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Local fixed point theorems for single-valued and multi-valued operators: Theory and applications

Doctoral Thesis - Summary

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Introduction

Developments in fixed point theory

Fixed point theory dates back to the late 19th and early 20th centuries. Some of the earliest results are Henri Poincaré's fixed-point-like result stated in 1886, and Luitzen E. J. Brouwer's fixed point theorem given in 1911. In 1922, Stefan Banach defined the contraction mapping principle, proving existence, as well as uniqueness of a fixed point for any contraction operator in a complete metric space. Also in 1922 George D. Birkhoff and Oliver D. Kellogg stated the first infinite dimensional fixed point theorem. Juliusz Schauder gave in 1927 and 1930 an extension to metric linear spaces and proved, within the framework of a Banach space, that every compact convex set fulfills the fixed point property if the mapping is continuous. Since then, many generalizations and extensions have been proved, such as Kannan, Bianchini, Berinde, Zamfirescu, Ćirić-Reich-Rus, Ćirić, and Hardy-Rogers.

Looking at the multi-valued case, the earliest fixed point result in this setting was published in 1969 by Sam B. Nadler Jr. in [41]. This important result was soon improved by Howard H. Covitz and Sam B. Nadler Jr. in [20], by removing the boundedness restriction. It was also extended and generalized in many articles, such as [56], [58], [61], [65], and [73]. Some notable generalizations are also studied throughout this work, such as Berinde, Ćirić, and Feng-Liu type operators.

The first mathematician to prove a local fixed point theorem for the Banach contraction case was Mark A. Krasnoselskii. Then Marlene Frigon and Andrzej Granas proved the multi-valued kind of the local fixed point result. These two theorems have inspired many other mathematicians to extend them over the past decades.

Motivation

The primary goal of this thesis is to extend local fixed point results to different generalized contractions, in both single- and multi-valued settings, such as Ćirić-Reich-Rus, Chatterjea, Berinde, etc. We also prove applications for the aforementioned results. These theoretical results can then be applied in many scenarios, such as the study of mathematical models governed by equations and differential or integral inclusions. Another use case could be to obtain the existence of optimal preferences in an abstract economy.

The results that follow in this work can prove useful in many fields, including engineering, economics, management, computer science, and telecommunication.

Thesis structure and original contributions

The general structure of the thesis is organized as follows. We open with a chapter dedicated to preliminaries. We follow with the two main chapters, each presenting in detail the main contributions of the research. Then, we have a dedicated section that presents the conclusions of our study, along with further research directions. The thesis is concluded with a list of references that were consulted throughout the research work.

In the first chapter, we set the notations that we will practice through the following chapters and we also recall some well-known definitions and results that will help us in the proofs to follow. Some notable concepts are Hausdorff-Pompeiu lemma and Cauchy lemma, along with the definitions of a Picard operator, weakly Picard operator, and essential operator.

The second chapter deals with proving local fixed point results for single-valued generalized contractions, specifically for Ćirić-Reich-Rus, Chatterjea, Berinde, and Ćirić generalized contractions. In the first section, we focus on the Ćirić-Reich-Rus type operator. We begin by recalling the definition of such an operator, the global fixed point theorem, and a local fixed point theorem. Further we prove another local result, which improves the previous one, then we define the notion of (α, β) -contractive family and we determine an applied result of the proven local fixed point theorem. The main results of this section are found in "**Local fixed point theorems and open mapping principles for generalized contractions**", *Annales Univ. Sci. Budapest., Sect. Math.*, **64**, (2021) and "**Some local fixed point theorems and**

applications to open mapping principles and continuation results", *Arabian Journal of Mathematics*, **10**, (2021).

The next section pays attention to Chatterjea type operators. We commence with recalling the operator's definition, and the global fixed point result. We then prove two local fixed point theorems, one self and one non-self, the latter considering a weaker condition than the former. We continue with defining the notion of a γ -contractive family and give an application of the presented local results. The central results discussed in this section are contained in "**Local fixed point theorems and open mapping principles for generalized contractions**", *Annales Univ. Sci. Budapest., Sect. Math.*, **64**, (2021) and "**Some local fixed point theorems and applications to open mapping principles and continuation results**", *Arabian Journal of Mathematics*, **10**, (2021).

In the third section, we select another generalization of the regular α -contraction, specifically the Berinde type operator. We start with restating the definition of a Berinde type operator and the global fixed point theorem. We will then give two local fixed point results, self and non-self, the second one improving the first by assuming a weaker condition. We proceed to define the concept of (α, L) -contractive family and prove an application of the given local fixed point theorems. The primary results of this section are found in "**Local fixed point theorems and open mapping principles for generalized contractions**", *Annales Univ. Sci. Budapest., Sect. Math.*, **64**, (2021) and "**Some local fixed point theorems and applications to open mapping principles and continuation results**", *Arabian Journal of Mathematics*, **10**, (2021).

The fourth section of the second chapter focuses on results for Ćirić operators. We open by recalling the notion of the aforementioned operator, as well as the global fixed point result. We continue with proving the local fixed point theorem with respect to Ćirić type operators and we close the section with an application of it, precisely a continuation theorem. The main results from this section are found in "**On some fixed point theorems for Ćirić operators**", *Miskolc Mathematical Notes*, **25**, (2024).

In the final section, we aim to obtain fixed point results for other types of non-self generalized contractions, such as the Hardy-Rogers type contraction. We first recall an extension of Conti's fixed point theorem proved by Petruşel, Precup, and Şerban and then proceed to introduce two local fixed point theorems, one for continuous Berinde type contractions and another for continuous Hardy-Rogers type contractions. The section ends with a local

fixed point theorem for Berinde type operators, where we address additional conditions. The original results from this section can be found in "**Iterative approximations for non-self operators**", *22nd International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)*, (2019).

The final chapter studies fixed point results, applications, and stability characteristics for the following types of multi-valued contraction generalizations: Ćirić-Reich-Rus, Chatterjea, Berinde, Ćirić, nonlinear graph φ -contractions and (φ, ψ) -contractions of Feng-Liu type. The first section handles multi-valued Ćirić-Reich-Rus type contractions, where we recall the definition of such a generalized contraction and the global fixed point theorem proved by Reich. We continue with the main results of the section, specifically a local fixed point result, a local strict fixed point theorem, and an homotopy result as an application to the first local theorem. The mentioned original contributions are part of the article "**Some local fixed point theorems for generalized multi-valued contractions with applications**", *Journal of Nonlinear and Convex Analysis*, **23**, (2022).

The second section of Chapter 3 investigates the case of multi-valued Chatterjea type contractions. We remind its definition and the global fixed point result, then we prove the three central results of the section: two local results regarding fixed points and strict fixed point, and an homotopy theorem. These new results are published in the article "**Some local fixed point theorems for generalized multi-valued contractions with applications**" *Journal of Nonlinear and Convex Analysis*, **23**, (2022).

The next section examines the context of multi-valued Berinde type contractions, where we follow the same structure as the two previous sections from above, by recalling the definition and already established global fixed point theorem. We continue with the core results, which are a local fixed point theorem and the corresponding homotopy result as an application, the nonlinear alternative, and the Leray-Schauder principle for this type of generalized contractions. The original results have appeared in "**Some local fixed point theorems for generalized multi-valued contractions with applications**" *Journal of Nonlinear and Convex Analysis*, **23**, (2022).

We move on to the fourth part of Chapter 3, wherein we consider multi-valued Ćirić type contractions. The same section structure is followed, we mention the well-known multi-valued Ćirić type contraction definition and then we include a fixed point result for such operators. The main developments from this section are local fixed point results and an homotopy theorem

as an application. The detailed proofs and other observations regarding the original results of this section have been documented in the article "**On some fixed point theorems for Ćirić operators**", *Miskolc Mathematical Notes*, **25**, (2024).

The succeeding section employs in the study of multi-valued nonlinear graph φ -contractions. Herein we recall the definitions of comparison functions, strong comparison functions, and multi-valued nonlinear graph φ -contractions. Then we give two fixed point existence results, considering both a generalized, as well as a classical metric space setting, followed by stating the data dependence, as well as stability in the sense of Ulam-Hyers properties of the fixed point set. The next illustrated results are a strict fixed point theorem, ensued by two local theorems. We close the section with a Maia type result concerning multi-valued nonlinear graph φ -contractions and a corollary that follows it. The main developments displayed in this section are issued from the article "**Fixed point and stability results for multi-valued nonlinear graph contractions in complete metric spaces**", *The Journal of Analysis*, (2025).

In the last section of Chapter 3, we engage in the study of multi-valued (φ, ψ) -contractions of Feng-Liu type. After stating the definition of the said type of contractions, we establish two theorems that prove the existence and localization of fixed points. We proceed by recalling, in the context of multi-valued (φ, ψ) -contractions of Feng-Liu type, the definitions for the following generalized stability concepts: Ulam-Hyers, Reich-Zaslavski, Ostrowski, data dependence, and the generalized strong retraction-displacement condition. We then prove that multi-valued (φ, ψ) -contractions of Feng-Liu type satisfy the strong retraction-displacement condition, which leads us to the last result of the section, which proves qualitative characteristics of the fixed point set. The results comprised in this section are established in the article "**Fixed point and stability results for multi-valued nonlinear graph contractions in complete metric spaces**", *The Journal of Analysis*, (2025).

Results dissemination

The results presented in the thesis have been disseminated in the following conferences.

1. 22nd International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), Timișoara, Romania, September 1 - 4, 2020. Title: Iterative approximations for local fixed point theorems and applications;
2. 13th Joint Conference on Mathematics and Computer Science (MaCS), Budapest, Hungary, October 1 - 3, 2020. Title: Local fixed point theorems and open mapping principles for generalized contractions;
3. 19th International Conference on Functional Equations and Inequalities, Bedlewo, Poland, September 11 - 18, 2021. Title: Some local fixed point results and applications for generalized contractions;
4. 23rd International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), Timișoara, Romania, December 7 - 10, 2021. Title: Local fixed point results and applications for multivalued generalized contractions;
5. 4th Romanian Itinerant Seminar on Mathematical Analysis and its Applications (RISMAA), Brașov, Romania, May 19 - 21, 2022. Title: Some applications of local fixed point theorems for some generalized contractions;
6. 4th International Conference on Mathematics and Computer Science, Brașov, Romania, September 15 - 17, 2022. Title: Fixed point theory for multi-valued nonlinear graph contractions in complete metric spaces (joint work with Mădălina Moga and Adrian Petrușel);
7. 24th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), Linz, Austria, September 12 - 15, 2022. Title: On some fixed point theorems for Ćirić operators (joint work with Mădălina Moga);
8. 14th International Conference on Fixed Point Theory and its Applications (ICFPTA), Brașov, Romania, July 11-14, 2023, Title: "Fixed points for nonlinear graph contractions with applications";

9. 6th Romanian Itinerant Seminar on Mathematical Analysis and its Applications (RISMAA), Cluj-Napoca, Romania, May 30 - 31, 2024. Title: Maia type theorems for some multi-valued generalized contractions;
10. 26th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), Timișoara, Romania, September 16 - 19, 2024, Title: Fixed points for Feng-Liu multi-valued operators with an application (joint work with Mădălina Moga).

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List of publications

The results presented throughout the thesis have been published in the following articles.

1. "**Iterative approximations for non-self operators**", *22nd International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC)*, (2019) (see [53]) – indexed in Scopus, and Web of Science.
2. "**Local fixed point theorems and open mapping principles for generalized contractions**", *Annales Univ. Sci. Budapest., Sect. Math.*, **64**, (2021) (see [74]) – indexed in MathSciNet, and ZbMath;
3. "**Some local fixed point theorems and applications to open mapping principles and continuation results**", *Arabian Journal of Mathematics*, **10**, (2021) (see [75]) – indexed in MathSciNet, Scopus, Web of Science, and ZbMath;
4. "**Some local fixed point theorems for generalized multi-valued contractions with applications**", *Journal of Nonlinear and Convex Analysis*, **23**, (2022) (see [54]) – indexed in MathSciNet, Web of Science (Impact Factor: 0.7), and ZbMath;
5. "**On some fixed point theorems for Ćirić operators**", *Miskolc Mathematical Notes*, **25**, (2024) (see [40]) – indexed in MathSciNet, Scopus, Web of Science (Impact Factor: 0.9), and ZbMath;
6. "**Fixed point and stability results for multi-valued nonlinear graph contractions in complete metric spaces**", *The Journal of Analysis*, **33**, (2025) (see [39]) – indexed in MathSciNet, Scopus, Web of Science (Impact Factor: 0.7), and ZbMath.

Chapter 1

Preliminaries

In this chapter, we aim to present the notations that will be used throughout the next chapters. We will also reminisce some well-known notions, definitions, as well as results. For other similar concepts and results, see, e.g., [6], [47], [66], and [69].

1.1 Notations

Let X be a non-empty set. Let $f: X \rightarrow X$ be a single-valued operator, and let $F: X \multimap X$ be a multi-valued operator. Throughout this work, we will use the following notations:

- $\mathcal{P}(X) := \{A \mid A \subset X\}$;
- $P(X) := \{A \subset X \mid A \neq \emptyset\}$;
- $P_b(X) := \{A \subset X \mid A \neq \emptyset, A \text{ is bounded}\}$;
- $P_{cl}(X) := \{A \subset X \mid A \neq \emptyset, A \text{ is closed}\}$;
- $P_{cp}(X) := \{A \subset X \mid A \neq \emptyset, A \text{ is compact}\}$;
- For $x_0 \in X$ and $r > 0$, we have
 - $B(x_0; r) := \{x \in X \mid d(x_0, x) < r\}$ denotes the open ball with center x_0 and radius r ;
 - $\tilde{B}(x_0; r) := \{x \in X \mid d(x_0, x) \leq r\}$ denotes the closed ball with center x_0 and radius r ;

- $Graph(f) := \{(x, y) \in X \times X \mid y \in f(x)\}$ – the graph of f ;
- $Graph(F) := \{(x, y) \in X \times X \mid y \in F(x)\}$ – the graph of F ;
- $Fix(f) := \{x \in X \mid f(x) = x\}$ – the set of all fixed points of f ;
- $f^0 := 1_X, f^1 := f, \dots, f^n := f \circ f^{n-1}$ – single-valued iterates of the operator f ;
- $F^1(Y) := F(Y), F^2(Y) := F(F(Y)), \dots, F^n(Y) := F(F^{n-1}(Y))$ – iterates of the multi-valued operator F ;
- We will denote $O(x, \infty) = \{x, f(x), \dots, f^n(x), \dots\}$, for every $x \in X$.
- $CR(Y, X)$ – the family of all contraction operators defined from Y to X ;
- $CR_{\delta Y}(Y, X) := \{f \in CR(Y, X) \mid f|_{\delta Y} : \delta Y \rightarrow X \text{ is fixed point free}\}$.

1.2 General notions

Let us now recall some of the main definitions and concepts from fixed point theory.

Definition 1.1. We say a metric space (X, d) is complete if each Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ converges.

Definition 1.2. We say a metric space (X, d) is connected if it is not the union of two disjoint non-empty open sets.

Definition 1.3. Consider (X, d) a metric space. We define the generalized diameter functional $diam: \mathcal{P}(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ as follows:

$$diam(Y) = \sup\{d(u, v) \mid u \in Y, v \in Y\}.$$

Definition 1.4. Consider the metric space (X, d) . Then:

- (i) the functional $D: P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ given by

$$D(Y, Z) = \inf\{d(y, z) \mid y \in Y, z \in Z\}$$

is called the gap functional. In particular, if $x_0 \in X$ then $D(x_0, Z) := D(\{x_0\}, Z)$ is the distance from x_0 to the set Z .

(ii) the functional $\rho: P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined as follows

$$\rho(Y, Z) = \sup\{D(y, Z) \mid y \in Y\}$$

is the excess functional of the set Y over the set Z .

(iii) the functional $H: P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined as

$$H(Y, Z) = \max\{\rho(Y, Z), \rho(Z, Y)\}$$

is called the Hausdorff-Pompeiu functional. It is known that H is a metric on $P_{b,cl}(X)$.

In addition, the pair $(P_{cl}(X), H)$ forms a generalized metric space, which means that H can also take infinite values.

We also recall one important property of the Hausdorff-Pompeiu functional.

Lemma 1.1. *We consider (X, d) a metric space and $\varepsilon > 0$. If Y, Z are two subsets of X with $H(Y, Z) < \varepsilon$ then for every $y \in Y$ there is $z \in Z$ for which we have $d(y, z) < \varepsilon$.*

Definition 1.5 (see [69]). Consider the pair of metric spaces (X, d) and (Y, ρ) . An operator $f: X \rightarrow Y$ is said to be:

(i) Lipschitz if there is an $l \in \mathbb{R}_+$ for which

$$d(f(x), f(y)) \leq ld(x, y), \text{ for every } x, y \in X;$$

(ii) α -contraction if there is an $\alpha \in [0, 1)$ for which f is α -Lipschitz (i.e., satisfies a Lipschitz condition using α).

Definition 1.6 (see [69]). Consider a metric space (X, d) . An operator $f: X \rightarrow X$ is a weakly Picard operator (abbreviated WPO) if $(f^n(x))_{n \in \mathbb{N}}$ is convergent for all $x \in X$ and the limit of this sequence, denoted by $f^\infty(x)$, is a fixed point of f .

Definition 1.7 (see [69]). If $f: X \rightarrow X$ is a WPO and $\text{Fix}(f) = \{x^*\}$, then f is a Picard operator (abbreviated PO).

Definition 1.8 (see [16], page 268). Consider a metric space (X, d) and an operator $f: X \rightarrow X$. Then X is said to be f -orbitally complete if each Cauchy sequence contained in $O(x, \infty)$ for some $x \in X$ converges in X .

By U we refer to a domain (open and connected) fixed in a complete metric space (X, d) . We denote by $\mathcal{K}^{CRR}(\bar{U}, X)$ (respectively $\mathcal{K}^C(\bar{U}, X)$ and $\mathcal{K}^B(\bar{U}, X)$) the set of all multi-valued generalized contractions $K: \bar{U} \rightarrow P(X)$ in the sense of Ćirić-Reich-Rus (CRR), respectively Chatterjea (C) and respectively Berinde (B). We also define

$$\mathcal{K}_0^\#(\bar{U}, X) = \{K \in \mathcal{K}(\bar{U}, X) : x \notin K(x) \text{ for all } x \in \partial U\}, \quad (1.1)$$

where $\#$ is replaced by CRR, C, and respectively B.

Definition 1.9. We say that $K \in \mathcal{K}_0^\#(\bar{U}, X)$ is essential if K has a fixed point. Otherwise, K is called nonessential.

Definition 1.10. A homotopy of multi-valued generalized contractions is a family of multi-valued generalized contractions $\{K_t\}_{t \in [0,1]}$ such that all K_t belong to $\mathcal{K}_0^\#(\bar{U}, X)$. Two operators S and T are said to be homotopic in $\mathcal{K}_0^\#(\bar{U}, X)$ if an homotopy of multi-valued generalized contractions $\{K_t\}_{t \in [0,1]}$ exists with the property $K_0 = S$ and $K_1 = T$.

Lemma 1.2 (Cauchy's Lemma). *Let $(a_n), (b_n)$ be two sequences of positive numbers such that $\sum_{n \geq 0} a_n < \infty$ and $\lim_{n \rightarrow \infty} b_n = 0$. Then*

$$\lim_{n \rightarrow \infty} \left(\sum_{k=0}^n a_{n-k} b_k \right) = 0.$$

Chapter 2

Local fixed point theorems for single-valued generalized contractions

In this chapter, we are focusing on proving local theorems in the single-valued generalized contractions case, as well as an application for each considered generalized contraction. Specifically, we will consider Ćirić-Reich-Rus, Chatterjea, Berinde, and Ćirić generalized type contractions.

These results enrich other original developments from papers such as [8], [13], [16], [18], [27], [59], and [72]. The main articles on which this chapter is based are [40], [74], and [75]. For further research directions, see, e.g., [1], [2], [9], [15], [61], [67], and [68].

2.1 Ćirić-Reich-Rus type generalized contractions

In Section 2.1 of Chapter 2, the studied fixed point equation $f(x) = x$ satisfies contraction conditions of Ćirić-Reich-Rus type. In the first part we recall the definition of such operators, then we prove the local fixed point theorem and we also give an application to it. The main existence and uniqueness results of this section are Theorem 2.3, and Theorem 2.4.

The Ćirić-Reich-Rus single-valued contraction was independently defined by Ljubomir Ćirić, Simeon Reich, and Ioan A. Rus in 1971.

Definition 2.1 (see [18], [59], and [72]). Suppose (X, d) is a metric space. An operator $f: Y \subseteq X \rightarrow X$ is a Ćirić-Reich-Rus type contraction if there exist $\alpha, \beta \in (0, 1)$ such that $\alpha + 2\beta < 1$ and for all $x, y \in X$ we have

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta [d(x, f(x)) + d(y, f(y))]. \quad (2.1)$$

In [72], I. A. Rus gave an example of such an operator. The global fixed point theorem given by Ioan A. Rus also in [72] states the following.

Theorem 2.1 (I. A. Rus, see Theorem 1, [72]). *Consider a complete metric space (X, d) and a mapping $f: X \rightarrow X$. We suppose there are $\alpha, \beta \in \mathbb{R}_+$, for which $\alpha + 2\beta < 1$, satisfying*

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta [d(x, f(x)) + d(y, f(y))]$$

for all $x, y \in X$. Then f admits exactly one fixed point.

The following local theorem was proved by Simeon Reich in [57], where the proof relies on Theorem 2.1.

Theorem 2.2 (Simeon Reich, see [57]). *Consider a complete metric space (X, d) , $x_0 \in X$, and $r > 0$. Let $f: B(x_0; r) \rightarrow X$ be a Ćirić-Reich-Rus type contraction. Then, if*

$$d(x_0, f(x_0)) < \frac{1 - \alpha - 2\beta}{1 + \beta} r, \quad (2.2)$$

we obtain $\text{Fix}(f) = \{x^*\}$ and the iterative sequence generated by successive approximations $(f^n(x))_{n \in \mathbb{N}}$ is convergent to x^* for all x in $B(x_0; r)$. Moreover, the following a priori error estimation is obtained:

$$d(f^n(x), x^*) \leq \frac{q^n}{1 - q} d(x, f(x)),$$

where $q := \frac{\alpha + \beta}{1 - \beta} < 1$, for all $x \in B(x_0; r)$.

We will now give another local fixed point theorem for Ćirić-Reich-Rus type contractions, which complements the global results Theorem 2.5 from [18], Theorem 3 from [59], Theorem 1 from [72], as well as the above recalled local theorem by considering a weaker condition. The proof relies on the method of the iterative sequence generated by successive approximations.

Theorem 2.3. Consider a complete metric space (X, d) , $x_0 \in X$, and $r > 0$. Let $f: B(x_0; r) \rightarrow X$ be a Ćirić-Reich-Rus type contraction. If

$$d(x_0, f(x_0)) < \frac{1 - \alpha - 2\beta}{1 - \beta}r, \quad (2.3)$$

then the iterative sequence generated by successive approximations $(f^n(x_0))_{n \in \mathbb{N}}$ starting from the center of the ball is convergent to the unique fixed point $x^* \in B(x_0; r)$ of the Ćirić-Reich-Rus type contraction f .

Let us now define the notion of (α, β) -contractive family, which we will use later in the application for the local fixed point theorem.

Definition 2.2. Consider a metric space (X, d) and a connected metric space (J, ρ) . We say that the sequence $(K_\lambda)_{\lambda \in J} \subset CR(Y, X)$ is an (α, β) -contractive family if there exist $\alpha \in (0, 1)$, $p \in (0, 1]$ and $M > 0$ such that

- (i) $d(K_\lambda(x_1), K_\lambda(x_2)) \leq \alpha d(x_1, x_2) + \beta [d(x_1, K_\lambda(x_1)) + d(x_2, K_\lambda(x_2))]$, for all $x_1, x_2 \in Y$ and $\lambda \in J$;
- (ii) $d(K_\lambda(x), K_\mu(x)) \leq M [\rho(\lambda, \mu)]^p$, for all $x \in Y$ and $\lambda, \mu \in J$.

As an application of Theorem 2.3 we give a continuation theorem, proved below.

Theorem 2.4. Consider (X, d) a complete metric space and $Y \in P_{cl}(X)$ such that $\text{int}Y$ is nonempty. Let (J, ρ) denote a connected metric space, and let $(K_\lambda)_{\lambda \in J}$ be an (α, β) -contractive family from $CR_{\delta Y}(Y, X)$. The following conclusions occur:

- (i) Assuming there is a point $\lambda_0^* \in J$, for which $K_{\lambda_0^*}(x) = x$ admits a solution, then $K_\lambda(x) = x$ admits a unique solution for any $\lambda \in J$;
- (ii) If $K_\lambda(x_\lambda) = x_\lambda$ for any $\lambda \in J$, then the operator defined as

$$\begin{aligned} j: J &\rightarrow \text{int}Y \\ j(\lambda) &= x_\lambda \end{aligned} \quad (2.4)$$

is continuous.

2.2 Chatterjea type generalized contractions

This section of Chapter 2 focuses on proving existence and uniqueness results for the fixed point equation $f(x) = x$ fulfilling Chatterjea type conditions. Theorem 2.6, Theorem 2.7 and Theorem 2.8 are the main results which improve the results presented in [13].

The definition of Chatterjea single-valued contractions was given in 1972 by Santi Chatterjea, as follows.

Definition 2.3 (see [13]). Let (X, d) be a metric space. An operator $f: Y \subseteq X \rightarrow X$ is a Chatterjea type contraction if there exists $\gamma \in [0, \frac{1}{2})$ such that for all $x, y \in X$ we have

$$d(f(x), f(y)) \leq \gamma [d(x, f(y)) + d(y, f(x))]. \quad (2.5)$$

Billy Rhoades has shown in [64] that the classical case of Banach contractions and Chatterjea contractions have independent conditions.

In [13], Chatterjea gives the global fixed point theorem, as outlined below.

Theorem 2.5 (S. K. Chatterjea, see [13]). *Consider a metric space (X, d) and a Chatterjea type contraction $f: X \rightarrow X$. Then, f admits a unique fixed point in X .*

The next local result enhances Chatterjea's fixed point theorem outlined above, while also relying on it for the proof.

Theorem 2.6. *Consider a complete metric space (X, d) , x_0 a point of X , and $r > 0$. Let $f: B(x_0; r) \rightarrow X$ be a Chatterjea type contraction. If*

$$d(x_0, f(x_0)) < \frac{1 - 2\gamma}{1 + \gamma} r, \quad (2.6)$$

then $\text{Fix}(f) = \{x^\}$ and also the iterative sequence generated by successive approximations $(f^n(x))_{n \in \mathbb{N}}$ is convergent to x^* for all points $x \in B(x_0; r)$. Additionally, the following error estimation occurs:*

$$d(f^n(x), x^*) \leq \frac{q^n}{1 - q} d(x, f(x)),$$

where $q := \frac{\gamma}{1 - \gamma} < 1$, for all $x \in B(x_0; r)$.

Furthermore, we improve Theorem 2.6 by assuming a weaker condition, as follows. As before, we utilize the method of constructing an iterative sequence of successive approximations for the proof.

Theorem 2.7. *Assume that (X, d) is a complete metric space, $x_0 \in X$, and $r > 0$. Let $f: B(x_0; r) \rightarrow X$ be a Chatterjea type contraction. If*

$$d(x_0, f(x_0)) < \frac{1 - 2\gamma}{1 - \gamma}r,$$

then the iterative sequence generated by successive approximations $(f^n(x_0))_{n \in \mathbb{N}}$ starting from the center of the ball is convergent to the unique fixed point $x^ \in B(x_0; r)$ of the Chatterjea type contraction f .*

We will now provide the definition of a γ -contractive family in order to set the context for our next result.

Definition 2.4. Let (X, d) be a metric space and (J, ρ) be a connected metric space. The sequence $(K_\lambda)_{\lambda \in J} \subset CR(Y, X)$ is a γ -contractive family if there exist $\gamma \in \left[0, \frac{1}{2}\right)$, $M > 0$ and $p \in (0, 1]$ such that

- (i) $d(K_\lambda(x_1), K_\lambda(x_2)) \leq \gamma [d(x_1, K_\lambda(x_2)) + d(x_2, K_\lambda(x_1))]$, for every $x_1, x_2 \in Y$, $\lambda \in J$;
- (ii) $d(K_\lambda(x), K_\mu(x)) \leq M [\rho(\lambda, \mu)]^p$, for all $x \in Y$ and $\lambda, \mu \in J$.

The last result of this section is the continuation theorem for Chatterjea type contractions.

Theorem 2.8. *Consider (X, d) a complete metric space and $Y \in P_{cl}(X)$ for which $\text{int}Y$ is nonempty. Let (J, ρ) denote a connected metric space, and let $(K_\lambda)_{\lambda \in J}$ be a γ -contractive family from $CR_{\delta Y}(Y, X)$. The following conclusions occur:*

- (i) *Assuming there is a point $\lambda_0^* \in J$, for which $K_{\lambda_0^*}(x) = x$ admits a solution, then $K_\lambda(x) = x$ admits a unique solution for any $\lambda \in J$;*
- (ii) *If $K_\lambda(x_\lambda) = x_\lambda$, for any $\lambda \in J$, then the operator defined as*

$$\begin{aligned} j: J &\rightarrow \text{int}Y \\ j(\lambda) &= x_\lambda \end{aligned} \tag{2.7}$$

is continuous.

2.3 Berinde type generalized contractions

The operator we study in this section was defined in 2004 in the article [8]. We will prove an existence result for the fixed point equation $f(x) = x$ with respect to a Berinde type contraction condition. Theorem 2.10, Theorem 2.11, and Theorem 2.12 are the main results comprised in this section, which further enrich developments from [8], [10], and [11].

Let us recall the definition of a Berinde type contraction.

Definition 2.5 (see [8]). Assume that (X, d) is a metric space. An operator $f: Y \subseteq X \rightarrow X$ is a Berinde type contraction if there exist $\alpha \in (0, 1)$ and $L \geq 0$ such that for all $x, y \in X$ we have

$$d(f(x), f(y)) \leq \alpha d(x, y) + Ld(x, f(y)). \quad (2.8)$$

An example of such an operator was further given by V. Berinde in [8]. In the same article, the following fixed point theorem for Berinde type single-valued contractions is given.

Theorem 2.9 (V. Berinde, see Theorem 1, [8]). *Consider a complete metric space (X, d) , and a Berinde contraction $f: X \rightarrow X$. Then*

- (1) $\text{Fix}(f) \neq \emptyset$;
- (2) For all $x_0 \in X$, the iterative sequence generated by successive approximations $(x_n)_{n \in \mathbb{N}}$ converges to some $x^* \in \text{Fix}(f)$;
- (3) The following estimates

$$d(x_n, x^*) \leq \frac{\alpha^n}{1 - \alpha} d(x_0, x_1), n = 0, 1, 2, \dots \quad (2.9)$$

$$d(x_n, x^*) \leq \frac{\alpha}{1 - \alpha} d(x_{n-1}, x_n), n = 0, 1, 2, \dots \quad (2.10)$$

hold.

We will now prove the first local theorem for single-valued Berinde type contractions. It calls upon the global fixed point theorem recalled above.

Theorem 2.10. *Assume that (X, d) is a complete metric space, x_0 in X , and $r > 0$. Let $f: B(x_0; r) \rightarrow X$ be a Berinde type contraction, and additionally we suppose that*

$$d(x_0, f(x_0)) < (1 - \alpha - L)r, \quad (2.11)$$

where $\alpha \in (0, 1)$, $L > 0$ such that $\alpha + L < 1$.

Then, the fixed point equation related to the Berinde type contraction f has a unique solution x^* , and the iterative sequence generated by successive approximations $(f^n(x))_{n \in \mathbb{N}}$ is convergent to a point x^* for every x in $B(x_0; r)$. Furthermore, the following a priori estimation takes place:

$$d(f^n(x), x^*) \leq \frac{\alpha^n}{1 - \alpha} d(x, f(x)),$$

for all $x \in B(x_0; r)$.

We now prove another local fixed point theorem for Berinde type contractions. This result enhances Theorem 2.9 from [8] without relying on it, and we also consider a weaker condition. Following the previous approach, the proof is established using the iterative sequence of successive approximations method.

Theorem 2.11. *Assume that (X, d) is a complete metric space, $x_0 \in X$, and $r > 0$. Let $f: B(x_0; r) \rightarrow X$ be a Berinde type contraction. If*

$$d(x_0, f(x_0)) < (1 - \alpha)r,$$

then the iterative sequence generated by successive approximations $(f^n(x_0))_{n \in \mathbb{N}}$ starting from the center of the ball is convergent to $x^ \in B(x_0; r)$ which is a fixed point of the Berinde type contraction f .*

Next, we present the definition of an (α, L) -contractive family.

Definition 2.6. Consider a metric space (X, d) and a connected metric space (J, ρ) . We define the (α, L) -contractive family as a sequence $(K_\lambda)_{\lambda \in J}$ included in $CR(Y, X)$ such that there exist $\alpha \in (0, 1)$, $M > 0$ and $p \in (0, 1]$ with

- (i) $d(K_\lambda(x_1), K_\lambda(x_2)) \leq \alpha d(x_1, x_2) + Ld(x_1, K_\lambda(x_2))$, for all $x_1, x_2 \in Y$, $\lambda \in J$;
- (ii) $d(K_\lambda(x), K_\mu(x)) \leq M [\rho(\lambda, \mu)]^p$, for all $x \in Y$ and $\lambda, \mu \in J$.

To conclude the section, we present a continuation theorem which serves as an application to the above local result.

Theorem 2.12. *Consider a complete metric space (X, d) and $Y \in P_{cl}(X)$ for which $\text{int}Y$ is nonempty. Consider (J, ρ) a connected metric space, and $(K_\lambda)_{\lambda \in J}$ an (α, L) -contractive family from $CR_{\delta Y}(Y, X)$. The following conclusions occur:*

- (i) Assuming there is a point $\lambda_0^* \in J$, for which $K_{\lambda_0^*}(x) = x$ admits a solution, then $K_\lambda(x) = x$ admits a solution which is unique for any $\lambda \in J$;
- (ii) If $K_\lambda(x_\lambda) = x_\lambda$, for any $\lambda \in J$ and also $\alpha + L < 1$, where $\alpha \in (0, 1)$ and $L > 0$, then the operator defined as

$$\begin{aligned} j: J &\rightarrow \text{int}Y \\ j(\lambda) &= x_\lambda \end{aligned} \tag{2.12}$$

is single-valued and continuous.

2.4 Ćirić type generalized contractions

The fourth section of Chapter 2 studies the fixed point equation $f(x) = x$ where f is a Ćirić type contraction. This type of generalization was first developed in 1974 within [16]. Theorem 2.14, and Theorem 2.15 are the main results of this section. For qualitative properties of such an operator, see [38].

We first recall the definition of a Ćirić type contraction.

Definition 2.7 ([16, Ćirić Definition 1]). Let (X, d) be a metric space. Then, $f: Y \subseteq X \rightarrow X$ is a single-valued Ćirić type contraction with constant q if there exists a number $q \in (0, 1)$, such that for all $x, y \in Y$ we have

$$d(f(x), f(y)) \leq q \cdot \max \{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}.$$

In [16], Lj. B. Ćirić additionally provides an instance of such a contraction. In the same article [16], L. B. Ćirić proved the following theorem.

Theorem 2.13 (Lj. B. Ćirić, see Theorem 1, [16]). *Let f be a Ćirić type contraction on a metric space X and let X be f -orbitally complete. Then*

- (a) f has a unique fixed point $x^* \in X$;
- (b) $\lim_{n \rightarrow \infty} f^n(x) = x^*$;
- (c) $d(f^n(x), x^*) \leq \frac{q^n}{1 - q} d(x, f(x))$, for every $x \in X$.

Let us now proceed with the local fixed point theorem for Ćirić type contractions, improving Theorem 1 from [16].

Theorem 2.14. *Assume that (X, d) is a complete metric space, $x_0 \in X$, and $r > 0$. We consider $f: B(x_0; r) \rightarrow X$ a single-valued Ćirić type contraction with constant $q \in (0, \frac{1}{2})$. We also suppose that*

$$d(x_0, f(x_0)) < \frac{1 - 2q}{1 - q}r.$$

Then the operator f has a unique fixed point $x^ \in B(x_0; r)$, $f^n(x_0) \in B(x_0; r)$, for all $n \in \mathbb{N}$ and the iterative sequence generated by successive approximations $(f^n(x_0))_{n \in \mathbb{N}}$ starting from x_0 converges to x^* as $n \rightarrow \infty$.*

Now, we focus on setting the background for an application to the previous result and give the definition below.

Definition 2.8. Let (X, d) be a metric space and (J, ρ) be a connected metric space. We say that $\{f_\lambda: \lambda \in J\} \subset CR(Y, X)$ is a family of single-valued Ćirić type contractions with constant $q \in (0, 1)$ if the conditions that follow are fulfilled: there exist $p \in (0, 1]$ and $M > 0$ such that

(i) for all $x_1, x_2 \in Y$ and $\lambda \in J$, we have

$$d(f_\lambda(x_1), f_\lambda(x_2)) \leq q \max \{d(x_1, x_2), d(x_1, f_\lambda(x_1)), d(x_2, f_\lambda(x_2)), d(x_1, f_\lambda(x_2)), d(x_2, f_\lambda(x_1))\};$$

(ii) for all $x \in Y$ and $\lambda, \mu \in J$, we have

$$d(f_\lambda(x), f_\mu(x)) \leq M [\rho(\lambda, \mu)]^p.$$

We can now state the continuation theorem for Ćirić type contractions, as follows.

Theorem 2.15. *Consider a complete metric space (X, d) and $Y \in P_{cl}(X)$ with a nonempty interior. Let (J, ρ) be a connected metric space and $\{f_\lambda: \lambda \in J\}$ be a family of single-valued Ćirić type contractions with constant $q \in (0, \frac{1}{2})$ from $CR_{\partial Y}(Y, X)$. Then the following conclusions occur:*

(i) *If $\lambda_0^* \in J$ exists such that the fixed point equation $f_{\lambda_0^*}(x) = x$ admits a solution, then $f_\lambda(x) = x$ has a unique solution for any $\lambda \in J$;*

(ii) If $f_\lambda(x_\lambda) = x_\lambda$, for any $\lambda \in J$, then the operator

$$\begin{aligned} j: J &\rightarrow \text{int}Y \\ j(\lambda) &= x_\lambda \end{aligned}$$

is continuous.

2.5 Fixed point theorems for other types of non-self generalized contractions

In the last section we will establish the existence of a fixed point using some ideas from [19], [50], [51], and [60]. The main results from this section, Theorem 2.17, Theorem 2.18, and Theorem 2.19, extend known results from the recent literature.

We firstly introduce the following result proved by Petruşel, Precup, and Şerban in [50] which extends Conti's fixed point theorem from [19].

Theorem 2.16. *Let (X, d) be a metric space, $Y \subset X$ be a closed set and $f: Y \rightarrow X$ be a continuous mapping. Assume that there exists a sequence $(x_n)_{n \geq 1}$ of elements from Y such that:*

(i) *the set $\{f(x_n) : n \geq 1\}$ is relatively compact;*

(ii) *$d(f(x_n), x_n) \rightarrow 0$ as $n \rightarrow \infty$.*

Then $\text{Fix}(f) \neq \emptyset$, and every limit point of the sequence $(x_n)_{n \geq 1}$ belongs to $\text{Fix}(f)$.

We establish a new fixed point result for non-self Berinde type contractions, in the following manner.

Theorem 2.17. *Assume that (X, d) is a complete metric space, $Y \subset X$ is a closed subset and $f: Y \rightarrow X$ is a continuous Berinde type contraction. If $\alpha + L < 1$ and there exists a sequence $(x_n)_{n \geq 1}$ of elements from Y such that*

$$d(f(x_n), x_n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then f has at least one fixed point in Y and each limit point of the sequence $(x_n)_{n \geq 1}$ is a fixed point of f . Moreover, we also have

$$d(x_n, x^*) \leq \frac{1}{1 - \alpha - L} d(f(x_n), x_n).$$

We will now give a fixed point theorem for non-self Hardy-Rogers type contractions.

Theorem 2.18. *Consider a metric space (X, d) which is complete, $Y \subset X$ is a closed subset and $f: Y \rightarrow X$ is a continuous Hardy-Rogers type generalized contraction, i.e., there exist $\alpha, \beta, \gamma \geq 0$ such that, for all $x, y \in Y$, we have*

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta [d(x, f(x)) + d(y, f(y))] \\ + \gamma [d(x, f(y)) + d(y, f(x))].$$

We suppose $\alpha + 2\gamma < 1$ and the existence of a sequence $(x_n)_{n \geq 1}$ of elements of Y such that

$$d(f(x_n), x_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Then f has at least one fixed point in Y and each limit point of the sequence $(x_n)_{n \geq 1}$ is a fixed point of f . Moreover, we also have

$$d(x_n, x^*) \leq \frac{1 + \beta + \gamma}{1 - \alpha - 2\gamma} d(f(x_n), x^*).$$

The concept of maximal displacement functional corresponding to a mapping $f: Y \rightarrow X$ was given in [50]. It refers to the functional denoted by $E_f: P(Y) \rightarrow \mathbb{R}_+ \cup \{\infty\}$ and defined by

$$E_f(A) := \sup \{d(x, f(x)) \mid x \in A\}. \quad (2.13)$$

Another fixed point theorem considering the case of non-self Berinde type contractions is given below.

Theorem 2.19. *Assume that (X, d) is a complete metric space, $Y \subset X$ is a nonempty closed set and $f: Y \rightarrow X$ is a continuous mapping. Suppose that the following conditions are satisfied:*

- (i) f is a Berinde type contraction with $\alpha + 2L < 1$;
- (ii) there exists a bounded sequence $(y_n)_{n \geq 1}$ in Y such that $f^n(y_n)$ is defined for every $n \geq 1$;
- (iii) $E_f(Y) < \infty$.

Then:

(a) f has at least one fixed point $x^* \in Y$;

(b) $f^{n-1}(y_n) \rightarrow x^*$ and $f^n(y_n) \rightarrow x^*$ as $n \rightarrow \infty$;

(c) $d(f^{n-1}(y_n), x^*) \leq \frac{1}{1 - \alpha - L} \left(\frac{\alpha + L}{1 - L} \right)^{n-1} d(f(y_n), y_n)$.

Chapter 3

Local fixed point theorems for multi-valued generalized contractions

The third chapter's focal point is to present and prove some fixed point results, such as strict fixed point, local theorems combined in some cases with an application, and some stability properties. These results take place in the context of multi-valued generalized contractions, particularly Ćirić-Reich-Rus, Chatterjea, Berinde, Ćirić, multi-valued nonlinear graph φ -contractions, and multi-valued (φ, ψ) -contractions of Feng-Liu type.

The proven results augment well-known theorems from the literature, e.g., [3], [4], [14], [16], [17], [21–23], [27], [29–31], [33–35], [37], [45], [51], [70], and [71]. Other extensions and generalizations can be found in [5], [12], [24], [26], [42], [55], [70], and [71].

3.1 Ćirić-Reich-Rus type generalized contractions

In the first section of Chapter 3 we will address operators that satisfy the Ćirić-Reich-Rus condition. We will first recall the definition of such operators and the global fixed point result. The main results comprised in this section are Theorem 3.2, Theorem 3.3, and Theorem 3.4, which can also be found in [54].

The definition of the well-known Ćirić-Reich-Rus type generalized contraction is given below.

Definition 3.1. Let (X, d) be a metric space. An operator $F: Y \subseteq X \rightarrow P_{cl}(X)$ is said to be a multi-valued Ćirić-Reich-Rus (α, β) -contraction if there exist $\alpha, \beta \in (0, 1)$ such that $\alpha + 2\beta < 1$ and for all $x, y \in X$, we have

$$H(F(x), F(y)) \leq \alpha d(x, y) + \beta [D(x, F(x)) + D(y, F(y))]. \quad (3.1)$$

The global fixed point result we will expand was proved by Simeon Reich in [57] and states the following.

Theorem 3.1. *Assume that (X, d) is a complete metric space, and $F: X \rightarrow P_{cl}(X)$ is a Ćirić-Reich-Rus type multi-valued operator. Then $Fix(F) \neq \emptyset$.*

By considering the operator's domain as a subset of the complete metric space, we obtained the theorem described herein.

Theorem 3.2. *Assume (X, d) is a complete metric space, $x_0 \in X$, and $r > 0$. Let $F: B(x_0; r) \rightarrow P_{cl}(X)$ be a multi-valued Ćirić-Reich-Rus (α, β) -contraction such that*

$$D(x_0, F(x_0)) < \frac{1 - \alpha - 2\beta}{1 - \beta} r. \quad (3.2)$$

Then, there exists a Picard type iterative sequence $\{x_n\}_{n \in \mathbb{N}} \subset B(x_0; r)$ which converges to a fixed point of F .

Another important result that was studied is the strict fixed point theorem for Ćirić-Reich-Rus operators.

Theorem 3.3. *Assume that (X, d) is a complete metric space, $x_0 \in X$, and $r > 0$. Let $F: B(x_0; r) \rightarrow P_{cl}(X)$ be a multi-valued Ćirić-Reich-Rus (α, β) -contraction. Suppose that $SFix(F) \neq \emptyset$. Then, $Fix(F) = SFix(F) = \{x^*\}$. If, additionally, $D(x_0, F(x_0)) < \frac{1 - \alpha - 2\beta}{1 - \beta} r$, then there exists a Picard type iterative sequence $\{x_n\}_{n \in \mathbb{N}} \subset B(x_0; r)$ which converges to x^* .*

Let us now give an application to Theorem 3.2. Specifically, we will prove an homotopy result, as Frigon and Granas did in [23]. We begin this part with the following definition of a family of multi-valued generalized (α, β) -contractions of Ćirić-Reich-Rus type.

Definition 3.2. Let (X, d) be a metric space. Then, $(K_t)_{t \in [0,1]}$ is a family of multi-valued generalized (α, β) -contractions of Ćirić-Reich-Rus type if

- (i) $H(K_t(x_1), K_t(x_2)) \leq \alpha d(x_1, x_2) + \beta [D(x_1, K_t(x_1)) + D(x_2, K_t(x_2))]$, for all $x_1, x_2 \in X, t \in [0, 1]$.
- (ii) $H(K_t(x), K_s(x)) \leq |\phi(t) - \phi(s)|$ for all $t, s \in [0, 1]$ and $x \in X$,

where $\phi: [0, 1] \rightarrow \mathbb{R}$ is strictly increasing and continuous.

The homotopy result for multi-valued Ćirić-Reich-Rus type contractions is given below.

Theorem 3.4. *Assume that (X, d) is a complete metric space, and S and T are homotopic in $\mathcal{K}_0^{CRR}(\bar{U}, X)$. If S is essential, then T is essential as well.*

Remark 3.5. If we consider $\beta = 0$ in Theorem 3.4, then we have the topological transversality theorem proved by Frigon and Granas (see Theorem 4.3 in [23]).

3.2 Chatterjea type generalized contractions

The second section handles another type of multi-valued contraction generalization, particularly the Chatterjea type. We will call to mind the definition for this specific generalization and the well-known global fixed point theorem. We will then proceed to state and prove a local fixed point theorem, a strict fixed point result, and an homotopy result serving as an application to the local theorem.

The three proven developments constitute the main results of the current section, explicitly Theorem 3.7, Theorem 3.8, and Theorem 3.9.

Definition 3.3. Let (X, d) be a metric space. An operator $F: Y \subseteq X \rightarrow P_{cl}(X)$ is a multi-valued Chatterjea γ -contraction if there exists $\gamma \in [0, \frac{1}{2})$ such that for all $x, y \in X$, we have

$$H(F(x), F(y)) \leq \gamma [D(x, F(y)) + D(y, F(x))]. \quad (3.3)$$

The result that follows is the global fixed point theorem for the above contraction type, proved by Santi Chatterjea.

Theorem 3.6. *Assume that (X, d) is a complete metric space and $F: X \rightarrow P_{cl}(X)$ is a multi-valued Chatterjea type operator. Then $Fix(F) \neq \emptyset$.*

We continue with proving the local fixed point result for multi-valued Chatterjea type generalized contractions.

Theorem 3.7. *Assume that (X, d) is a complete metric space, $F: B(x_0; r) \rightarrow P_{cl}(X)$ is a multi-valued Chatterjea γ -contraction, and*

$$D(x_0, F(x_0)) < \frac{1 - 2\gamma}{1 - \gamma}r. \quad (3.4)$$

Then there exists a Picard type iterative sequence $\{x_n\}_{n \in \mathbb{N}} \subset B(x_0; r)$ which converges to a fixed point of F .

The following theorem is a strict fixed point result concerning multi-valued Chatterjea type contractions.

Theorem 3.8. *Assume that (X, d) is a complete metric space, $x_0 \in X$, and $r > 0$. Let $F: B(x_0; r) \rightarrow P_{cl}(X)$ be a multi-valued Chatterjea γ -contraction. Suppose that $SFix(F) \neq \emptyset$. Then, $Fix(F) = SFix(F) = \{x^*\}$. If, additionally, $D(x_0, F(x_0)) < \frac{1-2\gamma}{1-\gamma}r$, then there exists a Picard type iterative sequence $\{x_n\}_{n \in \mathbb{N}} \subset B(x_0; r)$ which converges to x^* .*

The next result serves as an application to Theorem 3.7, based on Frigon and Granas' theorem from [23]. We will begin by defining the notion of family of multi-valued generalized (γ)-contractions of Chatterjea type.

Definition 3.4. Consider a metric space (X, d) . Then, $(K_\lambda)_{\lambda \in [0,1]}$ is a family of multi-valued generalized (γ)-contractions of Chatterjea type if

- (i) $H(K_t(x_1), K_t(x_2)) \leq \gamma [D(x_1, K_t(x_2)) + D(x_2, K_t(x_1))]$, for all $x_1, x_2 \in X$ and $t \in [0, 1]$.
- (ii) $H(K_t(x), K_s(x)) \leq |\phi(t) - \phi(s)|$, for all $t, s \in [0, 1]$ and $x \in X$,

where the function $\phi: [0, 1] \rightarrow \mathbb{R}$ is continuous and strictly increasing.

We will now give the homotopy result for multi-valued Chatterjea type contractions.

Theorem 3.9. *Assume that (X, d) is a complete metric space, and that S and T are homotopic in $\mathcal{K}_0^C(\bar{U}, X)$. If S is essential, then T is essential as well.*

3.3 Berinde type generalized contractions

In the third section, we will approach another type of multi-valued contraction, precisely the Berinde type contraction, initially referred to as multi-valued (α, L) -weak contraction. The main original results are Theorem 3.11, Theorem 3.12, Theorem 3.14, and Theorem 3.15.

Berinde and Berinde gave the definition for the studied multi-valued generalized contraction in [7], recalled below.

Definition 3.5. Let (X, d) be a metric space. An operator $F: Y \subseteq X \rightarrow P_{cl}(X)$ is a multi-valued Berinde (α, L) -contraction if there exist $\alpha \in (0, 1)$ and $L \geq 0$ such that for all $x, y \in X$, we have

$$H(F(x), F(y)) \leq \alpha d(x, y) + LD(x, F(y)). \quad (3.5)$$

The next result proving the existence of a fixed point for multi-valued Berinde type contractions in a general context was also introduced in [7].

Theorem 3.10. Consider a complete metric space (X, d) and a multi-valued Berinde type contraction $F: X \rightarrow P_{cl,b}(X)$. Then $Fix(F) \neq \emptyset$.

We will now extend the previous existence theorem by considering a weaker, local condition.

Theorem 3.11. Assume (X, d) is a complete metric space, and the operator $F: B(x_0; r) \rightarrow P_{cl}(X)$ is a multi-valued Berinde (α, L) -contraction. Suppose that

$$D(x_0, F(x_0)) < (1 - \alpha)r. \quad (3.6)$$

Then, there exists a Picard type iterative sequence $\{x_n\}_{n \in \mathbb{N}} \subset B(x_0; r)$ converging to a fixed point of F .

For setting the context of our next result, we will firstly define the concept of family of (α, L) -contractions of Berinde type.

Definition 3.6. Consider (X, d) is a metric space. Then, $(K_\lambda)_{\lambda \in [0,1]}$ is a family of multi-valued generalized (α, L) -contractions of Berinde type if

- (i) $H(K_t(x_1), K_t(x_2)) \leq \alpha d(x_1, x_2) + LD(x_1, K_t(x_2))$ for all $x_1, x_2 \in X$ and $t \in [0, 1]$.
- (ii) $H(K_t(x), K_s(x)) \leq |\phi(t) - \phi(s)|$, for all $t, s \in [0, 1]$ and $x \in X$,

where $\phi: [0, 1] \rightarrow \mathbb{R}$ is a strictly increasing and continuous function.

Let us now present the homotopy result for multi-valued contractions of Berinde type.

Theorem 3.12. *Assume that (X, d) is a complete metric space, and that S and T are homotopic in $\mathcal{K}_0^B(\bar{U}, X)$. If S is essential, then T is essential as well.*

Remark 3.13. If we consider $L = 0$ in Theorem 3.12, then we have the topological transversality theorem proved by Frigon and Granas in [23].

By using Theorem 3.12, a nonlinear alternative for multi-valued Berinde type contractions follows.

Theorem 3.14. *Assume that $(E, \|\cdot\|)$ is a Banach space, and the operator $F: \tilde{B}(0; r) \rightarrow P_{cl}(E)$ is a multi-valued Berinde (α, L) -contraction. Then, at least one of the following assertions holds:*

1. *there exists $x^* \in \tilde{B}(0; r)$ such that $x^* \in \text{Fix}(F)$;*
2. *there exists $u \in E$ with $\|u\| = r$ and $\lambda \in (0, 1)$ such that $u \in \lambda F(u)$.*

We now prove the following Leray-Schauder principle as a consequence of the previous result.

Theorem 3.15. *Let $(E, \|\cdot\|)$ be a Banach space and let $F: E \rightarrow P_{cl}(E)$ be a multi-valued operator. Suppose that for every $r > 0$ the restriction of F to $\tilde{B}(0; r) \subset E$ is a multi-valued Berinde (α, L) -contraction. Then, at least one of the following assertions holds:*

1. *$\text{diam}(\mathcal{F}_F) = \infty$, where $\mathcal{F}_F := \{x \in E \mid x \in \lambda F(x) \text{ for some } \lambda \in [0, 1]\}$;*
2. *$\text{Fix}(F) \neq \emptyset$.*

3.4 Ćirić type generalized contractions

The forthcoming section addresses results for multi-valued Ćirić type contractions, introduced by Ljubomir B. Ćirić in [16]. We commence by recalling the definition from the said article of a multi-valued Ćirić type contractions, as well as the global fixed point theorem which was proved by Mădălina Moga

in [40]. The central results comprised in this section are Theorem 3.16, Theorem 3.17, and Theorem 3.19.

Let us initiate the section by remembering the definition of the studied operator type, given by Ćirić in [16].

Definition 3.7 (see [16]). Consider a metric space (X, d) . The set-valued map $F: Y \subseteq X \rightarrow P_{b,d}(X)$ is said to be a multi-valued Ćirić type operator with constant k (named a k -set-valued quasi-contraction in [4]) if

$$H(F(x), F(y)) \leq k \max \{ d(x, y), D(x, F(x)), D(y, F(y)), \\ D(x, F(y)), D(y, F(x)) \},$$

for any $x, y \in X$, where $0 \leq k < 1$.

The global fixed point for multi-valued Ćirić type operators was proved by Mădălina Moga in [40]. The succeeding theorem proves the existence of a fixed point using a weaker, local condition.

Theorem 3.16. *Assume that (X, d) is a complete metric space, $x_0 \in X$, and $r > 0$. We consider the multi-valued operator $F: \tilde{B}(x_0; r) \rightarrow P_d(X)$ such that there exists $k \in \left(0, \frac{1}{2}\right)$ with*

$$H(F(x), F(y)) \leq k \max \{ d(x, y), D(x, F(x)), D(y, F(y)), \\ D(x, F(y)), D(y, F(x)) \}, \text{ for all } x, y \in \tilde{B}(x_0; r).$$

We also suppose that

$$D(x_0, F(x_0)) < \frac{1 - 2k}{1 - k} r.$$

Then there exists a Picard type iterative sequence $(x_n)_{n \in \mathbb{N}}$ starting from x_0 converging to a fixed point of F .

By extending the previously proved result, we obtain the following theorem.

Theorem 3.17. *Assume that (X, d) is a complete metric space, $x_0 \in X$ and $r > 0$. We consider the multi-valued operator $F: B(x_0; r) \rightarrow P_d(X)$ such that there exists $k \in \left(0, \frac{1}{2}\right)$ with*

$$H(F(x), F(y)) \leq k \max \{ d(x, y), D(x, F(x)), D(y, F(y)), D(x, F(y)), \\ D(y, F(x)) \}, \text{ for all } x, y \in B(x_0; r).$$

We also suppose that

$$D(x_0, F(x_0)) < \frac{1-2k}{1-k}r.$$

Then, there exists a Picard type iterative sequence $(x_n)_{n \in \mathbb{N}}$ starting from x_0 which converges to a fixed point of F .

Remark 3.18. It is an open question to obtain, by the above approach, a local fixed point theorem and related stability results for a multi-valued Ćirić type operators with constant $k \in (0, 1)$. For a different approach and a general existence result, see [28].

Let us now present the notion of family of multi-valued Ćirić type contractions.

Definition 3.8. Consider a metric space (X, d) . Then, the family $(K_t)_{t \in [0,1]}$ (where $K_t: Y \subseteq X \rightarrow P(X)$, for each $t \in [0, 1]$) is a family of multi-valued Ćirić type operators with constant k if $k \in (0, 1)$ and the following conditions are satisfied:

(i) for all $x_1, x_2 \in Y$ and $t \in [0, 1]$, we have

$$H(K_t(x_1), K_t(x_2)) \leq k \max \{d(x_1, x_2), D(x_1, K_t(x_1)), D(x_2, K_t(x_2)), D(x_1, K_t(x_2)), D(x_2, K_t(x_1))\};$$

(ii) for all $t, s \in [0, 1]$ and $x \in Y$, we have

$$H(K_t(x), K_s(x)) \leq |\phi(t) - \phi(s)|,$$

where $\phi: [0, 1] \rightarrow \mathbb{R}$ is strictly increasing and continuous.

As an application of the previously proved local results for multi-valued Ćirić type contractions, we employ to establish an homotopy principle for multi-valued Ćirić type contractions. It represents a generalization of the well-known result of Frigon and Granas in [23].

Theorem 3.19. Assume that (X, d) is a complete metric space, $U \subset X$ is an open set, and $K: [0, 1] \times \bar{U} \rightarrow P_{cl}(X)$ is a multi-valued operator with closed graph. We denote $K_t := F(t, \cdot)$, for $t \in [0, 1]$. We suppose:

(i) $(K_t)_{t \in [0,1]}$ is a family of multi-valued Ćirić type operators with a constant $k \in (0, \frac{1}{2})$;

(ii) $x \notin K_t(x)$, for all $(t, x) \in [0, 1] \times \partial U$.

Then K_0 has a fixed point if and only if K_1 has a fixed point.

3.5 Multi-valued nonlinear graph φ -contractions

In the fifth section, we will operate with another generalization of Nadler contractions, the multi-valued nonlinear graph φ -contraction. We will demonstrate several new existence results for operators fulfilling such a nonlinear condition, extending results from [46]. The central results covered over the course of this section are Theorem 3.21, Theorem 3.23, and Theorem 3.24.

In what follows (X, d) is a generalized metric space (in the sense that the metric d , which satisfies all the axioms of a metric, can also take the value $+\infty$). It is known that H is a generalized metric (in the sense that $H(A, B) \in \mathbb{R}_+ \cup +\infty$) on $P_{cl}(X)$.

Definition 3.9. A function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a comparison function if it satisfies:

- (i) φ is increasing;
- (ii) $(\varphi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \rightarrow \infty$, for all $t \in \mathbb{R}_+$.

If the condition (ii) is replaced by the condition:

- (iii) for any $t > 0$,

$$\sum_{k=0}^{\infty} \varphi^k(t) < \infty, \quad (3.7)$$

then φ is called a strong comparison function.

Let us now recall two important properties of the comparison and strong comparison functions, respectively.

Lemma 3.1 (see [69]). *If $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$, for any $t > 0$, $\varphi(0) = 0$ and φ is continuous at 0.*

Lemma 3.2 (see [69]). *If $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a strong comparison function, then the following hold:*

- (i) φ is a comparison function;
- (ii) the function $\zeta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, defined by

$$\zeta(t) = \sum_{k=0}^{\infty} \varphi^k(t),$$

is increasing and continuous at 0.

Magdaş has given in [36] some examples of functionals and their properties. We proceed to call to mind the definition of the considered multi-valued operator.

Definition 3.10. Let (X, d) be a generalized metric space. Then, an operator $F: Y \subseteq X \rightarrow P(X)$ is called a multi-valued nonlinear graph φ -contraction if there exists a strong comparison function $\varphi: [0, \infty) \rightarrow [0, \infty)$ such that

$$H(F(x), F(y)) \leq \varphi(d(x, y)), \text{ for every } (x, y) \in \text{Graph}(F). \quad (3.8)$$

Let us now consider the following two definitions.

Definition 3.11 (see [70], [71]). Assume that (X, d) is a metric space. Then the operator $F: X \rightarrow P(X)$ is said to be a multi-valued weakly Picard operator if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that

- (i) $x_0 = x, x_1 = y$;
- (ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;
- (iii) the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of F .

Definition 3.12 (see Petruşel, [44]). Let (X, d) be a metric space, and $F: X \rightarrow P(X)$ be a multi-valued weakly Picard operator. Then, we define the multi-valued operator $F^\infty: \text{Graph}(F) \rightarrow P(\text{Fix}(F))$ by the formula $F^\infty(x, y) = \{z \in \text{Fix}(F) \mid \text{there exists a Picard type iterative sequence of } F \text{ starting from } (x, y) \text{ that converges to } z\}$.

Another concept worth reminiscing is given further down.

Definition 3.13 (see Petruşel, [44]). Let (X, d) be a metric space and $F: X \rightarrow P(X)$ an multi-valued weakly Picard operator. Then, F is a φ -multi-valued weakly Picard operator if $\varphi: [0, \infty) \rightarrow [0, \infty)$ is increasing, continuous in 0 with $\varphi(0) = 0$ and there exists a selection f^∞ of F^∞ such that

$$d(x, f^\infty(x, y)) \leq \varphi(d(x, y)), \text{ for all } (x, y) \in \text{Graph}(F).$$

Remark 3.20. Notice that if $F: X \rightarrow P_{cl}(X)$ is an α -contraction, then F is a φ -MWP operator with $\varphi(t) = \frac{1}{1-\alpha}t$.

We will now visit an auxiliary lemma, which will demonstrate useful in proving our main results.

Lemma 3.3 (see S. B. Nadler Jr., [41]). *Let (X, d) be a generalized metric space, $F: X \rightarrow P(X)$ be a multi-valued operator and $q > 1$. Then, for every $x_0, x_1 \in X$ and $z_1 \in F(x_0)$ there exists $z_2 \in F(x_1)$ such that $d(z_1, z_2) \leq qH(F(x_0), F(x_1))$.*

A generalization of Wegrzyk's results from [77], as well as a particular case considering classical metric space can be found in [39]. The proofs belong to Mădălina Moga. In the same article, some qualitative results in regards of the fixed point set have been proved. Let us now recall the definition of Ulam-Hyers stability for a fixed point inclusion (see, e.g., [43]).

Definition 3.14. Let (X, d) be a metric space and $F: X \rightarrow P(X)$ be a multi-valued operator. Then, the fixed point problem

$$x \in F(x), x \in X, \quad (3.9)$$

is said to be generalized Ulam-Hyers stable if there exists an increasing mapping $\gamma: [0, \infty) \rightarrow [0, \infty)$ with $\gamma(0) = 0$ and γ is continuous at 0, such that for any $\varepsilon > 0$ and any ε -solution z of the fixed point inclusion (3.9) (i.e., $D(z, F(z)) \leq \varepsilon$) there exists $x^* \in Fix(F)$ such that

$$d(z, x^*) \leq \gamma(\varepsilon).$$

We also recollect two other stability properties as follows.

Definition 3.15. Consider a metric space (X, d) , and a multi-valued operator $F: X \rightarrow P(X)$. Suppose that $Fix(F)$ is nonempty and there exists $r: X \rightarrow Fix(F)$ a set retraction. Then, the fixed point inclusion $x \in F(x)$ is said to be well-posed if for every $x^* \in Fix(F)$ and for any sequence $\{u_n\}_{n \in \mathbb{N}} \subset r^{-1}(x^*)$ such that

$$D(u_n, F(u_n)) \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

we have that

$$u_n \longrightarrow x^* \text{ as } n \longrightarrow \infty.$$

Definition 3.16. Consider a metric space (X, d) and a multi-valued operator $F: X \rightarrow P(X)$. Suppose that $\text{Fix}(F) \neq \emptyset$ and there exists $r: X \rightarrow \text{Fix}(F)$ a set retraction. Then, the fixed point inclusion $x \in F(x)$ has the Ostrowski stability property if for any sequence $\{v_n\}_{n \in \mathbb{N}} \subset r^{-1}(x^*)$ such that

$$D(v_{n+1}, F(v_n)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

we have that

$$v_n \rightarrow x^* \text{ as } n \rightarrow \infty.$$

We continue with a local fixed point result for multi-valued nonlinear graph φ -contractions. This theorem expands some results introduced in [14] to the multi-valued case.

Theorem 3.21. *Assume that (X, d) is a complete metric space, $x_0 \in X$ and $r > 0$. Let $F: \tilde{B}(x_0; r) \rightarrow P(X)$ be a multi-valued operator satisfying the following assumptions:*

- (i) *Graph(F) is closed;*
- (ii) *for every $(x, y) \in \text{Graph}(F)$ there exists $z \in F(y)$ such that $d(y, z) \leq \varphi(d(x, y))$;*
- (iii) *there exists $x_1 \in F(x_0)$ such that $\sum_{k=0}^{\infty} \varphi^k(d(x_0, x_1)) \leq r$.*

Then, the following conclusions hold:

- (a) $\text{Fix}(F) \neq \emptyset$;
- (b) *there exists in $\tilde{B}(x_0; r)$ a Picard type iterates equence $(x_n)_{n \in \mathbb{N}}$ for F starting from (x_0, x_1) which converges to a fixed point of F .*

Remark 3.22. Notice that the condition (ii) in the above theorem implies that

$$\rho(F(x), F(y)) \leq \varphi(d(x, y)), \text{ for every } (x, y) \in \text{Graph}(F).$$

For the above assumption see also [76].

Another theorem similar to the one introduced above is about to be specified.

Theorem 3.23. *Assume that (X, d) is a complete metric space, $x_0 \in X$, and $R > 0$. Let $F: B(x_0; R) \rightarrow P(X)$ be a multi-valued operator which satisfies the following assumptions:*

- (i) *Graph(F) is closed;*
- (ii) *for every $(x, y) \in \text{Graph}(F)$ there exists $z \in F(y)$ such that $d(y, z) \leq \varphi(d(x, y))$;*
- (iii) *there is $x_1 \in F(x_0)$ for which $\sum_{k=0}^{\infty} \varphi^k(d(x_0, x_1)) < R$.*

Then, $\text{Fix}(F) \neq \emptyset$ and there exists in $B(x_0; R)$ a Picard type iterative sequence $(x_n)_{n \in \mathbb{N}}$ for F starting from (x_0, x_1) which is convergent to a fixed point of F .

When we consider the case of a nonempty set endowed with two metrics, the upcoming Maia type theorem for multi-valued nonlinear graph φ -contractions arises.

Theorem 3.24. *Let $X \neq \emptyset$, d and ρ be two metrics on the set X . Let $F: X \rightarrow P(X)$ be a multi-valued operator. We suppose:*

- (i) *(X, d) is a generalized complete metric space;*
- (ii) *(X, ρ) is a generalized metric space;*
- (iii) *there exists $R > 0$ such that $d(x, y) \leq R\rho(x, y)$, for each $x, y \in X$;*
- (iv) *Graph(F) is closed with respect to d ;*
- (v) *F is a multi-valued nonlinear graph φ -contraction with respect to ρ , i.e., there exists a strong comparison function $\varphi: [0, \infty[\rightarrow [0, \infty[$ such that*

$$H_\rho(F(x), F(y)) \leq \varphi(\rho(x, y)), \text{ for every } (x, y) \in \text{Graph}(F). \quad (3.10)$$

Then, for every $x_0 \in X$ one of the following two assertions holds:

- (a) *each Picard type iterative sequence $(x_n)_{n \in \mathbb{N}}$ for the operator F starting from x_0 has the property $d(x_n, x_{n+1}) = \infty$ for all $n \in \mathbb{N}$;*

- (b) there exists a Picard type iterative sequence $(x_n)_{n \in \mathbb{N}}$ for the operator F starting from x_0 which converges to a fixed point of F .

For the above result, the following consequence occurs.

Corollary 3.25. *Consider a complete metric space (X, d) and (X, ρ) be a metric space. Let $F: X \rightarrow P(X)$ be a multi-valued nonlinear graph φ -contraction. We assume that $\text{Graph}(F)$ is closed with respect to d and there is $R > 0$ such that $d(x, y) \leq R\rho(x, y)$, for every $x, y \in X$. Then we have:*

- (a) there is $x^* \in X$ for which $x^* \in \text{Fix}(F^n)$, for every $n \in \mathbb{N}^*$;
- (b) for every pair $(x, y) \in \text{Graph}(F)$ there exists a Picard type iterative sequence $(x_n)_{n \in \mathbb{N}}$ for F starting from (x, y) that is convergent to a fixed point of the operator F ;
- (c) F is a ψ -multi-valued weakly Picard operator, i.e., there is a selection $f^\infty: \text{Graph}(F) \rightarrow \text{Fix}(F)$ of F^∞ for which

$$d(x, f^\infty(x, y)) \leq \psi(\rho(x, y)), \text{ for every pair } (x, y) \in \text{Graph}(F),$$

where

$$\psi(t) := t + \sum_{k=0}^{\infty} \varphi^k(st), \text{ with } s > 1 \text{ arbitrarily chosen.} \quad (3.11)$$

3.6 Multi-valued (φ, ψ) -contraction of Feng-Liu type

The final section of Chapter 3 approaches the case of multi-valued (φ, ψ) -contractions of Feng-Liu type and construct two fixed point theorems and prove various stability results. We begin the section by looking back on the definition of the aforementioned type of multi-valued contractions. The key results found in this section are Theorem 3.26, Theorem 3.27, and Theorem 3.29.

Let us commence by evoking the definition of the considered type of operators. It was introduced in [48].

Definition 3.17 (see Definition 2.4, [48]). Let (X, d) be a metric space, $F: Y \subseteq X \rightarrow P(X)$ be a multi-valued operator and $\psi: [0, \infty) \rightarrow [0, \infty)$ be an increasing mapping such that $\psi(t) > t$ for any $t > 0$ and $\psi(0) = 0$. For each $x \in X$ we consider the set

$$I_\psi^x(d) := \{y \in F(x) : d(x, y) \leq \psi(D_d(x, F(x)))\}. \quad (3.12)$$

The operator F is called a multi-valued nonlinear (φ, ψ) -contraction of Feng-Liu type with respect to the metric d if there exists an increasing mapping $\varphi: [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t) < t$ for each $t \in (0, \infty)$ and $\varphi(0) = 0$, such that

- (a) $\psi \circ \varphi$ is a strong comparison function;
- (b) for each $x \in X$ there exists $y \in I_\psi^x(d)$ such that

$$D_d(y, F(y)) \leq \varphi(d(x, y)). \quad (3.13)$$

The first new theorem introduced in the current section is a fixed point existence result that extends Theorem 2.5 in [48] to the case of a space endowed with two metrics.

Theorem 3.26. *Assume that X is a nonempty set endowed with two metrics d and ρ , and let $F: X \rightarrow P_{cl}(X)$. We assume the following:*

- (i) (X, d) is a complete metric space;
- (ii) there exists $R > 0$ such that $d(x, y) \leq R\rho(x, y)$ for all $x, y \in X$;
- (iii) $\text{Graph}(F)$ is closed with respect to the metric ρ ;
- (iv) F is a multi-valued (φ, ψ) -contraction of Feng-Liu type with respect to ρ .

Then, we have that $\text{Fix}(F) \neq \emptyset$ and for each $x_0 \in X$ there exists $(x_n)_{n \in \mathbb{N}}$ an Picard type iterative sequence convergent to a point $x^* \in \text{Fix}(F)$. Moreover, the following retraction-displacement condition holds

$$d(x_0, x^*(x_0)) \leq \sum_{k=0}^{\infty} (\psi \circ \varphi)^k(t_0), \quad (3.14)$$

where $t_0 := d(x_0, x_1)$, $x_1 \in I_\psi^{x_0}(\rho)$.

A particular case of Theorem 3.26 is given below.

Theorem 3.27. *Assume that X is a nonempty set endowed with two metrics d and ρ . Let $F: X \rightarrow P_{cl}(X)$ and $\psi: [0, \infty) \rightarrow [0, \infty)$ be an increasing mapping such that $\psi(t) > t$ for any $t > 0$ and $\psi(0) = 0$. We assume that:*

- (*i) (X, d) is a complete metric space;
- (*ii) there exists $R > 0$ such that $d(x, y) \leq R\rho(x, y)$ for all $x, y \in X$;
- (*iii) $\text{Graph}(F)$ is closed with respect to the metric ρ ;
- (*iv) there exists an increasing mapping $\varphi: [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t) < t$ for each $t \in (0, \infty)$ and $\varphi(0) = 0$, such that $\psi \circ \varphi$ is a strong comparison mapping and the following relation holds

$$D_\rho(y, F(y)) \leq \varphi(\rho(x, y)), \text{ for every } (x, y) \in \text{Graph}(F).$$

Then, we have $\text{Fix}(F) \neq \emptyset$ and for each $x_0 \in X$ there exists $(x_n)_{n \in \mathbb{N}}$ a Picard type iterative sequence convergent to a point $x^* \in \text{Fix}(F)$. Moreover, the following retraction-displacement condition holds

$$d(x_0, x^*(x_0)) \leq \sum_{k=0}^{\infty} (\psi \circ \varphi)^k(t_0), \quad (3.15)$$

where $t_0 := d(x_0, x_1)$, $x_1 \in I_\psi^{x_0}(\rho)$.

We proceed with considering the extensions of some stability properties with regards to the fixed point inclusion in the context of multi-valued non-linear (φ, ψ) -contractions of Feng-Liu type in a space equipped with two metrics. Similar stability concepts can be found in [32] and [49]. Let us firstly go through the Ulam-Hyers stability, see e.g., [43].

Definition 3.18. Let X be a nonempty set endowed with two metrics d and ρ . Consider a multi-valued operator $F: X \rightarrow P(X)$. The multi-valued fixed point inclusion $x \in F(x)$, $x \in X$ has the generalized (d, ρ) -Ulam-Hyers stability if there exists an increasing function $\mu: [0, \infty) \rightarrow [0, \infty)$ with $\mu(0) = 0$ and μ is continuous at 0, such that for every $\varepsilon > 0$ and for each ε -fixed point $z \in X$ of F (in the sense that $D_\rho(z, F(z)) < \varepsilon$), there exists $x^* \in \text{Fix}(F)$ such that

$$d(z, x^*) \leq \mu(\varepsilon).$$

We also recall the well-posedness property of the fixed point inclusion. The single-valued case corresponding properties can be found in [25], [63], and [62].

Definition 3.19. Let X be a nonempty set endowed with two metrics d and ρ . Let $F: X \rightarrow P(X)$ be a multi-valued operator such that $Fix(F) \neq \emptyset$ and $r: X \rightarrow Fix(F)$ be a set retraction. Then, the fixed point inclusion $x \in F(x), x \in X$ is generalized (d, ρ) -well-posed in the sense of Reich and Zaslavski if for each $x^* \in Fix(F)$ and for any sequence $(y_n)_{n \in \mathbb{N}} \subset r^{-1}(x^*)$ such that

$$D_\rho(y_n, F(y_n)) \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

we have that

$$y_n \xrightarrow{d} x^* \text{ as } n \longrightarrow \infty.$$

We continue with calling to mind the Ostrowski stability property, see [46] and [52].

Definition 3.20. Let $X \neq \emptyset$ be equipped with two metrics d and ρ . Consider a multi-valued operator $F: X \rightarrow P(X)$ such that $Fix(F)$ is nonempty and a set retraction $r: X \rightarrow Fix(F)$. We say the multi-valued operator possesses the generalized (d, ρ) -Ostrowski stability property if and only if for each $x^* \in Fix(F)$ and for any sequence $(z_n)_{n \in \mathbb{N}} \subset r^{-1}(x^*)$ such that

$$D_\rho(z_{n+1}, F(z_n)) \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

we have that

$$z_n \xrightarrow{d} x^* \text{ as } n \longrightarrow \infty.$$

The last definition we will recall here is the data dependence property.

Definition 3.21. Let $X \neq \emptyset$ be equipped with two metrics d and ρ . Consider a multi-valued operator $F: X \rightarrow P(X)$. We say that the fixed point inclusion $x \in F(x), x \in X$ has the generalized data dependence property related to the pair of metrics (d, ρ) if and only if, for any multi-valued operator $G: X \rightarrow P(X)$ having:

- (i) $Fix(G)$ is nonempty;
- (ii) there is $\eta > 0$ for which $H_\rho(F(x), G(x)) \leq \eta$, for all $x \in X$,

then there is an increasing function $\mu: [0, \infty) \rightarrow [0, \infty)$ with $\mu(0) = 0$ and μ is continuous at 0, such that for every $y^* \in \text{Fix}(G)$ there exists $x^* \in \text{Fix}(F)$ such that

$$d(x, y) \leq \mu(\eta).$$

In the proofs of our next results, the following generalized strong retraction-displacement condition will be useful.

Definition 3.22. Let $X \neq \emptyset$ be equipped with two metrics d and ρ . Consider a multi-valued operator $F: X \rightarrow P(X)$ for which $\text{Fix}(F) \neq \emptyset$. We say that F fulfills the generalized strong retraction-displacement condition if there are an increasing function $\mu: [0, \infty) \rightarrow [0, \infty)$ with $\mu(0) = 0$ and μ is continuous at 0, and a set retraction $r: X \rightarrow \text{Fix}(F)$ for which

$$d(x, r(x)) \leq \mu(D_\rho(x, F(x))), \text{ for every } x \in X. \quad (3.16)$$

We will now reproduce an abstract result, which was proved in [49].

Theorem 3.28. *Assume that $X \neq \emptyset$ is equipped with two metrics d and ρ . Consider a multi-valued operator $F: X \rightarrow P(X)$ such that $\text{Fix}(F)$ is nonempty. We assume that F fulfills the generalized strong retraction-displacement condition. Then the fixed point inclusion $x \in F(x), x \in X$ has the generalized (d, ρ) -Ulam-Hyers stability property, it is generalized (d, ρ) -well-posed in the sense of Reich-Zaslavski and satisfies the generalized data dependence property with respect to the metrics pair (d, ρ) .*

The next result proves the generalized strong retraction-displacement property for multi-valued nonlinear (φ, ψ) -contractions of Feng-Liu type on a space endowed with two metrics.

Theorem 3.29. *Assume that $X \neq \emptyset$ is equipped with two metrics d and ρ , and let $F: X \rightarrow P_d(X)$. We assume the following:*

- (i) (X, d) is a complete metric space;
- (ii) there exists $R > 0$ such that $d(x, y) \leq R\rho(x, y)$ for all $x, y \in X$;
- (iii) $\text{Graph}(F)$ is closed with respect to the metric ρ ;
- (iv) F is a multi-valued (φ, ψ) -contraction of Feng-Liu type with respect to ρ .

Then, F fulfills the generalized strong retraction-displacement condition.

Finally, we fuse Theorem 3.28 and Theorem 3.29 into the subsequent result.

Theorem 3.30. *Assume that X is a nonempty set equipped with two metrics d and ρ , and let $F: X \rightarrow P_{cl}(X)$. We assume the following:*

- (i) (X, d) is a complete metric space;
- (ii) there exists $R > 0$ such that $d(x, y) \leq R\rho(x, y)$ for all $x, y \in X$;
- (iii) $\text{Graph}(F)$ is closed with respect to the metric ρ ;
- (iv) F is a multi-valued (φ, ψ) -contraction of Feng-Liu type with respect to ρ .

Then, the fixed point inclusion $x \in F(x), x \in X$ has the generalized Ulam-Hyers stability property, it is well-posed in the sense of Reich-Zaslavski and has the generalized data dependence property.

Conclusions and further research

As we have seen through this thesis, we developed several fixed point results for different generalizations of the classical single-valued and multi-valued contractions. In addition, we presented applications for most of the proven theorems, as well as stability properties of the fixed point problem. These results enhance the current literature and also provide means for further research.

In terms of future research, the results presented in the thesis offer new directions of study. One direction could be laboring to obtain other local fixed point results, as well as homotopy or continuation results, by verifying the considered hypotheses and following the same proof method for other types of contraction generalizations, such as Sehgal, Zamfirescu, Singh, and others. Rhoades' article [64] from 1977 gives proven implications among several types of generalized contractions.

A second research trajectory is to try to operate in the context of generalized metric spaces. One could check if the proof methods used in the above chapters can be adapted to obtain similar fixed point theorems. A more practical research approach is to investigate some applications to differential and integral inclusions.

In conclusion, the proven results unveiled in this thesis represent advancements in fixed point theory, focused on non-self operators. They are improvements to the theoretical literature, but also contribute to more practical use cases.

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