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Extended Variants for Fixed Point Theorems related to Classes of Generalized Contractions with Applications

PhD Thesis - Summary

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Introduction

Consideration on fixed point theory

Among the most dynamic fields of Mathematics in recent years, which has many applications in diverse areas, such as physics, computer science, biology, or economics, is the *fixed point theory*. The existence and uniqueness of the solutions, stability, and properties of the solution set of differential equations or inclusions are fundamental problems that can be studied using the fixed point theory. The first proof of the existence of solutions for a problem with initial conditions was given by Augustin-Louis Cauchy in 1884. In the same period, Émile Picard proposed the method of successive approximations as a method for approximating the solutions. These concepts will be further used to construct an abstract method, proved and published by Stefan Banach in the year 1922, see [9]. He proved that any contraction mapping f on a complete linear normed space E admits a unique fixed point, and this fixed point is the limit of the sequence of successive approximations generated by f . This means that there exists a unique point $x^* \in E$, such that $f(x^*) = x^*$ and, for every $x_0 \in E$, the sequence $(f^n(x_0))_{n \in \mathbb{N}}$ converges to x^* . This result was extended by Renato Caccioppoli in the year 1930. His main contribution was to generalize Banach's theorem mentioned above from complete linear normed spaces to complete metric spaces. This result is named today as *The Contraction Principle of Banach-Caccioppoli*. After Caccioppoli's extension, the fixed point theorem can also be applied to spaces without a linear structure. This theorem also has applications in both numerical analysis and iterative methods for various equations, fractal generation, and so on. In addition, it is used for solving various problems connected with integral and differential equations.

In the following years, the contributions to the study of the fixed point

theory became more significant, new types of contraction conditions for operators appeared, and the existing working spaces were extended. For example, the concept of a Meir-Keeler operator on a complete metric space was introduced by Amram Meir and Emmett Keeler in the year 1969, in the paper [39]. The fixed point result given by A. Meir and E. Keeler is an extension of the results of some authors like M. Edelstein [22] or E. Rakotch [60].

It is worth noting that, in 1972, the Ćirić-Reich-Rus fixed point theorem was independently discovered by three important mathematicians, Ljubomir B. Ćirić, Simeon Reich, and Ioan A. Rus. The results extended Banach's contraction principle, introducing the notion of generalized contractions, see [16]. Later, in 1974, in the paper [17], Ćirić has results that show that the condition of quasi-contractions implies all the conclusions of Banach's contraction principle. This means that his fixed point existence and uniqueness theorem is regarded among the most general results in metric fixed point theory.

Discussing the generalizations of the contraction principle of Banach-Caccioppoli, we should also mention that in the paper from 1996 [13], Theodore A. Burton formulated the concept of large contraction and proved several fixed point theorems for such mappings. Moreover, a Krasnoselkii-type fixed point theorem for the sum of two operators involving a large contraction is also obtained.

In 1968, M.G. Maia [36] proved a fixed point theorem for a single-valued operator $f : X \rightarrow X$ on a set X endowed with two metrics which meet the conditions of Banach-Caccioppoli's contraction principle. In the context of multi-valued operators, the exploration of a fixed point inclusion on a set endowed with two metrics was introduced later, see [47] and [56]. A similar approach was followed in this thesis, but in the context of nonlinear graph φ -contractions.

In parallel, the results mentioned above for the fixed-point equation $x = f(x)$ were extended to the fixed-point inclusion $x \in F(x)$. Here $f : X \rightarrow X$ is a single-valued operator on the complete metric space X , while F is a multivalued operator, assigning to each $x \in X$ the set $F(x) \subset X$. In a paper published in 1969, Sam B. Nadler Jr. demonstrated a fixed point theorem for what are known as multi-valued contraction mappings, see [46]. Nadler contraction mapping uses the Hausdorff-Pompeiu distance to measure the distance between two sets. This theorem has applications in differential inclusions, optimization, or game theory.

An extension of the fixed point theorem of Nadler was given by Covitz and Nadler in 1970 in the paper [19]. Later, in 2001, it was proved that any Nadler contraction (see, e.g., [5]) is a multi-valued Meir-Keeler operator. In the paper [55], A. Petruşel, G. Petruşel, and J.-C. Yao presented a new generalization of of Nadler's contraction principle, in the context of multi-valued graph contraction in complete metric spaces.

There are many situations where fixed points are applied. For example, Volterra type integral equations can be studied by fixed point methods. The study of these problems is significant, since the Volterra integral equations are used in physics, biology, and economics.

Reasons for Selecting the Thesis Theme

The main purpose of this doctoral thesis is to consider the fixed point problem (in the single-valued setting, as well as in the multi-valued case) and to give extended versions of some fixed point theorems for various classes of operators. The extended version means here not only existence, uniqueness, and approximation for the fixed point, but also localization and different stability properties (such as Ulam-Hyers stability, well-posedness according to Reich and Zaslavski, and Ostrovski stability) for the fixed point equation or the fixed point inclusion. The classes of operators we considered are: single-valued and multi-valued quasi-contractions in the sense of Ćirić, Meir-Keeler operators, multi-valued nonlinear graph φ -contractions, and large contractions in the sense of Burton. Several examples illustrate the developments obtained here, and some applications to integral equations of Volterra, as well as an application for surjectivity theorems, are given.

Structure of the thesis and main original contributions

The doctoral thesis is divided into three chapters, followed by the reference list of publications. Each chapter has sections and subsections presented below.

Chapter 1: *Notions and preliminary results*

The primary objective of this chapter is to remind the reader of some basic notions and results that are used in the following chapters of the thesis. We are referring to notions specific to Nonlinear Analysis. In this chapter, we have the following sections:

1. *Single-valued operators*

In this section, we will give some important concepts referring to the fixed points of the single-valued operators. Definitions for open ball and closed ball, Lipschitz operator, nonexpansive operator, the sequence of Picard iterates, α -contraction, φ -contraction, the concept of well-posedness as defined by Reich and Zaslavski, Ostrowski stability, and Ulam-Hyers stability are given. Also, the Cauchy-Toeplitz lemma, the contraction principle of Banach (1922), and Caccioppoli (1930) are mentioned. This subsection also provides the definition of the Meir-Keeler operator and the extension of Banach's Contraction Principle in the sense of a Meir-Keeler operator. The following references were used in this section: V. Berinde, M. Păcurar [10], [11], Lj. B. Ćirić [17], Granas-Dugundji [25], T. A. Lazăr, A. Petruşel and N. Shazhad [33], A. Magdaş [37] and I. A. Rus [69], [70], [76].

2. *Multi-valued operators*

In this section, basic notions from the theory of multi-valued operators are reviewed. The following notions are recalled: Pompeiu-Hausdorff distance between two sets, graph α -contraction, multi-valued weakly Picard operators, quasi-bounded operators, continuity concepts, and some stability properties. An example of a multi-valued contraction and another one of a multi-valued graph contraction are given. The following references are the basis for the presented notions: R. Iannacci [26], T. C. Lim [34], G. Petruşel [59], I.A. Rus, A. Petruşel, G. Petruşel [72] and I.A. Rus, A. Petruşel, A. Sîntămărian, [74], [75].

Chapter 2: *Qualitative properties of solution for the fixed point equation*

This chapter consists of three sections. This chapter is devoted to the study of some qualitative properties of the fixed point equation (existence, uniqueness, data dependence) for Ćirić type operators, Meir-Keeler type operators, and multi-valued nonlinear graph φ -contractions in complete metric spaces. Moreover, in the case of a multi-valued nonlinear (ϕ, ψ) -contraction of Feng-Liu type, an application to an integral inclusion of Volterra type is provided.

Also, some results for Maia type fixed point theorems are presented. The structure of this chapter includes the following sections:

1. *Fixed point equation with Ćirić type operators*

This section is based on the article by the author, *M. Moga, On some qualitative properties of Ćirić's fixed point theorem* published in *Studia Universitatis Babeş-Bolyai Mathematica*, see [42]. In addition, this section references collaborative work from the article of *M. Moga and R. Truşcă, On some fixed point theorems for Ćirić operators*, published in *Miskolc Mathematical Notes*, see [44]. In this chapter, the stability properties for the fixed point equation and the fixed point inclusion are studied from another point of view.

The most general extension of the Banach Caccioppoli contraction principle was given by Ćirić's in 1974. In this section, the equation $x = f(x)$ and the inclusion $x \in F(x)$ with operators satisfying Ćirić type condition are studied from the following perspective: existence, uniqueness, data dependence, and multiple stability types (well-posedness according to Reich and Zaslavski, Ulam-Hyers and Ostrowski stability).

This section includes the following three subsections:

The first subsection is named *The notion of Ćirić type operator*. There are defined the notions of generalized contraction and Ćirić type operator from [17]. A relevant example, in which a Ćirić type operator, which is not a generalized contraction, is outlined.

The second subsection is titled *Main results for single-valued Ćirić type operators*. Here, the work is focused on the existence and uniqueness theorem in the single-valued case. The localization result, which generalizes Ćirić's theorem and the homotopy result, was proved by Radu Truşcă. The definition of the retraction-displacement condition is used to establish the well-posedness property. The primary contribution of the main Theorem 2.1.2 of the section is the result of my own work. It demonstrates that the Ćirić type operator f has the following properties: data dependence, well-posedness in the framework proposed by Reich and Zaslavski, Ulam-Hyers stability, Ostrowski stability, graphic $\frac{q}{1-q}$ -contraction, and that f is quasi-contraction.

The title of the third subsection is *Main results for multi-valued Ćirić type operators*. In this part, the research is centered on the multi-valued

case. It starts with an example of a multi-valued Ćirić type operator which is not considered a multi-valued generalized contraction. In the next step, using the Theorem of A. Amini-Harandi [4], my achievement was to give a constructive proof with data dependence and stability results of the fixed point problem $x \in F(x)$. Additionally, a theorem is presented in which the fixed point set coincides with the strict fixed point set of a multi-valued Ćirić type operator. For the subsequent local fixed point theorem, the primary contribution is attributed to Radu Trușcă. In the paper of Radu Trușcă [80], theorems and applications to open mapping principles and continuation results are given.

The results generalize and complement the theorems given in V. Berinde, Șt. Mărușter, I.A. Rus [12], Lj. B. Ćirić [17], A. Petrușel [51], I.A. Rus [68], [70], and I.A. Rus, A. Petrușel, G. Petrușel [76].

2. *Fixed point equation for Meir-Keeler type operator*

The reference of this section is the article of the author, *M. Moga, Some properties of the fixed point equation with Meir-Keeler operator* published in *Annales Universitatis Scientiarum Budapestinensis*, see [40]. In addition to the regular existence and uniqueness results, different stability properties of the fixed point equation are given.

The second section of this chapter has the following two subsections:

The first subsection is entitled *The notion of Meir-Keeler operator in the metric space*. An important observation, along with an example, is that any contraction is a Meir-Keeler operator, but the reverse implication does not generally hold. The definitions of an L -function and φ -contraction are also given. A notable theorem in this subsection is Lim's theorem, which makes the connection between Meir-Keller's condition and the nonlinear contraction condition.

The second subsection refers to *Main results for single-valued Meir-Keeler operators*. Using Lim's characterization, the qualitative properties of the fixed point are proved. The main Theorem 2.2.4 of the section refers to a Meir-Keeler operator f and the L -function ϕ . The existence, well-posedness according to Reich and Zaslavski, and Ulam-Hyers stability of the fixed point are proved.

The most used references in this section are the following: T. C. Lim [35], A. Meir and E. Keeler [39], and A. Petrușel [49].

3. *Fixed point results for multi-valued nonlinear graph φ -contractions in complete metric spaces*

The last section of this chapter has as a reference the joint work from the article, *M. Moga, Radu Trușcă, Fixed point and stability results for multi-valued nonlinear graph contractions in complete metric spaces* published in The Journal of Analysis, see [45]. A notable generalization of the multi-valued contraction is the multi-valued graph φ -contraction. This section presents a generalization of the multi-valued graph φ -contraction, and its associated existence and stability results.

The section contains three subsections, presented below.

The title of the first subsection is *The notion of multi-valued nonlinear graph φ -contraction*. This subsection serves as an introduction to the notions of multi-valued nonlinear graph φ -contraction, and φ -multi-valued weakly Picard operator, which are later used.

The second subsection discusses the *Fixed point result for multi-valued nonlinear graph φ -contraction*. We begin this section with a fixed point theorem for a multi-valued nonlinear graph φ -contraction. This theorem also refers to a retraction-displacement condition. Additionally, an example of a multi-valued function F that fulfills the condition for a multi-valued nonlinear graph φ -contraction, while it does not satisfy the contraction condition, is given. Other original contributions relate to the theorem for the data dependence of the fixed point set, Ulam-Hyers stability of the fixed point inclusion, and the strict fixed point property for the multi-valued nonlinear graph φ -contractions. The section continues with the result proved by Radu Trușcă, related to the local fixed point theorem for multi-valued nonlinear graph φ -contractions.

The third subsection, created in collaboration with Radu Trușcă, is entitled *Maia type fixed point theory for multi-valued nonlinear graph contractions*. This subsection presents the Maia type theorem for nonlinear graph φ -contractions, in the case of a nonempty set, endowed with two metrics. The concept of multi-valued (φ, ψ) -contraction of Feng-Liu type, as introduced in [54], is presented. A further original contribution relates to an application presented in this section. The integral inclusion of the Volterra type is considered. For the hypotheses outlined in the subsection, Theorem 2.3.14 demonstrates an existence result. The last part of the subsection refers to Radu Trușcă's main

results concerning the extension of several stability properties for the fixed point inclusion, within the framework of multi-valued nonlinear Feng-Liu type (φ, ψ) -contraction in a space endowed with two metrics.

This section aims to present a generalization of the multi-valued nonlinear principle of Węgrzyk [82], as well as of the multi-valued graph contraction principle given by A. Petruşel, G. Petruşel and J.C. Yao [55].

Chapter 3: *Surjectivity theorems for classes of contractive operators*

This chapter contains two sections. This chapter aims to present some surjectivity theorems for Meir-Keeler operators and for large contractions. Both of the cases consist of results proven for single-valued and multi-valued operators. The main objective is to prove that the operators are norm-contractions and, under adequate conditions, surjectivity takes place. The chapter is structured as follows:

1. *Surjectivity theorems with Meir-Keeler operators*

The content of this section is based on the article *M. Moga, Meir-Keeler operators and applications to surjectivity theorems* published in [41]. This section establishes that every Meir-Keeler operator is a norm contraction in both the single-valued and multi-valued frameworks

The three subsections are presented as follows:

The first subsection is named *Surjectivity theorems for single-valued Meir-Keeler operators*. The result obtained in this section was a new surjectivity theorem for a single-valued Meir-Keeler operator. It refers to the field generated by the Meir-Keeler operator, which is a surjective operator. For the proof of the theorem, the method relies on the norm-contraction operator theory and a theorem of A. Granas [24]. The cases when x is part of the closed ball and when it is outside the closed ball are considered.

The second subsection of this section is titled *Surjectivity theorems for multi-valued Meir-Keeler operators*. The definition of the multi-valued Meir-Keeler operator given by H.K. Xu [83] is presented. Theorem 3.1.3 is an original contribution that proves the multi-valued form of the theorem given by T. Suzuki in 2007, [78]. This result and R. Iannacci's theorem from 1978, [26], lead to the main theorem of this subsection. This result relates to the surjectivity of the multi-valued field, $1_Y - F$, generated by the Meir-Keeler operator.

The last subsection of this section is about *An application to a nonlinear integral equation*. More precisely, some existence results for nonlinear integral equations are given. The steps of the proof indicate that the operator F is quasi-bounded, which implies that F is also a norm-contraction. Hence, applying the theorem of A. Granas [24] for norm contractions, we conclude that the integral equation admits at least one solution. The approach is continued by demonstrating that the operator F is a Meir-Keeler operator. Proving all the steps above, the results lead to the Theorem 3.1.12, which underlines that the integral equation admits at least one solution.

Some of the most relevant references are F. Aldea [1], A. Granas, J. Dugundji [25], Meir and E. Keeler [39], A. Petruşel [49], G. Petruşel [59] and I. A. Rus [69].

2. *Surjectivity theorems for large contractions*

The last section of this chapter has in the spotlight the joint work with Adrian Petruşel *Large Contractions and Surjectivity in Banach Spaces* in the Springer Proceedings in Mathematics and Statistics, Optimization and Soft Computing, see [43]. The purpose of this section is to show that any large contraction is a norm-contraction in the sense of A. Granas. Then, the surjectivity results are proved to extend and complement several surjectivity results.

The first subsection of this section is titled *The notion of large contraction*. The notion of a large contraction introduced by T.A. Burton in 1996, [13], is presented. Also, it is recalled the remark of J. Jachymski from [28] about the contractive condition from the definition of large contraction. Regarding this, the definition for a generalized contraction, introduced by M.A. Krasnoselskii, is specified.

The title of the second subsection is *A surjectivity theorem for single-valued large contractions*. The theorem 3.2.1 of T.A. Burton is the starting point of this part. Using this characterization theorem, an example of a large contraction that is not a contraction in the sense of Banach is given. One more example is about the consideration that any Meir-Keeler operator on a nonempty and convex subset is a large contraction. The main theorem of the subsection refers to the field $1_X - f$ generated by f that is a surjective operator.

The last subsection is named *A surjectivity theorem for multi-valued*

large contractions. The approach is similar to subsection two. The own contribution relates to the definition of multi-valued large contraction and the proof of a surjectivity theorem for the field $1_X - F$ generated by the multi-valued operator large contraction F . A fixed point result for a multi-valued large contraction in a complete metric space is given.

The useful literature is referenced A. T. Burton [13], A. T. Burton, I. K. Purnaras [14], A. Granas [24], R. Iannacci [26], I. A. Rus [69], I. A. Rus, A. Petruşel, G. Petruşel [72].

Original Results of the Thesis and Publications List

The author's contributions to the doctoral thesis involve studying different bibliographic sources about this subject. Significant contributions are made to extend variants of Banach's Contraction Principle to more general classes of operators.

The results obtained in collaboration with Radu Truşcă are not included in the summary and appear only in the thesis.

Those contributions are referenced in the bibliography and have been published in the following scientific journals:

1. Moga, M.: *Some properties of the fixed point equation with Meir-Keeler operator*, Annales Univ. Sci. Budapest., Sec. Math. vol. **64** (2021), 109-115. (Indexed in MathSciNet, ZbMath).
2. Moga, M.: *Meir-Keeler operators and applications to surjectivity theorems*, J. Nonlinear and Convex Anal., vol. **23**, no. 3 (2022), 625-634. (Indexed in MathSciNet, Web Of Science with Impact factor: 0.7, ZbMath).
3. Moga, M.: *On some qualitative properties of Ćirić's fixed point theorem*, Stud. Univ. Babeş-Bolyai Math., vol. **67**, no. 1 (2022), 47-54. (Indexed in MathSciNet, Scopus, Web Of Science with Impact factor: 0.3, ZbMath).
4. Moga, M., Petruşel, A.: *Large Contractions and Surjectivity in Banach Spaces*. In: Som, T., Ghosh, D., Castillo, O., Petrusel, A., Sahu, D. (eds)

Applied Analysis, Optimization and Soft Computing. ICNAAO 2021. Springer Proceedings in Mathematics & Statistics, vol. **419**, Springer, Singapore (2023), 3-12. (Indexed in MathSciNet, Scopus, ZbMath), [10.1007/978-981-99-0597-3_1](https://doi.org/10.1007/978-981-99-0597-3_1).

5. Moga, M., Trușcă, R.: *On some fixed point theorems for Ćirić operators*. Miskolc Mathematical Notes, vol. **25**, no. 2 (2024), 871-885. (Indexed in MathSciNet, Scopus, Web Of Science with Impact factor: 0.9, ZbMath).
6. Moga, M., Trușcă, R.: *Fixed point and stability results for multi-valued nonlinear graph contractions in complete metric spaces*. J. Anal., vol. 33, (2025), 717-742. (Indexed in MathSciNet, Scopus, Web Of Science with Impact factor: 0.7, ZbMath), [10.1007/s41478-024-00858-6](https://doi.org/10.1007/s41478-024-00858-6).

Communication and Dissemination of Results

The conferences where the original results from the doctoral thesis were presented, and which brought a lot of knowledge, are worth mentioning here.

1. Within the context of the conference *22nd International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)*, held in Timișoara, Romania, between 1st and 4th September 2020, I presented the work with the title: *An extended version of Meir-Keeler theorem*.
2. During the *13th Joint Conference on Mathematics and Computer Science (MACS)*, organized in Budapest, Hungary, between 1st and 3rd October 2020, I presented the talk with the title: *Some properties of the fixed point equation with Meir-Keeler operator*. The article that serves as the basis of this presentation was published in the conference proceedings.
3. On the occasion of *19th International Conference on Functional Equations and Inequalities (ICFEI)*, hosted in Bedlewo, Poland, between 12th and 18th September, 2021, I presented the work with the title: *On some properties of Meir-Keeler operators*.
4. At the *23rd International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)*, organized in Timișoara,

Romania, between 7th and 10th December, 2021, I presented the paper with the title: *On some qualitative properties of Ćirić's fixed point theorem*.

5. Within the context of 4th *Romanian Itinerant Seminar on Mathematical Analysis and its Applications* (RISMAA), held in Braşov, Romania, between 19th and 21st May, 2022, I presented the paper with the title: *Fixed point and surjectivity results for large contractions in Banach spaces*.
6. At the 4th *International Conference on Mathematics and Computer Science (MACOS)*, held in Braşov, Romania, between 15th and 17th September, 2022, I presented the work with the title: *Fixed point theory for multi-valued nonlinear graph contractions in complete metric spaces* (joint work with Radu Truşcă and Adrian Petruşel).
7. During the 24th *International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)*, organized Linz, Austria, between 12th and 15th September, 2022, I gave a talk with the title: *On some fixed point theorems for Ćirić's operators* (joint work with Radu Truşcă).
8. At the 14th *International Conference on Fixed Point Theory and its Applications (ICFPTA)*, hosted in Braşov, Romania, between 11th and 14th July, 2023, I presented the paper with the title: *Fixed point and surjectivity results for large contractions in Banach spaces*.
9. Within the context of 6th *Romanian Itinerant Seminar on Mathematical Analysis and its Applications* (RISMAA), held in Cluj-Napoca, Romania, between 30th and 31st May, 2024, I presented the paper with the title: *Fixed point and stability results for multi-valued nonlinear graph contractions in complete metric spaces*.
10. At the 26th *International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)* organized Timişoara, Romania, between 16th and 19th September, 2024, I gave a talk with the title: *Fixed points for Feng-Liu multi-valued operators with an application* (joint work with Radu Truşcă).

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Above all, I am deeply thankful to my family for their patience and encouragement throughout the preparation of the thesis. I would also like to take this opportunity to thank my family, my friends and my colleagues for their understanding and support throughout the entire duration of my doctoral journey.

Chapter 1

Notions and preliminary results

This chapter is devoted to establishing the foundational notions and results that will serve as prerequisites for the chapters that follow in this Ph.D. thesis. This chapter has two sections: the single-valued operators and the multi-valued operators. Important notions for the Fixed Point theory as metric, metric space, Cauchy sequence, open and closed ball, are recalled. Also, we review specific notions from the Nonlinear Analysis as generalized metric space, Banach space, α -contraction, etc. In order to write the *Notions and preliminary results*, the following works were studied: V. Berinde, M. Păcurar [11], Granas-Dugundji [25], T. A. Lazăr, A. Petruşel and N. Shazhad [33], G. Petruşel [59], I.A. Rus [70], [76], I.A. Rus, A. Petruşel, G. Petruşel [72] and [73].

1.1 Single-valued operators

In the first part, we recall some useful notions, as the sequence of Picard iterates, α -contraction, φ -contraction, quasi-bounded operator, Reich and Zaslavski notion of well-posedness, Ostrowski stability, Ulam-Hyers stability, etc., which will be used in the proofs of the main theorems.

From the book, see, e.g., of I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, [73], the following notions are written.

Let X be a nonempty set and let $d : X \times X \rightarrow \mathbb{R}$ be a metric. We say that d with infinite-values is called a generalized metric $d : X \times X \rightarrow \mathbb{R}_+ \cup \{+\infty\}$

if the axioms of the metric are met.

Let x_0 be a given point in X and $r > 0$. The set $B(x_0; r)$ is the open ball of center x_0 and radius r , where

$$B(x_0; r) := \{x \in X : d(x_0, x) < r\},$$

while $\tilde{B}(x_0; r)$ is the closed ball of center x_0 and radius r , where

$$\tilde{B}(x_0; r) := \{x \in X : d(x_0, x) \leq r\}.$$

A subset Y of X is bounded if there exist $x \in X$ and $r > 0$ such that Y is contained in the ball $B(x_0; r)$, meaning $Y \subset B(x_0; r)$.

Definition 1.1.1 (R. Węgrzyk [82]). *The metric space (X, d) is metrically convex if for every distinct $x_1, x_2 \in X$ there exist $x \in X$ such that*

$$d(x_1, x_2) = d(x_1, x) + d(x, x_2) \text{ and } x_1 \neq x \neq x_2.$$

Let $(X, \|\cdot\|)$ be a Banach space. Then, by $P(X)$, we will denote the family of all nonempty subsets of X .

$$\mathcal{P}(X) := \{Y | Y \subset X\}, \quad P(X) := \{Y \subset \mathcal{P}(X) | Y \neq \emptyset\},$$

1_X – the identity operator.

We will also use the following families of sets:

$$P_b(X) := \{Y \in P(X) | Y \text{ is bounded}\},$$

$$P_{cp}(X) := \{Y \in P(X) | Y \text{ is compact}\},$$

$$P_{cl}(X) := \{Y \in P(X) | Y \text{ is closed}\}, \quad P_{cv}(X) := \{Y \in P(X) | Y \text{ is convex}\}.$$

Lemma 1.1.1 (Cauchy-Toeplitz Lemma). *Let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ be two sequences of positive numbers such that $\sum_{n \geq 0} a_n < \infty$ and $\lim_{n \rightarrow \infty} b_n = 0$. Then*

$$\lim_{n \rightarrow \infty} \left(\sum_{k=0}^n a_{n-k} b_k \right) = 0.$$

Definition 1.1.2 (Granas-Dugundji [25]). *Let X, Y be two normed spaces. Then:*

- (i) *a map $f : X \rightarrow Y$ is called compact if $f(X)$ is contained in a compact subset of Y (or equivalently, $f(X)$ is relatively compact);*
- (ii) *a map $f : X \rightarrow Y$ is called completely continuous if it is continuous, and for each $S \in P_b(X)$, we have that $f(S)$ is relatively compact.*

The following notions are used during the thesis, see, e.g., the book of I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, [73].

Definition 1.1.3. *Let (X, d) and (Y, \tilde{d}) be metric spaces, and let $f : X \rightarrow Y$.*

Continuity and compactness properties.

We define f to be:

- (i) *continuous if for every sequence $(x_n)_{n \in \mathbb{N}} \subset X$, with $x_n \rightarrow x$ as $n \rightarrow \infty$, we have $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$;*
- (ii) *with closed graph if, for every sequence $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \rightarrow x$ and $f(x_n) \rightarrow y$ as $n \rightarrow \infty$ it follows that $y = f(x)$. Equivalently, the graph $G(f) \subset X \times Y$ is a closed subset;*
- (iii) *bounded if $S \in P_b(X)$ implies $f(S) \in P_b(Y)$;*
- (iv) *compact if $S \in P_b(X)$ implies $\overline{f(S)} \in P_b(Y)$;*
- (v) *completely continuous if it is both compact and continuous.*

Lipschitz-type properties.

We define f to be:

- (vi) *Lipschitz if there exists a constant $l \in \mathbb{R}_+$ for which*

$$d(f(x), f(y)) \leq l d(x, y), \text{ for every } x, y \in X;$$

- (vii) *contraction if there exists $l \in [0, 1[$ such that f is l -Lipschitz (i.e. Lipschitz with the constant l);*

(viii) *contractive if*

$$d(f(x), f(y)) < d(x, y), \text{ for all } x, y \in X, x \neq y;$$

(ix) *nonexpansive if it is 1-Lipschitz;*

(x) *expansive if*

$$d(f(x), f(y)) > d(x, y), \text{ for all } x, y \in X, x \neq y.$$

Consider X be a nonempty set and $f : X \rightarrow X$ be an operator. We recall that,

$$\text{Fix}(f) = \{x \in X \mid x = f(x)\}$$

is the fixed point set of f .

Let us use the notation $(f^n(x))_{n \in \mathbb{N}}$ for the sequence of Picard iterates (also referred to as the sequence of successive approximations) of f starting at $x_0 \in X$, where $f^n = f \circ f \circ \dots \circ f$ for n -times. Notice that the sequence of Picard iterates of f from $x_0 \in X$ can be recursively defined by the formula $x_{n+1} = f(x_n)$, for $n \in \mathbb{N}$, where $x_n := f^n(x_0)$, $n \in \mathbb{N}$.

An operator f is called a weakly Picard operator (WPO) if, for every $x \in X$, the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges and its limit, denoted by $f^\infty(x)$, is a fixed point of f . In this case, we define f^∞ by

$$f^\infty : X \rightarrow X, f^\infty(x) := \lim_{n \rightarrow \infty} f^n(x).$$

Consider f be given as a WPO and a constant $c > 0$. The operator f is known as c -WPO when

$$d(x, f^\infty(x)) \leq cd(x, f(x)), \text{ for every } x \in X.$$

Let (X, d) be a metric space and $f : X \rightarrow X$ be an operator. For $A \subset X$, let $\delta(A) := \sup \{d(a, b) : a, b \in A\}$ the diameter of the set A . For each $x \in X$, we denote:

$$O(x, n) = \{x, f(x), \dots, f^n(x)\}, n = 1, 2, \dots,$$

$$O(x, \infty) = \{x, f(x), \dots, f^n(x), \dots\}.$$

Definition 1.1.4 (Lj. B. Ćirić [17]). *Let (X, d) be a metric space and $f : X \rightarrow X$ be an operator. Then X is said to be f -orbitally complete if every Cauchy sequence which is contained in $O(x, \infty)$, for some $x \in X$, converges in X .*

The following classes of operators in a metric space (X, d) are important for our approach.

Definition 1.1.5. *Let (X, d) be a metric space and $f : X \rightarrow X$ be an operator. Then f is said to be an α -contraction if there exists $\alpha \in [0, 1)$ such that*

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for all } x, y \in X. \quad (1.1.1)$$

Definition 1.1.6 (I.A. Rus [70]). *Consider a metric space (X, d) and $f : X \rightarrow X$ an operator. The operator f is called a graphic α -contraction if there exists $\alpha \in [0, 1)$ satisfying*

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X. \quad (1.1.2)$$

Through this paper we denote $\mathbb{N} := \{0, 1, 2, \dots\}$ the set of all natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Definition 1.1.7 (I.A. Rus [76]). *A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a comparison function if it satisfies:*

- (i) φ is strictly increasing;
- (ii) $(\varphi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \rightarrow \infty$, for all $t \in \mathbb{R}_+$.

If the condition (ii) is replaced by the condition:

$$(iii) \sum_{k=0}^{\infty} \varphi^k(t) < \infty, \text{ for any } t > 0, \quad (1.1.3)$$

then φ is called a strong comparison function.

Lemma 1.1.2. *If $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$, for any $t > 0$, $\varphi(0) = 0$ and φ is continuous at 0.*

Lemma 1.1.3. *If $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a strong comparison function, then the following hold:*

- (i) φ is a comparison function;
- (ii) the function $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined by

$$\zeta(t) = \sum_{k=0}^{\infty} \varphi^k(t),$$

is increasing and continuous at 0.

Lemma 1.1.4 (V. Berinde, M. Păcurar [11]). *Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a strong comparison function and $a_n \in \mathbb{R}_+$, $n \in \mathbb{N}$ such that $a_n \rightarrow 0$ as $n \rightarrow \infty$. Then*

$$\sum_{k=0}^n \varphi^{n-k}(a_k) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Definition 1.1.8 (T. A. Lazăr, A. Petruşel and N. Shazhad [33]). *Let (X, d) be a complete metric space and $f : X \rightarrow X$ be an operator. We say that f is a φ -contraction if*

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for each } x, y \in X,$$

where $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a comparison function, i.e., φ is increasing and satisfies $\lim_{n \rightarrow \infty} \varphi^n(f) = 0$, for any $f \geq 0$.

In particular, if $\varphi(f) = kf$, $f \in \mathbb{R}_+$ (for some $k \in (0, 1)$), then f is called a k -contraction.

Definition 1.1.9 (Meir-Keeler [39]). *Let (X, d) be a metric space. An operator $f : X \rightarrow X$ is said to be a Meir-Keeler operator if for each $\varepsilon > 0$ there exists $\delta > 0$ such that the following implication holds:*

$$x, y \in X, \varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(f(x), f(y)) < \varepsilon.$$

A result established by Meir and Keeler in 1969 is presented here. Actually, in [39], the notion of the Meir-Keeler operator is introduced on a complete metric space, and then the subsequent fixed point theorem is established.

Theorem 1.1.5 (Meir-Keeler [39]). *Consider a complete metric space (X, d) and a Meir-Keeler operator $f : X \rightarrow X$, then f is a Picard operator, i.e.,*

$$(i) \text{Fix}(f) = \{x^*\};$$

(ii) *the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges to x^* , for any $x \in X$.*

Definition 1.1.10 (A. Granas, see, e.g., I. A. Rus [69]). *Let $(X, \|\cdot\|)$ be a normed space. An operator $f : X \rightarrow X$ is called quasi-bounded if there exist two numbers $a, b \in]0, \infty[$ such that*

$$\|f(x)\| \leq a\|x\| + b, \text{ for all } x \in X. \quad (1.1.4)$$

In the literature, the notion of quasi-bounded is also referred to as linear growth.

In this setting, if f is quasi-bounded, then the quasi-norm of f is expressed by

$$|f| = \inf \{a > 0 \mid \text{there exists } b > 0 \text{ such that the relation (1.1.4) holds}\}.$$

Definition 1.1.11 (A. Granas, see, e.g., I. A. Rus [69]). *Let $(X, \|\cdot\|)$ be a normed space and $f : X \rightarrow X$ be a quasi-bounded operator. If the quasi-norm of f is strictly less than one, then f is called a norm-contraction.*

Below we present a number of definitions that will be used to describe various stability properties.

Definition 1.1.12 (S. Reich and A. J. Zaslavski [64]). *Let (X, d) be a metric space and $f : X \rightarrow X$ be an operator. Then, the fixed point equation is well-posed according to Reich and Zaslavski if $\text{Fix}(f) = \{x^*\}$ and if $(y_n)_{n \in \mathbb{N}} \in X$ is a sequence such that*

$$d(y_n, f(y_n)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then we have that

$$y_n \rightarrow x^* \text{ as } n \rightarrow \infty.$$

Definition 1.1.13 (V. Berinde, M. Păcurar [10]). *Let (X, d) be a metric space and an operator $f : X \rightarrow X$ be such that $\text{Fix}(f) = \{x^*\}$. We consider*

that f has the Ostrowski stability property if the following implication holds: for any sequence $(y_n)_{n \in \mathbb{N}} \subset X$ having the property

$$d(y_{n+1}, f(y_n)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

we have that

$$\lim_{n \rightarrow \infty} y_n = x^*.$$

Definition 1.1.14 (P. T. Petru, A. Petruşel, J.-C. Yao [48]). Let (X, d) be a metric space and $f : X \rightarrow X$. We say that the fixed point equation $f(x) = x$ is generalized Ulam-Hyers stable if for any $\varepsilon > 0$ and for any ε -solution y^* of the fixed point equation (i.e., $d(y^*, f(y^*)) \leq \varepsilon$) there exists a unique solution x^* of the fixed point equation such that

$$d(x^*, y^*) \leq \psi(\varepsilon),$$

where $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a function which is increasing, continuous at 0, and $\psi(0) = 0$.

Remark 1.1.6. In particular, the fixed point equation is said to be Ulam-Hyers stable if there exists $c > 0$ such that $\psi(f) = cf, f \in \mathbb{R}_+$.

The main fixed point principle on a metric space is the one given by Banach (1922) and Caccioppoli (1930).

Theorem 1.1.7 (Banach (1922) - Caccioppoli (1930) Contraction Principle). Let (X, d) be a complete metric space. Let $f : X \rightarrow X$ be an α -contraction. The following holds:

- (i) $\text{Fix}(f) = \{x^*\}$;
- (ii) for every $x \in X$ the following sequence of successive approximations (i.e. $x_0 = x, x_n := f^n(x)$ for $n \geq 1$) of f starting from x converges to x^* ;
- (iii) $d(x_n, x^*) \leq \frac{\alpha^n}{1 - \alpha} d(x_0, f(x_0))$, for every $n \in \mathbb{N}$.

1.2 Multi-valued operators

This chapter aims to present the fundamental notions of the theory of multi-valued operators on metric spaces. Notions such as the Pompeiu-Hausdorff distance, quasi-bounded operator, and continuity are presented.

The notation below will be used throughout the paper. (see, e.g., [59], [72]).

Let (X, d) be a metric space. Then, it follows that:

- (i) the gap between the two sets S_1 and S_2 is the functional

$$D(S_1, S_2) : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}, \text{ defined by}$$

$$D(S_1, S_2) = \inf \{d(x_1, x_2) | x_1 \in S_1, x_2 \in S_2\}.$$

- (ii) the excess of a set S_1 over a set S_2 is the functional

$$\rho(S_1, S_2) : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}, \text{ defined by}$$

$$\rho(S_1, S_2) = \sup \{D(x_1, S_2) | x_1 \in S_1\}.$$

- (iii) the Pompeiu-Hausdorff distance between two sets S_1 and S_2 is the functional

$$H(S_1, S_2) : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}, \text{ defined by}$$

$$H(S_1, S_2) = \max \{\rho(S_1, S_2), \rho(S_2, S_1)\}.$$

- (iv) the diameter between two sets S_1 and S_2 is the functional

$$\Delta(S_1, S_2) : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\},$$

$$\Delta(S_1, S_2) = \sup \{d(x_1, x_2) | x_1 \in S_1, x_2 \in S_2\}.$$

For a multi-valued operator $F : X \rightarrow P(X)$, its fixed point set is denoted by

$$Fix(F) := \{x \in X | x \in F(x)\},$$

while the graph of F is the set

$$Graph(F) := \{(x, y) \in X \times X : y \in F(x)\}.$$

Definition 1.2.1 (see, e.g., A. Petruşel, G. Petruşel, J.-C. Yao [55]). *Let (X, d) be a generalized metric space and $F : X \rightarrow P(X)$. Then, the multi-valued operator F is called:*

- (a) α -Lipschitz if $\alpha > 0$ and $H(F(x_1), F(x_2)) \leq \alpha d(x_1, x_2)$, for all $(x_1, x_2) \in X \times X$;
- (b) α -contraction if it is α -Lipschitz with $\alpha < 1$;
- (c) graph α -Lipschitz if $\alpha > 0$ and $H(F(x_1), F(x_2)) \leq \alpha d(x_1, x_2)$, for all $(x_1, x_2) \in \text{Graph}(F)$;
- (d) graph α -contraction if it is graph α -Lipschitz with $\alpha < 1$.

It is obvious that (a) implies (c) and (b) implies (d), but not vice versa.

Definition 1.2.2 (I. A. Rus, [74], [75]). *Let (X, d) be a metric space. Then $F : X \rightarrow P(X)$ is called a multi-valued weakly Picard (MWP) operator if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X such that*

- (i) $x_0 = x, x_1 = y$;
- (ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;
- (iii) the sequence $\{x_n\}_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of F .

Any sequence fulfilling conditions (i) and (ii) is termed a Picard-type sequence of F starting from $(x, y) \in \text{Graph}(F)$.

Assume that $F : X \rightarrow P(X)$ is an MWP operator. Accordingly, the multivalued operator is introduced as

$$F^\infty : \text{Graph}(F) \rightarrow P(\text{Fix}(F)) \text{ by}$$

$$F^\infty(x, y) := \{z \in \text{Fix}(F) \mid \text{there exists a sequence of successive approximations of } F \text{ starting from } (x, y) \text{ that converges to } z\},$$

for any $(x, y) \in \text{Graph}(F)$.

Remark 1.2.1. Notice that if $F : X \rightarrow P_{cl}(X)$ is an α -contraction, then F is a φ -MWP operator with $\varphi(t) = \frac{1}{1-\alpha}t$.

Definition 1.2.3 (R. Iannacci [26]). Let $(X, \|\cdot\|)$ be a Banach space and $F : X \rightarrow P_b(X)$. Then F is called quasi-bounded if there exists $m, M \in]0, \infty[$ such that

$$\|y\| \leq m\|x\| + M, \text{ for each } (x, y) \in \text{Graph}(F). \quad (1.2.1)$$

The quasi-norm of F is defined by

$$|F| = \inf \{m > 0 \mid \text{there exists } M > 0 \text{ such that the relation (1.2.1) holds}\}.$$

If the quasi-norm of F is less than one (i.e., $|F| < 1$), then F is said to be a multi-valued norm-contraction. In the above setting, we will denote

$$\|F(x)\| := H(F(x), \{0\}), \text{ for any } x \in X.$$

Definition 1.2.4 (see, e.g., M. Kisielewicz [31]). A multi-valued operator $F : X \rightarrow P(X)$ is said to be compact if $F(X)$ is relatively compact. A multi-valued operator $F : X \rightarrow P(X)$ is said to be completely continuous if it is upper semi-continuous and for each $A \in P_b(X)$ we have that $F(A)$ is relatively compact.

The following notions and results are well known.

Definition 1.2.5 (see, e.g., T. C. Lim [34]). Let $(X, \|\cdot\|)$ be a normed space. Then $F : X \rightarrow P(X)$ is said to be nonexpansive if

$$H(T(x), T(y)) \leq \|x - y\|, \text{ for each } x, y \in X.$$

We now recall the notion of Ulam-Hyers stability, well-posedness according to Reich and Zaslavski, and Ostrowski stability for a fixed point inclusion in the multi-valued case.

Definition 1.2.6 (P. T. Petru, A. Petruşel, J.-C. Yao [48]). Let (X, d) be a metric space and $F : X \rightarrow P(X)$ be a multi-valued operator. Then, the fixed point problem

$$x \in F(x), x \in X, \quad (1.2.2)$$

is said to be *generalized Ulam-Hyers stable* if there exists an increasing mapping $\gamma : [0, \infty[\rightarrow [0, \infty[$ with $\gamma(0) = 0$ and γ is continuous at 0, such that for any $\varepsilon > 0$ and any ε -solution z of the fixed point inclusion (1.2.2) (i.e., $D(z, F(z)) \leq \varepsilon$) there exists $x^* \in \text{Fix}(F)$ such that

$$d(z, x^*) \leq \gamma(\varepsilon).$$

As defined in [72], for a nonempty set X and a subset $Y \subset X$, a set retraction of X onto Y is a mapping $r : X \rightarrow Y$, for which the restriction of r to Y equals the identity operator.

The following notion of well-posedness, according to Reich and Zaslavski, in the context of a multi-valued operator was introduced in [64] as follows:

Definition 1.2.7 (S. Reich and A. J. Zaslavski [64]). *Let (X, d) be a metric space and let $F : X \rightarrow P(X)$ be a multi-valued operator. Suppose that $\text{Fix}(F) \neq \emptyset$ and there exists a set retraction $r : X \rightarrow \text{Fix}(F)$. Then, the fixed point inclusion $x \in F(x)$ is said to be well-posed in the Reich and Zaslavski sense if for every $x^* \in \text{Fix}(F)$ and given any sequence $\{u_n\}_{n \in \mathbb{N}} \subset r^{-1}(x^*)$ such that*

$$D(u_n, F(u_n)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

we have that

$$u_n \rightarrow x^* \text{ as } n \rightarrow \infty.$$

Definition 1.2.8 (S. Reich and A. J. Zaslavski [64]). *Let (X, d) be a metric space and let $F : X \rightarrow P(X)$ be a multi-valued operator. Suppose that $\text{Fix}(F) \neq \emptyset$ and there exists a set retraction $r : X \rightarrow \text{Fix}(F)$. Then, the fixed point inclusion $x \in F(x)$ satisfies the Ostrowski stability property if for any sequence $\{v_n\}_{n \in \mathbb{N}} \subset r^{-1}(x^*)$ such that*

$$D(v_{n+1}, F(v_n)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

we have that

$$v_n \rightarrow x^* \text{ as } n \rightarrow \infty.$$

For more details about multi-valued Picard operators, weakly Picard operators, generalized contractions, fixed point theorems, and stability properties, see, e.g., [30], [48], [50], [53], [57], and [58].

Chapter 2

Qualitative properties of solution for the fixed point equation

This chapter is devoted to the stability properties of the fixed point for different operator classes. The approach taken in the first section is on expanding the properties for the Ćirić type operator. The stability and fixed point results are demonstrated on both single-valued and multi-valued cases. In the second section, the Meir-Keeler operator is considered. The main result in this section addresses the single-valued operators. The last section is about multi-valued nonlinear graph φ -contraction in complete metric spaces. The discussion focuses on fixed point results and Maia-type theorems.

2.1 Fixed point equation with Ćirić type operators

It is well known that of all the extensions of the Banach-Caccioppoli Contraction Principle, the most general result was established by Ćirić in 1974. In this chapter, we will present some results related to Ćirić type operator in complete metric spaces. Existence and uniqueness are recalled, and several stability properties (data dependence and Ostrowski stability property) are

proved. Using the retraction-displacement condition, we will establish the Ulam-Hyers stability property of the fixed point equation $x = f(x)$ and the well-posedness in the Reich and Zaslavski sense.

2.1.1 The notion of Ćirić type operator

We review several definitions and well-known results that will be needed throughout the thesis.

Definition 2.1.1 (Lj. B. Ćirić [17]). *An operator $f : X \rightarrow X$ is said to be a generalized contraction if and only if for every $x, y \in X$ there exist non-negative numbers $q, r, s,$ and $t,$ which may depend on both x and $y,$ such that $\sup \{q + r + s + 2t : x, y \in X\} < 1$ and*

$$d(f(x), f(y)) \leq q \cdot d(x, y) + r \cdot d(x, f(x)) + s \cdot d(y, f(y)) + t \cdot [d(x, f(y)) + d(y, f(x))].$$

Definition 2.1.2 (Lj. B. Ćirić [17]). *Let (X, d) be a metric space and $f : X \rightarrow X$ be an operator. Then f is said to be a Ćirić type operator (named a quasi-contraction in the original paper [17]) if there exists a number $q \in (0, 1),$ such that*

$$d(f(x), f(y)) \leq q \cdot \max \{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}, \quad (2.1.1)$$

for all $x, y \in X.$

It is well known from the paper of B. E. Rhoades, see [67], that of all the extensions according to the Banach-Caccioppoli Contraction Principle, the most general result was established by Ćirić in 1974 for the above class of operators.

In the following example, we present a Ćirić type operator, which is not a generalized contraction.

Example 2.1.1. *Let*

$$X_1 = \left\{ \frac{m}{n} : m = 0, 1, 2, 4, 6, \dots; n = 1, 3, 7, \dots, 2k + 1, \dots \right\},$$

$$X_2 = \left\{ \frac{m}{n} : m = 1, 2, 4, 6, 8, \dots; n = 2, 5, 8, \dots, 3k + 2, \dots \right\},$$

where $k \in \mathbb{N}$ and let $X = X_1 \cup X_2$. Let us define $f : X \rightarrow X$ by

$$f(x) = \begin{cases} \frac{2}{3}x, & x \in X_1, \\ \frac{1}{5}x, & x \in X_2. \end{cases}$$

The mapping f is a Ćirić type operator with $q = \frac{2}{3}$. If both x and y are in X_1 or in X_2 , then

$$d(f(x), f(y)) \leq \frac{2}{3}d(x, y).$$

Thus, we have that f satisfies the following condition:

$$d(f(x), f(y)) \leq \frac{2}{3} \max \{d(x, y), d(x, f(y)), d(y, f(x))\}$$

and, hence, it is Ćirić type operator.

Having the following comparison, we get that $d(f(x), f(y)) = \frac{17}{30} > \frac{32}{60}$. Hence, f is not a generalized contraction.

Our results generalize and complement some theorems given in [2] [12], [17], [51], [68], [71], [73], [70].

2.1.2 Main results for single-valued Ćirić type operators

In this section, we will consider a metric space (X, d) and $f : X \rightarrow X$ a Ćirić type operator. Besides the usual properties, which are proved by Ćirić in [17], we will prove some other stability properties. More precisely, we will establish the continuous data dependence property of the fixed point and the Ostrowski stability property for the operator f . Moreover, using the retraction-displacement condition, we also prove that the fixed point equation

$x = f(x)$ is well-posed according to Reich and Zaslavski and Ulam-Hyers stable.

The following notion is essential in our approach.

Definition 2.1.3 (I. A. Rus [73]). *Let (X, d) be a metric space and $f : X \rightarrow X$ be an operator such that $Fix(f) \neq \emptyset$. The retraction-displacement condition is satisfied by f if one can find $c > 0$ and a set retraction $\rho : X \rightarrow Fix(f)$ for which*

$$d(x, \rho(x)) \leq cd(x, f(x)), \text{ for all } x \in X. \quad (2.1.2)$$

If $Fix(f) = \{x^*\}$ then we have

$$d(x, x^*) \leq cd(x, f(x)), \text{ for all } x \in X.$$

For instance, if $f : X \rightarrow X$ is an α -contraction on a complete metric space (X, d) , then f fulfills the retraction-displacement property:

$$d(x, x^*) \leq \frac{1}{1 - \alpha} d(x, f(x)), \text{ for every } x \in X.$$

On the same lines, if $f : X \rightarrow X$ is a graphic α -contraction then it meets the retraction-displacement property:

$$d(x, \rho(x)) \leq \frac{1}{1 - \alpha} d(x, f(x)), \text{ for all } x \in X,$$

where $\rho : X \rightarrow Fix(f)$ is specified by

$$\rho(x) = \lim_{n \rightarrow \infty} f^n(x).$$

Theorem 2.1.1 (Lj. B. Ćirić [17]). *Let (X, d) be a metric space and $f : X \rightarrow X$ be a Ćirić type operator. Suppose that X is f -orbitally complete. Then:*

1. f has a unique fixed point x^* in X and $\lim_{n \rightarrow \infty} f^n(x) = x^*$, i.e., f is a Picard operator;
2. $d(f^n(x), x^*) \leq \frac{q^n}{1 - q} d(x, f(x))$, for every $x \in X$ and every $n \in \mathbb{N}^*$.

An essential part of the proof is based on the following two relations:

(i) if $n \in \mathbb{N}^*$, then for every $x \in X$ it follows that

$$d(f^i(x), f^j(x)) \leq q\delta(O(x, n)), \text{ for every } i, j \in \mathbb{N}^*;$$

(ii) for every $x \in X$ we have that $\delta(O(x, \infty)) \leq \frac{1}{1-q}d(x, f(x))$.

A second result in [17] shows that if there exists $p \in \mathbb{N}$ with $p \geq 2$ such that f^p is a Ćirić type operator, then f is a Picard operator.

The next theorem, proving the stability properties of the fixed point, stands as the central contribution of this part.

Theorem 2.1.2. *Let (X, d) be a metric space, $f : X \rightarrow X$ be a Ćirić type operator and suppose that X is f -orbitally complete. Let $x^* \in X$ denote the unique fixed point of f . Then the subsequent conclusions are obtained:*

1. *the fixed point $x = f(x)$ equation is said to have the data dependence property, i.e., for any operator $g : X \rightarrow X$ such that $\text{Fix}(g) \neq \emptyset$ and*

$$d(f(x), g(x)) \leq \eta,$$

for every $x \in X$ and a constant $\eta > 0$, we have

$$d(x^*, u^*) \leq \frac{1+q}{1-q}\eta,$$

for every $u^ \in \text{Fix}(g)$.*

2. *the fixed point equation has the well-posedness property, i.e., for every sequence $(u_n)_{n \in \mathbb{N}} \subset X$ for which*

$$d(u_n, f(u_n)) \rightarrow 0,$$

as $n \rightarrow \infty$, it holds that $u_n \rightarrow x^$, as $n \rightarrow \infty$;*

3. *the fixed point equation is Ulam-Hyers stable, i.e., there exists $c > 0$ such that for any $\varepsilon > 0$ and any $u^* \in X$ an ε -solution of the fixed point equation (in the sense that $d(u^*, f(u^*)) \leq \varepsilon$), thus*

$$d(u^*, x^*) \leq c \cdot \varepsilon.$$

4. if $q < \frac{1}{2}$, then the fixed point equation has the Ostrowski stability property, i.e., for any sequence $(u_n)_{n \in \mathbb{N}} \subset X$ with $d(u_{n+1}, f(u_n)) \rightarrow 0$ as $n \rightarrow \infty$, it holds that $u_n \rightarrow x^*$;
5. if $q < \frac{1}{2}$, then f is a graphic $\frac{q}{1-q}$ -contraction;
6. if $q < \frac{1}{3}$, then the operator f is a quasi-contraction, in the sense that there exists $\beta := \frac{q}{1-2q} < 1$ for which

$$d(f(x), x^*) \leq \beta d(x, x^*), \text{ for every } x \in X.$$

2.1.3 Main results for multi-valued Ćirić type operators

We first consider some notions related to our main results. After that, we will underline the main theorem of this section, which extends, with some stability results, the well-known theorem of Ćirić. The approach will be for multi-valued Ćirić operators, which were considered by A. Amini-Harandi in the paper [4], from 2011.

Definition 2.1.4. An operator $F: X \rightarrow P_{cl}(X)$ is said to be a multi-valued generalized contraction if for every $x, y \in X$ there exist non-negative numbers p, q, r , which may depend on both x and y , such that $\sup \{p + 2q + 2r \mid x, y \in X\} < 1$ and

$$H(F(x), F(y)) \leq p \cdot d(x, y) + q \cdot [D(x, F(x)) + D(y, F(y))] + r \cdot [D(x, F(y)) + D(y, F(x))].$$

Definition 2.1.5 (A. Amini-Harandi [4]). Let (X, d) be a metric space. The set-valued map $F: Y \subseteq X \rightarrow P_{b,cl}(X)$ is called a multi-valued Ćirić type operator with constant k (named a k -set-valued quasi-contraction in [4]) if

$$H(F(x), F(y)) \leq k \max \{d(x, y), D(x, F(x)), D(y, F(y)), D(x, F(y)), D(y, F(x))\},$$

for any $x, y \in X$, where $0 \leq k < 1$.

If (X, d) is a metric space and $F: X \rightarrow P(X)$ is a multi-valued operator, then the sequence $(x_n)_{n \in \mathbb{N}}$ from X is called a sequence of Picard type starting from $(x, y) \in \text{Graph}(F)$ if $x_0 = x, x_1 = y$ and $x_n \in F(x_{n-1}), n \in \mathbb{N}^*$.

Theorem 2.1.3. (A. Amini-Harandi Theorem 2.2 [4]) *Let (X, d) be a complete metric space. Let $F: X \rightarrow P_{b,cl}(X)$ be a multi-valued Ćirić type operator with constant $k < \frac{1}{2}$. Then, F has a fixed point.*

In what follows, we will present some data dependence and stability results for the fixed point problem $x \in F(x)$.

Theorem 2.1.4. *Let (X, d) be a complete metric space. Let $F: X \rightarrow P_{cl}(X)$ be a multi-valued Ćirić type operator with constant $k < \frac{1}{2}$. Then:*

- (i) *Fix(F) $\neq \emptyset$ and for every $(x, y) \in \text{Graph}(F)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of Picard type starting from $x_0 := x, x_1 := y$ which converge to a fixed point x^* of F ;*
- (ii) *the fixed point equation $x \in F(x)$ satisfies the data dependence property, i.e., for any $x^* \in \text{Fix}(F)$ and any $G: X \rightarrow P(X)$ such that $\text{Fix}(G) \neq \emptyset$ and the inequality $H(F(x), G(x)) \leq \eta$ holds for all $x \in X$ and some $\eta > 0$, there is $u^* \in \text{Fix}(G)$ such that*

$$d(x^*, u^*) \leq \frac{(1+k)q}{1-k}\eta,$$

where $1 < q < \frac{1}{2k}$;

- (iii) *the fixed point equation is well-posed in the Reich and Zaslavski sense, i.e., for every sequence $(u_n)_{n \in \mathbb{N}} \subset X$ such that*

$$D(u_n, F(u_n)) \rightarrow 0,$$

as $n \rightarrow \infty$, we have that $u_n \rightarrow x^*$, as $n \rightarrow \infty$.

- (iv) *if $q < \frac{1}{2}$, then the fixed point equation has the Ostrowski stability property, i.e., for any sequence $(u_n)_{n \in \mathbb{N}} \subset X$ with $D(u_{n+1}, F(u_n)) \rightarrow 0$ as $n \rightarrow \infty$, we have that $u_n \rightarrow x^*$.*

We will now give a theorem that shows that, under an additional condition, the fixed point set and the strict fixed point set of a multi-valued Ćirić type operator coincide.

Theorem 2.1.5. *Let (X, d) be a complete metric space. Let $F: X \rightarrow P_{cl}(X)$ be a multi-valued Ćirić type operator with constant $k < 1$. Suppose that $SFix(F) \neq \emptyset$. Then $Fix(F) = SFix(F) = \{x^*\}$.*

2.2 Fixed point equation for Meir-Keeler type operator

The principal objective of this study is to establish results connected to the Meir-Keeler theorem. Besides the existence and uniqueness results (which are well-known), we give several stability properties (data dependence, Ulam-Hyers stability, well-posedness in the Reich and Zaslavski sense, and Ostrowski stability property) of the fixed point equation. The starting points of this study are the papers of Meir and Keeler [39] and Lim [35], as well as the seminal paper of Rus [70]. For related results see [49] and [73].

2.2.1 The notion of Meir-Keeler operator in the metric space

The definition of the Meir-Keeler operator 1.1.9 and the Meir-Keeler theorem 1.1.5 can be found in the first chapter, single-valued section, since they are also used in Chapter 3.

Theorem 1.1.5 is an extension of the famous Banach's Contraction Principle since any contraction mapping is a Meir-Keeler operator. Moreover, any Meir-Keeler operator is contractive in the sense that

$$d(f(x), f(y)) < d(x, y),$$

for every $x, y \in X$ with $x \neq y$.

Remark 2.2.1. *Notice that any contraction is a Meir-Keeler operator, but, in general, the converse implication fails.*

In what follows, we introduce the first result of T. C. Lim [35], a key ingredient in this chapter. We start by recalling the notion of an L -function.

Definition 2.2.1 (T. C. Lim [35]). *A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is an L -function if $\phi(0) = 0$, $\phi(s) > 0$, for each $s > 0$ and for any $u > 0$, there exists $u > s$ such that $\phi(t) \leq s$ whenever $t \in [s, u]$.*

Every L -function satisfies $\phi(s) \leq s$, for each $s \geq 0$.

We recall now Lim's theorem from [35].

Theorem 2.2.2. (T. C. Lim [35]) *Let (X, d) be a metric space and let $f : X \rightarrow X$. The following assertions are equivalent:*

- (i) *f satisfies Meir-Keeler's condition;*
- (ii) *there exists an L -function $\phi : [0, \infty) \rightarrow [0, \infty)$ such that*

$$d(f(x), f(y)) < \phi(d(x, y)),$$

for any $x, y \in X$ with $x \neq y$.

In (ii), one can choose ϕ to be also nondecreasing and right continuous.

Our approach is based on the equivalence between (i) and (ii).

2.2.2 Main results for single-valued Meir-Keeler operators

In this section, we will consider the fixed point equation $x = f(x)$, $x \in X$, where (X, d) is a complete metric space and f is a Meir-Keeler operator. In addition to the existence, uniqueness, and approximation for the fixed point of f , we will study (using Lim's characterization) some qualitative properties of the fixed point equation, like data dependence property, Ulam-Hyers stability, well-posedness in the Reich and Zaslavski sense, and Ostrowski stability property.

Lemma 2.2.3 (I. A. Rus [68]). *Let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing function such that*

$$\lim_{t \rightarrow \infty} (t - \phi(t)) = \infty.$$

Then, the function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

$$\beta(\eta) := \sup \{t \in \mathbb{R}_+ | t - \phi(t) \leq \eta\}$$

is well-defined and has the following properties:

- (i) β is increasing;
- (ii) $\beta(0) = 0$;
- (iii) β is continuous in 0.

The principal theorem of this section, stated below, establishes several qualitative properties of the fixed point equation.

Theorem 2.2.4. *Let (X, d) be a complete metric space. Let the mapping $f : X \rightarrow X$ be a Meir-Keeler operator. Assume that the L-function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ guaranteed by Theorem 2.2.2 fulfills the condition $\lim_{t \rightarrow \infty} (t - \phi(t)) = \infty$. Then we obtain the following conclusions:*

- (i) whenever $g : X \rightarrow X$ is a given operator and there exists $\eta > 0$ for which $d(f(x), g(x)) \leq \eta$, for any $x \in X$, satisfying

$$d(x^*, u^*) \leq \beta(\eta), \text{ for every } u^* \in \text{Fix}(g),$$

where $\beta(\eta) := \sup \{t \in \mathbb{R}_+ | t - \phi(t) \leq \eta\}$;

- (ii) the fixed point equation is well-posed in the sense of Reich and Zaslavski, i.e., for every sequence $(u_n)_{n \in \mathbb{N}} \subset X$ for which $d(u_n, f(u_n)) \rightarrow 0$ as $n \rightarrow \infty$ we obtain that $u_n \rightarrow x^*$, as $n \rightarrow \infty$;
- (iii) the fixed point equation is generalized Ulam-Hyers stable, i.e., there exists an increasing function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\beta(0) = 0$ and β is continuous in 0, such that for any $\varepsilon > 0$ and any $u^* \in X$ an ε -solution of the fixed point equation (in the sense that $d(u^*, f(u^*)) \leq \varepsilon$), there exists a fixed point $x^* \in X$ for which

$$d(u^*, x^*) \leq \beta(\varepsilon).$$

In order to prove another stability result for Meir-Keeler operators, we need the definition of a strong comparison function 1.1.7 and the lemma 1.1.4 regarding this notion.

Theorem 2.2.5. *Let (X, d) be a complete metric space. Let the mapping $f : X \rightarrow X$ be a Meir-Keeler operator. Suppose that the L-function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ guaranteed by Theorem 2.2.2 is a subadditive, strong comparison function and it fulfills the condition*

$$\lim_{t \rightarrow \infty} (t - \phi(t)) = \infty.$$

Then the fixed point equation satisfies the Ostrowski stability condition.

2.3 Fixed point results for multi-valued nonlinear graph φ -contractions in complete metric spaces

In 1969, Nadler proved the most important metric fixed point theorem for multi-valued operators. In the following year, Covitz and Nadler published a refinement of Nadler's fixed point principle for multi-valued contractions. Various authors have considered further extensions of this principle to different classes of generalized metric spaces and multi-valued generalized contractions; see, e.g., [3], [23], [27], [55], [61], [62], [63], [81], [82]. Additionally, some fixed point set properties (closeness, compactness, absolute retract property, etc.) were obtained, see, e.g., [20], [65], [66], [76], [77].

2.3.1 The notion of multi-valued nonlinear graph φ -contraction

The following notions are essential for our approach in order to establish the main result. Let us give the definition for multi-valued nonlinear graph φ -contraction, the sequence of Picard iterates of F , and the φ -multi-valued weakly Picard operator.

Definition 2.3.1. Let (X, d) be a generalized metric space. Then, an operator $F: X \rightarrow P(X)$ is said to be a multi-valued nonlinear graph φ -contraction if there exists a strong comparison function $\varphi: [0, \infty) \rightarrow [0, \infty)$ such that

$$H(F(x), F(y)) \leq \varphi(d(x, y)), \text{ for every } (x, y) \in \text{Graph}(F). \quad (2.3.1)$$

Definition 2.3.2 (A. Petruşel [50]). Let (X, d) be a metric space and consider a multi-valued weakly Picard operator $F: X \rightarrow P(X)$. Then, we introduce the multi-valued operator $F^\infty: \text{Graph}(F) \rightarrow P(\text{Fix}(F))$ given by $F^\infty(x, y) = \{z \in \text{Fix}(F) \mid \text{there exists a sequence of Picard iterates of } F \text{ starting from } (x, y) \text{ that converges to } z\}$.

Definition 2.3.3 (A. Petruşel [50]). Let (X, d) be a metric space and consider a multi-valued weakly Picard operator $F: X \rightarrow P(X)$. Then, F is a φ -multi-valued weakly Picard operator if $\varphi: [0, \infty) \rightarrow [0, \infty)$ is increasing, continuous in 0 with $\varphi(0) = 0$ and there a selection f^∞ of F^∞ exists such that

$$d(x, f^\infty(x, y)) \leq \varphi(d(x, y)), \text{ for all } (x, y) \in \text{Graph}(F).$$

Remark 2.3.1. Notice that if $F: X \rightarrow P_{cl}(X)$ is an α -contraction, then F is a φ -MWP operator with $\varphi(f) = \frac{1}{1-\alpha}f$.

2.3.2 Fixed point result for multi-valued nonlinear graph φ -contraction

The first main theorem in the current section is a fixed point result for a multi-valued nonlinear graph φ -contraction. We first recall an auxiliary result.

Lemma 2.3.2 (S. B. Nadler Jr. [46]). Let (X, d) be a generalized metric space, a multi-valued operator $F: X \rightarrow P(X)$, and $q > 1$. Then, for every $x_0, x_1 \in X$ and $z_1 \in F(x_0)$ there exists $z_2 \in F(x_1)$ such that

$$d(z_1, z_2) \leq qH(F(x_0), F(x_1)).$$

The results below are extensions of a classical theorem of R. Węgrzyk, see Theorem 2.1 and Theorem 2.2 in [82].

In the context of a classical metric space, the result below is obtained.

Theorem 2.3.3. *Let (X, d) be a complete metric space and consider $F: X \rightarrow P(X)$ to be a multi-valued nonlinear graph φ -contraction with $\text{Graph}(F)$ closed. Hence, we obtain the following consequences:*

- (a) *there exists a point $x^* \in X$ with the property that $x^* \in \text{Fix}(F^n)$, for every $n \in \mathbb{N}^*$;*
- (b) *for each $(x, y) \in \text{Graph}(F)$, there exists a sequence of Picard iterates $(x_n)_{n \in \mathbb{N}}$ for F with (x, y) as the starting point that converges to a fixed point of F ;*
- (c) *F is called a ψ -multi-valued weakly Picard operator, i.e., it requires the existence of a selection $f^\infty: \text{Graph}(F) \rightarrow \text{Fix}(F)$ of F^∞ for which*

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)), \text{ for every } (x, y) \in \text{Graph}(F),$$

where

$$\psi(t) := t + \sum_{k=0}^{\infty} \varphi^k(st), \text{ with } s > 1 \text{ chosen arbitrarily.} \quad (2.3.2)$$

Remark 2.3.4. *Taken together, (a) and (b) imply that F represents a multi-valued weakly Picard operator. The property stated in conclusion (c) is named the retraction–displacement condition. From (a)–(c) together with Definition 2.3.3, it follows that F is a ψ -multi-valued weakly Picard operator.*

Theorem 2.3.5 (C. K. Jung [29]). *Let (X, d) be a generalized metric space. Then the relation \tilde{d} defined as*

$$x \tilde{d} y \Leftrightarrow d(x, y) < \infty \text{ for } x, y \in X, \quad (2.3.3)$$

is an equivalence relation and if $X_s : s \in S$ are the equivalence classes under ρ then $d(x, y) = \infty$ whenever $x \in X_s, y \in X_t, s \neq t$. Also, if we let $d_s := d|_{X_s \times X_s}$, then (X_s, d_s) is a metric space (for each $s \in S$).

Theorem 2.3.6 (R. Węgrzyk [38]). *Let (Y, d) be a metrically convex complete metric space and let (X, \tilde{d}) be an arbitrary metric space. Moreover, let $f: Y \rightarrow X$ be an arbitrary function. Then there is a function $\varphi: [0, \infty) \rightarrow [0, \infty)$, $\varphi(t) < t$, for $t > 0$ and*

$$\tilde{d}(f(x), f(y)) \leq \varphi(d(x, y)), \text{ with } x, y \in Y \quad (2.3.4)$$

which is strictly increasing, concave, and continuously differentiable (i.e., its derivative exists and is itself a continuous function) in $[0, \infty)$.

Let us denote by $C^n[I]$ the class of functions f , which have continuous derivatives up to order n in I .

By S_ξ^n we write the class of functions f which belongs to $C^n[I]$ (I arbitrary interval) and fulfill the conditions:

$$(f(x) - x)(\xi - x) > 0 \text{ for } x \in I, x \neq \xi,$$

$$(f(x) - \xi)(\xi - x) < 0 \text{ for } x \in I, x \neq \xi.$$

Theorem 2.3.7 (M. Kuczma [32]). *If $f \in S_0^n$, then for every $x_0 \in I$ the sequence $(f^n(x_0))_{n \in \mathbb{N}}$ is increasing (and strictly increasing whenever $x_0 \neq 0$) and*

$$\lim_{n \rightarrow \infty} f^n(x_0) = 0.$$

Corollary 2.3.8. *Suppose (X, d) is a metrically convex complete metric space and let $F : X \rightarrow P(X)$ be a multi-valued operator with closed graph fulfilling the condition (2.3.1) with a function φ such that $\varphi(t) < t$ for $t > 0$, then:*

(a) *for every $x_0 \in X$ and for every fixed point $x \in X$ of F there exists a sequence of iterations of F starting from x_0 which converges to x ,*

(b) *moreover, if there exists a $s_0 > 0$ such that the function φ is strictly increasing in the interval $[0, s_0]$ and for which the series $\sum_{k=0}^{\infty} \varphi^k(s_0)$ converges, then the set of fixed points of F is nonempty.*

As a consequence of Theorem 2.3.3, we obtain the following qualitative results regarding the fixed point set.

Theorem 2.3.9. (Data dependence of the fixed point set) *Let (X, d) be a complete metric space and let $F : X \rightarrow P(X)$ be a multi-valued nonlinear graph φ -contraction whose graph is closed. Hence, the results below hold:*

(a) $Fix(F) \neq \emptyset$;

(b) *if $G : X \rightarrow P_d(X)$ represents a multi-valued nonlinear graph β -contraction with closed graph, and if $\eta > 0$ satisfies*

$$H(F(x), G(x)) < \eta, \text{ for each } x \in X,$$

it follows that

$$H(\text{Fix}(F), \text{Fix}(G)) \leq \max \left\{ \psi(\eta), \tilde{\psi}(\eta) \right\},$$

where, for any $s > 1$, we denote

$$\psi(t) = t + \sum_{k=1}^{\infty} \varphi^k(s \cdot t)$$

and

$$\tilde{\psi}(t) = t + \sum_{k=1}^{\infty} \beta^k(s \cdot t).$$

Regarding the Ulam–Hyers stability of the fixed point inclusion, we obtain the following result.

Theorem 2.3.10. *Let (X, d) be a complete metric space and $F: X \rightarrow P(X)$ be a multi-valued nonlinear graph φ -contraction. Then, the fixed point problem (1.2.2) is Ulam-Hyers stable.*

Remark 2.3.11. *It is an open question to establish well-posedness according to Reich and Zaslavski, and Ostrowski stability property of the fixed point inclusion $x \in F(x)$ with a multi-valued nonlinear graph φ -contraction F .*

The strict fixed point property constitutes a special case in the study of multi-valued operators within fixed point theory. Accordingly, the following strict fixed point theorem holds for multi-valued nonlinear graph φ -contractions.

Theorem 2.3.12. *Let (X, d) be a complete metric space and $F: X \rightarrow P(X)$ be a multi-valued nonlinear graph φ -contraction with closed graph. Suppose:*

- (i) $F(F(x)) \subset F(x)$, for each $x \in X$;
- (ii) if $A \in P_{cl}(X)$ with $F(A) = A$, then A is a singleton.

Then, $\text{Fix}(F) = S\text{Fix}(F) \neq \emptyset$.

2.3.3 Maia type fixed point theory for multi-valued nonlinear graph φ -contractions

The following Maia-type theorem holds for a nonlinear graph φ -contractions on a nonempty set endowed with two metrics. The Maia type result was proved by R. Trușcă, in the paper [45]. An application for the mentioned result is given by the author in the case of the Volterra-type integral inclusion.

We will now extend the notion of multi-valued nonlinear (φ, ψ) -contraction of Feng-Liu type, introduced in the paper by A. Petrușel, G. Petrușel and J.-C. Yao [54].

Definition 2.3.4 (A. Petrușel, G. Petrușel, J.-C. Yao [54]). *Let (X, d) be a metric space, $F: X \rightarrow P(X)$ be a multi-valued operator and $\psi: [0, \infty) \rightarrow [0, \infty)$ be an increasing mapping such that $\psi(t) > t$ for any $t > 0$ and $\psi(0) = 0$. For each $x \in X$, we consider the set*

$$I_{\psi}^x(d) := \{y \in F(x) : d(x, y) \leq \psi(D_d(x, F(x)))\}. \quad (2.3.5)$$

The operator F is called a multi-valued nonlinear (φ, ψ) -contraction of Feng-Liu type with respect to the metric d if there exists an increasing mapping $\varphi: [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t) < t$ for each $t \in (0, \infty)$ and $\varphi(0) = 0$, such that:

- (a) $\psi \circ \varphi$ is a strong comparison function;
- (b) for each $x \in X$ there exists $y \in I_{\psi}^x(d)$ such that

$$D_d(y, F(y)) \leq \varphi(d(x, y)). \quad (2.3.6)$$

An extension of Theorem 2.5 in [54] is established for spaces equipped with two metrics. The following two theorems are Radu Trușcă's results, and the proof can be found in the article [45].

Theorem 2.3.13. *Let X be a nonempty set endowed with two metrics, d and ρ . Let $F: X \rightarrow P_{cl}(X)$ and $\psi: [0, \infty) \rightarrow [0, \infty)$ be an increasing mapping such that $\psi(t) > t$ for any $t > 0$ and $\psi(0) = 0$. We assume that:*

- (i) (X, d) is a complete metric space;

- (ii) there exists $R > 0$ such that $d(x, y) \leq R\rho(x, y)$ for all $x, y \in X$;
- (iii) $\text{Graph}(F)$ is closed with respect to the metric ρ ;
- (iv) there exists an increasing mapping $\varphi: [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t) < t$ for each $t \in (0, \infty)$ and $\varphi(0) = 0$, such that $\psi \circ \varphi$ is a strong comparison mapping and the following relation holds

$$D_\rho(y, F(y)) \leq \varphi(\rho(x, y)), \text{ for every } (x, y) \in \text{Graph}(F).$$

Then, we have that $\text{Fix}(F) \neq \emptyset$ and for each $x_0 \in X$ there exists an iterative sequence of Picard type $(x_n)_{n \in \mathbb{N}}$ convergent to a point $x^* \in \text{Fix}(F)$. Moreover, the operator satisfies the following retraction-displacement condition:

$$d(x_0, x^*(x_0)) \leq \sum_{k=0}^{\infty} (\psi \circ \varphi)^k(t_0), \quad (2.3.7)$$

where $t_0 := d(x_0, x_1)$, $x_1 \in I_\psi^{x_0}(\rho)$.

As author's contribution in this subsection is an application of the above result. We can consider the following integral inclusion of the Volterra type.

Let us introduce the notation for the interval $I := [a, b]$.

$$x(t) \in \int_a^t L(s, x(s))ds + l(t), \quad t \in I, \quad (2.3.8)$$

where $L : I \times \mathbb{R}^n \rightarrow P_{cl,cv}(\mathbb{R}^n)$ and $l : I \rightarrow \mathbb{R}^n$ satisfy the presented below hypotheses:

- (A₁) for every $u \in C(I, \mathbb{R}^n)$ the multi-valued operator $L_u : I \rightarrow P_{cl,cv}(\mathbb{R}^n)$, defined by $L_u(s) := L(s, u(s))$, is measurable and integrable bounded, i.e., an integrable mapping exists $m : I \rightarrow \mathbb{R}_+$ for which $L(s, u(s)) \subset B(0, m(s))$ a.e. $s \in I$;
- (A₂) there exist two increasing mappings $\psi, \varphi: [0, \infty) \rightarrow [0, \infty)$ with $\psi(t) > t$ for any $t > 0$, $\psi(0) = 0$, $\varphi(t) < t$ for each $t \in (0, \infty)$ and $\varphi(0) = 0$, such that $\psi \circ \varphi$ is a strong comparison function;
- (A₃) l is continuous;

(A₄) there exist $\kappa : I \rightarrow \mathbb{R}_+$ measurable in s and an increasing function $\varphi : [0, \infty[\rightarrow [0, \infty)$, for which φ is a strong comparison function and

$$D(v, L(s, v)) \leq \kappa(s)\varphi(|u - v|),$$

for all $(s, u) \in I \times \mathbb{R}^n$ and $v \in S(s, u)$.

By a solution of (2.3.8) we mean a continuous mapping $u \in C(I, \mathbb{R}^n)$ that fulfills the above inclusion for each $t \in I$. We will denote by $\|\cdot\|_B$ the following Bielecki type norm in $C(I, \mathbb{R}^n)$, i.e.,

$$\|x\|_B := \sup_{t \in I} [|x(t)|e^{-\tau K(t)}],$$

where $K(t) := \int_a^t k(s)ds$ and $\tau > 1$.

The following theorem gives an existence result for the integral inclusion (2.3.8).

Theorem 2.3.14. *Consider the integral inclusion (2.3.8) and suppose that the hypotheses (A₁) – (A₄) hold. It follows that there is at least one solution of (2.3.8).*

Remark 2.3.15. *Theorem 2.3.13 extends a result given by A. Petruşel, in [52] (see Theorem 2.8.11), where a stronger contraction type condition on the kernel L is imposed.*

Chapter 3

Surjectivity theorems for classes of contractive operators

This chapter is divided into two sections, one is about the surjectivity theorems for Meir-Keeler operators, and the other is about surjectivity theorems for large contractions. The first step in proving the surjectivity theorems in the following subsections is to prove that the operator is a norm-contraction. Using this result, we will prove the surjectivity property. Important theorems used later in the following subsections are the one given by A. Granas in 1957, see [24], and the one shown by T. Suzuki in 2007, see [78].

Theorem 3.0.1 (A. Granas [24]). *Let $(X, \|\cdot\|)$ be a Banach space, and let $f : X \rightarrow X$ be a complete continuous operator. If, additionally, f is a norm-contraction, then $\text{Fix}(f) \neq \emptyset$.*

Theorem 3.0.2 (T. Suzuki [78]). *Let $(X, \|\cdot\|)$ be a Banach space and C be a nonempty and convex subset of X . Assume that $f : C \rightarrow C$ is a Meir-Keeler operator. Then, for every $\varepsilon > 0$, there exists $r_\varepsilon \in (0, 1)$ for which*

$$\|x - y\| \geq \varepsilon \text{ implies } \|f(x) - f(y)\| \leq r_\varepsilon \|x - y\|,$$

for all $x, y \in C$.

3.1 Surjectivity theorems for Meir-Keeler operators

This section first aims to establish that a Meir–Keeler operator f on a Banach space X , single-valued and multi-valued, is a norm contraction. Then, using this result, we will give sufficient conditions assuring that the field $1_X - f$, generated by f , is surjective. The next two sections contain surjectivity theorems for single-valued and, respectively, multi-valued Meir-Keeler operators. The last subsection is devoted to an application of the previous results in the single-valued setting. Our results generalize some well-known theorems of this type for Banach/Nadler type contractions, see [5] and [25], as well as other results of this type for generalized contractions, see [1], [6], [59], [69], [79].

3.1.1 Surjectivity theorems for single-valued Meir-Keeler operators

The definition of the Meir-Keeler operator 1.1.9 and the Meir-Keeler theorem 1.1.5 are provided in the *Notions and preliminary results* chapter, single-valued section, since they are also used in Chapter 2.

In this section, we will present a new surjectivity theorem for a single-valued Meir-Keeler operator. The approach is based on the norm-contraction operator theory and on a well-known theorem of A. Granas [24], Theorem 3.0.1. Also, the result proved by T. Suzuki in 2007 is used in our approach, Theorem 3.0.2, mentioned above in the introduction of the chapter.

The following theorem provides our first main result concerning the surjectivity of Meir–Keeler operators in Banach spaces. For the proof of this theorem, we will use Definition 1.1.11, and consider two cases: first, when x is inside the closed ball $\tilde{B}(x_0; R)$, and second, when x is outside the closed ball $\tilde{B}(x_0; R)$.

Theorem 3.1.1. *Let $(X, \|\cdot\|)$ be a Banach space and Y be a nonempty, closed, and convex subset of X . Let us consider the operator $f : Y \rightarrow Y$ that satisfies the following conditions:*

- (i) f restricted to any bounded set in Y is compact;

(ii) f represents a Meir-Keeler operator.

Then $1_Y - f$ is a surjective operator.

3.1.2 Surjectivity theorems for multi-valued Meir-Keeler operators

This section begins by recalling some preliminary notions and results that will let us prove a new surjectivity theorem for the multi-valued field generated by a multi-valued Meir-Keeler operator. In 2000, H. K. Xu, [83] provided the definition of a multi-valued Meir-Keeler operator. No corresponding multi-valued theorem was given, similar to the one given by Suzuki for the single-valued case. This gap motivated the proposal of a multi-valued version, which is presented as Theorem 3.1.3.

Definition 3.1.1 (see, e.g., H. K. Xu [83]). *Let $(X, \|\cdot\|)$ be a Banach space. Then, $F : X \rightarrow P(X)$ is referred to as a multi-valued Meir-Keeler operator if for all $\varepsilon > 0$ there exists $\delta > 0$ for which*

$$\varepsilon \leq \|x - y\| < \varepsilon + \delta \text{ implies } H(F(x), F(y)) < \varepsilon.$$

Lemma 3.1.2. *Let $(X, \|\cdot\|)$ be a normed space. If $Y, Z \in P_{cp}(X)$, then given any $y \in Y$ one can find $z \in Z$ with*

$$\|y - z\| \leq H(Y, Z).$$

To establish the principal theorem, we will first establish a new result regarding the following multi-valued version of Suzuki's single-valued Theorem 3.0.2.

Theorem 3.1.3. *Let a Banach space $(X, \|\cdot\|)$ and $Y \subseteq X$ be a convex subset of X . Let $F : Y \rightarrow P_{cp}(Y)$ be a multi-valued Meir-Keeler operator. Then, for every $\varepsilon > 0$ there exists $r_\varepsilon \in (0, 1)$ satisfying*

$$x, y \in X, \|x - y\| \geq \varepsilon \text{ implies } H(F(x), F(y)) \leq r_\varepsilon \|x - y\|.$$

To prove our main surjectivity theorem for multi-valued Meir-Keeler operators, we require the following known result, established by R. Iannacci [26] in 1978.

Theorem 3.1.4 (R. Iannacci [26]). *Let $(X, \|\cdot\|)$ be a Banach space and $F : X \rightarrow P_{cv}(X)$. Suppose that F is a completely continuous multi-valued norm-contraction. Then, the field $1_X - F$ generated by F is surjective.*

To establish our main surjectivity theorem for multi-valued Meir-Keeler operators, we require the following known result, established by R. Iannacci [26] in 1978.

Theorem 3.1.5. *Let $(X, \|\cdot\|)$ be a Banach space and Y be a nonempty, closed, and convex subset of X . Let us consider the multi-valued operator $F : Y \rightarrow P_{cp,cv}(Y)$ fulfilling the conditions below:*

- (i) *for each $A \in P_b(X)$ the set $F(A)$ is relatively compact;*
- (ii) *F represents a Meir-Keeler operator.*

Then, the field $1_Y - F$ generated by F is surjective.

Remark 3.1.6. *It is easy to see that any Nadler contraction (see, e.g., [5]) is a multi-valued Meir-Keeler operator, and any multi-valued Meir-Keeler operator is contractive (i.e., $H(F(x), F(y)) < \|x - y\|$, for every distinct $x, y \in X$). It remains an open question whether a similar surjectivity theorem holds for multi-valued contractive operators.*

3.1.3 An application to a nonlinear integral equation

To apply the theory developed above, we establish several existence results for the nonlinear integral equation presented below.

We shall denote the interval $[0, T]$ by I .

$$x(t) = g(x(t)) + \int_0^T K(t, s, x(s))ds, \quad t \in I, \quad (3.1.1)$$

where $K \in C(I \times I \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C(\mathbb{R}^n, \mathbb{R}^n)$.

If we define the operator $F : C(I, \mathbb{R}^n) \rightarrow C(I, \mathbb{R}^n)$,

$$Fx(t) := g(x(t)) + \int_0^T K(t, s, x(s))ds, \quad t \in I, \quad (3.1.2)$$

then the equation (3.1.1) takes the form of a fixed point equation

$$x = Fx, \quad x \in C(I, \mathbb{R}^n). \quad (3.1.3)$$

First, we show that under some appropriate conditions, the operator F is a norm-contraction.

Lemma 3.1.7. *Let us suppose that:*

1. $K \in C(I \times I \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C(\mathbb{R}^n, \mathbb{R}^n)$;
2. there exists $a > 0$ such that $\|g(u)\|_{\mathbb{R}^n} \leq a\|u\|_{\mathbb{R}^n}$, for any $u \in \mathbb{R}^n$;
3. one can find $p, q, \alpha, \beta, \eta \in]0, \infty[$ fulfilling

$$\|K(t, s, u)\|_{\mathbb{R}^n} \leq \alpha t^p + \beta s^q + \eta\|u\|_{\mathbb{R}^n}, \quad \text{for all } t, s \in I, u \in \mathbb{R}^n;$$

4. $a + \eta T < 1$.

Then the operator F defined in (3.1.2) is a norm-contraction on $C(I, \mathbb{R}^n)$.

Remark 3.1.8. *Lemma 3.1.7 also holds if 3. is replaced by the following condition*

- 3'. there exist $\eta, G \in]0, \infty[$ such that

$$\|K(t, s, u)\|_{\mathbb{R}^n} \leq G + \eta\|u\|_{\mathbb{R}^n}, \quad \text{for any } u \in \mathbb{R}^n;$$

By Lemma 3.1.7 and Theorem 3.0.1, we get the following result.

Theorem 3.1.9. *We consider the integral equation (3.1.1). We suppose that:*

1. $K \in C(I \times I \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in C(\mathbb{R}^n, \mathbb{R}^n)$;
2. there exists $a > 0$ for which $\|g(u)\|_{\mathbb{R}^n} \leq a\|u\|_{\mathbb{R}^n}$, for any $u \in \mathbb{R}^n$;
3. there exists $p, q, \alpha, \beta, \eta \in (0, \infty)$ fulfilling

$$\|K(t, s, u)\|_{\mathbb{R}^n} \leq \alpha t^p + \beta s^q + \eta\|u\|_{\mathbb{R}^n}, \quad \text{for every } t, s \in I, u \in \mathbb{R}^n;$$

4. $a + \eta T < 1$.

Therefore, the integral equation (3.1.1) admits at least one solution.

Corollary 3.1.10. *The conclusion of Theorem 3.1.9 also holds for the hypotheses 1, 2, 3' and 4, mentioned in the Remark 3.1.8.*

In our second approach, we will show that the operator F is a norm-contraction by imposing appropriate conditions such that F is a Meir-Keeler operator.

Lemma 3.1.11. *We suppose that:*

1. $K \in C(I \times I \times \mathbb{R}^n, \mathbb{R}^n)$ is Lipschitz in the third variable, and $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a continuous function;
2. there exists $a > 0$ satisfying

$$\|g(u) - g(v)\|_{\mathbb{R}^n} \leq a\|u - v\|_{\mathbb{R}^n},$$

for any $u, v \in \mathbb{R}^n$;

3. there exists $L_K > 0$ such that, for every $\varepsilon > 0$ there exists $\delta > 0$ with

$$\delta < \frac{1 - a - L_K T}{a + L_K T} \varepsilon,$$

for which the following implication holds:

$$\|u - v\|_{\mathbb{R}^n} < \varepsilon + \delta \Rightarrow \|K(t, s, u) - K(t, s, v)\|_{\mathbb{R}^n} < \varepsilon,$$

for any $u, v \in \mathbb{R}^n$;

4. $L_K < \frac{1 - a}{T}$.

Then, F defined by (3.1.2) is a Meir-Keeler operator on $C(I, \mathbb{R}^n)$.

Theorem 3.1.12. *We consider the integral equation (3.1.1). We suppose that:*

1. $K \in C(I \times I \times \mathbb{R}^n, \mathbb{R}^n)$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is bounded;
2. there exists $a > 0$ for which

$$\|g(u) - g(v)\|_{\mathbb{R}^n} \leq a\|u - v\|_{\mathbb{R}^n},$$

for any $u, v \in \mathbb{R}^n$

3. there exists $L_K > 0$ such that, given any $\varepsilon > 0$ there exists $\delta > 0$ with

$$\delta < \frac{1 - a - L_K T}{a + L_K T} \varepsilon,$$

for which the following implication holds:

$$\|u - v\|_{\mathbb{R}^n} < \varepsilon + \delta \Rightarrow \|K(t, s, u) - K(t, s, v)\|_{\mathbb{R}^n} < \varepsilon,$$

for any $u, v \in \mathbb{R}^n$;

4. $L_K < \frac{1 - a}{T}$.

Hence, the integral equation (3.1.1) admits at least one solution.

3.2 Surjectivity theorems for large contractions

The concept of a large contraction was introduced for the single-valued case by T.A. Burton in 1996. In this section, we show that each single-valued large contraction $f : X \rightarrow X$ is a norm-contraction. Using this result, we can prove some surjectivity theorems for the field $1_X - f$ generated by f . In the second subsection, we will establish the definition of multi-valued large contraction and, using a similar approach, we prove the surjectivity of the field $1_X - F$ generated by a multi-valued operator F from X to X . The results of this section generalize and strengthen several theorems from the previous studies; see Lemma in [13] and the surjectivity theorems in [41], [59] and [72].

3.2.1 The notion of a large contraction

The following concept was introduced by T.A. Burton in the framework of a metric space.

Definition 3.2.1 (T. A. Burton [13]). *Let $(X, \|\cdot\|)$ be a normed space. An operator $f : X \rightarrow X$ is defined as a large contraction if*

1. f satisfies the contractive condition, that is,

$$\|f(x) - f(y)\| < \|x - y\| \text{ for all distinct } x, y \in X,$$

2. for all $\varepsilon > 0$, there exists $\delta < 1$ with

$$x, y \in X, \|x - y\| \geq \varepsilon \Rightarrow \|f(x) - f(y)\| \leq \delta \|x - y\|.$$

J. Jachymski in [28] noted that the contractive condition in the above definition could be avoided and, as a consequence, an operator $f : X \rightarrow X$ is a large contraction if, for every $\varepsilon > 0$, one can find $\delta(\varepsilon) \in]0, 1[$ for which

$$x, y \in X, \|x - y\| \geq \varepsilon \Rightarrow \|f(x) - f(y)\| \leq \delta(\varepsilon) \|x - y\|.$$

For related equivalences involving classes of generalized contractions, see [28].

The following concept, introduced by M.A. Krasnoselskii, is related to the above definition.

Definition 3.2.2 (M.A. Krasnoselskii, see, e.g., Y.-Z. Chen [18]). *Let $(X, \|\cdot\|)$ be a Banach space. An operator $f : X \rightarrow X$ is called a generalized contraction if for any $0 < a < b < \infty$, there exists $\delta(a, b) \in (0, 1)$ such that*

$$\|f(x) - f(y)\| \leq \delta(a, b) \|x - y\|,$$

for all $x, y \in X$ satisfying $a < \|x - y\| < b$.

3.2.2 A surjectivity theorem for single-valued large contractions

In this subsection, we will prove the property that the field $1_X - f$ is surjective, where f is a single-valued large contraction. The proof is based on T.A.

Burton's large contraction principle [13] and the theorem of A. Granas [24], Theorem 3.0.1. On the one hand, a large contraction is a generalized contraction, see [18]. On the other hand, there exist large contractions which are not (Banach) contractions.

T. A. Burton gave the principal fixed point theorem for large contractions.

Theorem 3.2.1 (T. A. Burton [13]). *Let (X, d) be a complete metric space and $f : X \rightarrow X$ be a large contraction. Suppose there exist $x \in X$ and $L > 0$, such that*

$$d(x, f^n(x)) \leq L,$$

for all $n \geq 1$. Then, f has a unique fixed point in X .

The following result is an immediate consequence of the preceding theorem.

Corollary 3.2.2. *Let a complete metric space (X, d) and let $f : X \rightarrow X$ an operator for which one can find $n_0 \in \mathbb{N}$, $n_0 \geq 2$ such that f^{n_0} is a large contraction. Suppose there is an $x \in X$ and $L > 0$ for which $d(x, f^n(x)) \leq L$ for all $n \geq 1$. Then, f has a unique fixed point.*

The theorem below provides the principal result established in this part of the work.

Theorem 3.2.3. *Let $(X, \|\cdot\|)$ be a Banach space and the operator $f : X \rightarrow X$ satisfying the following assumptions:*

- (i) f is a large contraction;
- (ii) $f(A)$ is relatively compact, for each $A \in P_b(X)$.

Then, the field $1_X - f : X \rightarrow X$ generated by f is a surjective operator.

3.2.3 A surjectivity theorem for multi-valued large contractions

We begin by recalling some preliminary notions and results that are useful in proving the main theorem of this section. The subsequent theorem was established in 1978 by R. Iannacci.

From the *Notions and preliminary results* chapter, multi-valued section, the definition of multi-valued norm-contraction is known.

Theorem 3.2.4 (R. Iannacci [26]). *Let $(X, \|\cdot\|)$ be a Banach space, and a multi-valued completely continuous operator $F : X \rightarrow P_{cv}(X)$. Suppose that F is a multi-valued norm-contraction. Then, the field $1_X - F$ generated by F is surjective.*

Definition 3.2.3. *Let $(X, \|\cdot\|)$ be a Banach space. Then $F : X \rightarrow P(X)$ is said to be a multi-valued large contraction if for all $\varepsilon > 0$, exists $\delta(\varepsilon) \in]0, 1[$ such that*

$$x, y \in X, \|x - y\| \geq \varepsilon \Rightarrow H(F(x), F(y)) \leq \delta(\varepsilon)\|x - y\|.$$

It follows easily that a multi-valued large contraction is contractive. Therefore, the subsequent theorem represents the principal surjectivity result for the multi-valued large contraction operator discussed in this section.

Theorem 3.2.5. *Let $(X, \|\cdot\|)$ be a Banach space, and let a multi-valued operator $F : X \rightarrow P_{cp,cv}(X)$ that satisfies the assumptions listed below:*

1. F represents a multi-valued large contraction;
2. $F(U)$ is relatively compact, for each $U \in P_b(X)$.

Then, the field $1_X - F$ generated by F is surjective.

Conclusions

The primary objective of the thesis is to give extended variants for the fixed point theorems related to Ćirić type operators, Meir-Keeler type operators and multi-valued nonlinear graph φ -contractions. Additionally, we gave surjectivity theorems for Meir-Keeler operators and large contractions in the sense of Burton. An application to a nonlinear integral equation is given related to the Meir-Keeler type operators. More precisely, several qualitative properties of the fixed point problem for single-valued and multi-valued Ćirić type operators were obtained. Also, for the multi-valued case, some strict fixed point theorems are proved. The stability properties were also proved for the Meir-Keeler operators and for the multi-valued nonlinear graph φ -contraction. Moreover, the study also focuses on sets that admit two metrics, following the idea introduced in the Maia type theorems. The results can also be developed for spaces carrying two metrics, following the approach used in the Maia-type theorems.

Possible directions for future development would be to extend the framework of the problems to different generalized metric spaces, to prove some qualitative properties of the solutions for the fixed point problems. Also, the studies can be enhanced for the case of a set endowed with two metrics, as in the case of the Maia type theorems.

Future work may involve applying the abstract fixed point result to some problems generated by the models from the natural science area and economics.

New studies may refer to the problem with initial values:

$$\begin{cases} x'(t) \in K(t, x(t)) \\ x(0) = x^0 \end{cases}$$

that models the dynamic evolution of some complex systems of material points/ data/ populations, and so forth. The following references provide valuable background for this study, see, e.g., [8], [7], [15], [21] and [31].

Another study can focus on the existence and stability results for the following integral inclusion with delay:

$$\begin{cases} x(t) \in \int_{t-T}^t K(s, x(s))ds, & t \in [0, T] \\ x(t) = \varphi(t), & t \in [-\tau, 0] \end{cases}$$

which is modeling the evolution (growth) of the large (undetermined) systems in a periodic environment, see A. Petruşel [52].

In conclusion, the study offers numerous potential applications and opens up broad directions for future research by introducing new concepts and theorems. The presented work contributes to a deeper theoretical understanding of nonlinear fixed point concepts while also supporting the application perspective.

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