders satisfying some desirable properties are completely determined abstraction making of equivalent orders.

I have many plans to continue the investigations in approximation problems and others. Recently we have extended approximations of fuzzy numbers by considering approximations in the space of so called 1-knot piecewise linear fuzzy numbers ([43]). These fuzzy numbers are piecewise linear, each side having two segments split on the same α cut. In addition they depend on 6 parameters and hence the approximation capability increases. In the future, together with the authors of paper [43], we will extend our research by considering fuzzy numbers with piecewise linear sides having n knots. Then, in the paper [20] we consider a general trapezoidal approximation problem, more exactly we study the trapezoidal approximation which preserves a linear characteristic given in a general form. This approach can be generalized by considering the preservation of multiple characteristics. As a project of my own, I have become interested in the study of quadratic programming. I have observed that using Hilbert space theory, the study of trapezoidal approximation operators can be reduced to the study of the solutions of quadratic programs depending on some parameters. The study of such problems belongs to the so called sensitivity analysis. My idea is to study in detail the properties of the so called solution function of such quadratic programs because any important result will have its corresponding result in the case of trapezoidal or parametric approximation operators. In this research I have been mostly inspired by the paper [86] and by the monograph [66]. In paper [86] the author proves the Lipschitz continuity of the solution function for canonical quadratic programs depending on two parameters c and λ where c appears in the quadratic function while λ is a parameter in the right-hand side of the polyhedral constraints. In the paper [42] I have obtained the same conclusion but in a more general setting where standard and even general quadratic programs are investigated. Moreover, in the same paper it is proved that there exists a piecewise additive and positively homogenous relation between the parameters and the solution. I am also studying the problem of the approximation of fuzzy numbers, by using the $F-$transform (see [48]) introduced recently by Perfilieva (see [70]). Finally, I would like to mention that there are many other interesting problems that should be investigated with respect to max-product kind operators. I will mention here the saturation and inverse results where we already have an accepted paper (see [45]).

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