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Control problems for Kolmogorov type systems

Ph.D. Thesis Summary

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Contents

Introduction

Numerous mathematical models that describe real-world processes are formulated through equations and differential systems. These usually include a set of parameters, some of which are fixed, while others are associated with the variable quantities in the model and can be adjusted to achieve a certain objective, defined by a controllability condition.

Parameter modification is mathematically realized by introducing control parameters, whose expressions can, in many cases, be expressed as a function of the state variables. Once incorporated into the model equations, these lead to functional-differential equations, whose study can be reduced to the analysis of fixed point problems. Thus, the fixed point method becomes a fundamental tool in control problems. The application of this method varies depending on the specifics of each problem, as detailed in J. M. Coron's monograph [7].

In this thesis, we study control problems related to Kolmogorov type differential systems. The classical Kolmogorov system is of the form:

$$\begin{cases} x' = xf(x, y) \\ y' = yg(x, y) \end{cases} \tag{1}$$

and represents the mathematical model of population dynamics where f(x, y) and g(x, y) represent the per capita rates of the two populations. The most well-known example of such systems is the Lotka-Volterra system:

$$\begin{cases} x' = \alpha x - \beta xy \\ y' = \delta xy - \gamma y. \end{cases} \tag{2}$$

In addition to first-order Kolmogorov systems, the thesis also defines and analyzes second-order Kolmogorov type systems of the form:

$$\begin{cases} \left(\frac{x'}{x}\right)' = f(x, y) \\ \left(\frac{y'}{y}\right)' = g(x, y). \end{cases} \tag{3}$$

In the language of population dynamics, $f(x, y)$ describes the change in *per capita* rate, and $g(x, y)$ expresses the change in *per capita* rate $\frac{y'}{y}$.

Regarding these systems, several control problems are formulated, where the control parameters can be real numerical values, vectors, or time functions. Also, these parameters can be either additive or multiplicative quantities that modify the growth rates, or terms that appear directly in the nonlinear structure of the equations.

The method used throughout the thesis is the fixed point method, which consists of reducing the control problem to a fixed point equation. For this purpose, various results from fixed point theory are applied, including Banach's contraction principle, Perov's vectorial fixed point theorem, as well as Schauder's, Krasnoselskii's, and Avramescu's fixed point theorems. In the case of multivalued problems, we use Nadler's and Bohnenblust-Karlin's fixed point theorems. The thesis is structured into six chapters, each containing several sections and subsections. In the following, we will detail the results obtained in each chapter.

Chapter 1: Preliminaries

Chapter 1 is dedicated to the essential preliminary concepts and fundamental results that are used throughout the thesis. In Section 1.1, we introduce the notion of a general control problem. Section 1.2 refers to the general form of the first-order Kolmogorov system, where, by analogy, the form of the second-order Kolmogorov system is also deduced. In Section 1.3, the definition of a zero-convergent matrix is given, and the properties of such matrices are mentioned. We continue with Section 1.4, where we present the notion of the Pompeiu-Hausdorff metric. The last three sections present the necessary results used throughout the thesis, namely, the fixed-point theorems used, the Bielecki type norm, and the Arzelà-Ascoli theorem.

Chapter 2: Control problems for Kolmogorov type systems

In Chapter 2, we study three Kolmogorov type control problems, each with specific initial and controllability conditions. In Section 2.1, a Kolmogorov system is analyzed in which both populations are influenced by the same control parameter. The problem consists of finding a solution such that the ratio between the two populations follows a desired evolution. By applying Banach's fixed-point theorem together with Bielecki type norms and imposing sufficient conditions, we obtain the existence (and uniqueness) of a solution, both on the entire space and in a ball.

The second section is dedicated to the study of a system in which the control influences the *per capita* rate of one of the two populations, with the objective of reaching a predetermined threshold within a fixed time interval. The existence of the solution is demonstrated by applying Schauder's fixed-point theorem.

In the third section, we apply Banach's fixed-point theorem to demonstrate the existence of a solution to a control problem where the *per capita* rate is modified only for one of the two populations, with the objective of reaching a predetermined level of the total population at the final time.

Finally, Section 2.4 presents three examples of Kolmogorov systems applied in biology, illustrating the usefulness of the theoretical results obtained. These examples include models from population dynamics and epidemiology.

Our contributions in this chapter are as follows: In Section 2.1: Theorem 2.1 and Theorem 2.2. In Section 2.2: Theorem 2.3. In Section 2.3: Theorem 2.4. In the last Section 2.4: Example 1, Example 2 and Example 3.

All these results were included in the work of A. Hofman and R. Precup [13].

Chapter 3: Vectorial approach through fixed-point methods for control problems of Kolmogorov differential systems

In this chapter, we analyze three Kolmogorov type control problems with initial conditions. The method used is correlated with the type of system considered, providing an approach tailored to each case. The method used is the vectorial one, which allows the use of more precise constants, eliminating the dependence on the type of norm used.

The system analyzed in Section 3.1 imposes a control over each *per capita* rate of the two populations. The existence of the solution is demonstrated by applying Perov's theorem, under the hypothesis of Lipschitz type conditions, and Schauder's theorem, by imposing logarithmic growth conditions.

In Section 3.2, we study a control problem involving changes in growth rates. The solution is guaranteed by Perov's fixed point theorem.

Section 3.3 combines the two problems analyzed in the previous sections, considering a system in which, for one population, control is applied to the growth rate, while for the other population, it is imposed on the *per capita* rate.

Finally, Section 3.4 presents four applications of the results obtained in the previous three sections. These applications illustrate the usefulness of the proposed method and validate the applicability of fixed point theorems in diverse contexts.

Our contributions in this chapter are presented below. In Section 3.1: Theorem 3.1, Theorem 3.2. In Section 3.2: Theorem 3.3. In Section 3.3: Theorem 3.4, Remark 3.1. In the last Section 3.4: Example 1, Example 2, Example 3 and Example 4.

All these results are original and have been included in the work of A. Hofman and R. Precup [15].

Chapter 4: Control problems for second-order Kolmogorov differential equations and systems

In Chapter 4, we present second-order Kolmogorov differential equations and systems. We investigate several control problems with fixed finite time T and fixed final state x_T, with additive or multiplicative control. The controllability of these problems is demonstrated by applying fixed-point techniques, the theorems of Banach, Schauder, Krasnoselskii, Avramescu and Perov.

In Section 4.1, we study problems with additive control related to second-order Kolmogorov equations. By imposing a Lipschitz condition and using Banach's fixed-point theorem, we demonstrate the existence and uniqueness of the solution. We observe that if the Lipschitz conditions are relaxed and replaced by logarithmic growth conditions, then using Schauder's fixed-point theorem, it is obtained that the control problem has at least one solution.

The last result in this section combines the first two previous results and is based on Krasnoselskii's fixed-point theorem for a sum of two operators.

We then continue with problems with multiplicative control, where we use Banach's contraction principle to demonstrate controllability.

In Section 4.2, we focus on control problems for a second-order Kolmogorov system. The first result guarantees the existence and uniqueness of the solution using Perov's fixed-point theorem. Subsequently, an existence result is obtained based on the application of Schauder's fixed-point theorem. At the end of this chapter, we present an application to the control problem of Avramescu's fixed-point theorem.

Our contributions in this chapter are as follows. In Section 4.1: Theorem 4.1, Remark 4.1, Theorem 4.2, Remark 4.2, Theorem 4.3, Theorem 4.4. In Section 4.2: Theorem 4.5, Theorem 4.6, Theorem 4.7.

All these results are original and have been included in the work of A. Hofman and R. Precup [14].

Chapter 5: Fixed point methods with multi-valued operators for control problems

In Chapter 5, we focus on applying fixed point methods with multivalued operators in solving a control problem for first-order Kolmogorov type equations. Here, the controllability condition is expressed as an inclusion, and to demonstrate the existence of solutions, fixed point theorems for multivalued operators are used, such as Nadler's theorem, the Bohnenblust-Karlin theorem and the multivalued version of Krasnoselskii's theorem.

Section 5.1 is dedicated to solving the control problem by applying Nadler's fixed point theorem in a ball of a given radius of the space $C[0,T]$. In Section 5.2, we solve the control problem using Bohnenblust-Karlin's fixed point theorem, under the assumption of logarithmic growth conditions. In the last section, representing the fixed point operator as the sum of two operators, we obtain a solution by applying the multivalued version of Krasnoselskii's theorem.

Our contributions in this chapter are as follows. In Section 5.1: Lemma 5.1, Theorem 5.1. In Section 5.2: Theorem 5.2, Lemma 5.2. In Section 5.3: Theorem 5.3.

All the results are original and have been included in the work A. Hofman [12].

Chapter 6: Algorithms for solving control problems related to Kolmogorov systems

This chapter is dedicated to the development and analysis of theoretical algorithms for solving control problems associated with Kolmogorov type systems, both first-order and second-order. The proposed algorithms are based on the method of sub and super solutions, which allows constructing a sequence of approximate solutions that, under certain conditions, converges to the exact solution of the control problem.

In the two sections of the chapter, we establish sufficient conditions for the algorithm to be convergent. These conditions help to determine a unique solution depending on the control condition, using Perov's fixed point theorem, along with other relevant results. Also, in Section 6.2, an algorithm for obtaining an approximate solution to the problem is presented.

Our contributions in this chapter are as follows: Lemma 6.1, Lemma 6.2, Theorem 6.1, Theorem 6.2, Theorem 6.3, Theorem 6.4, Theorem 6.5.

All results are original and have been included in the works A. Hofman [10, 11].

Author's publications:

1. A. Hofman, R. Precup. *On some control problems for Kolmogorov type systems.* Mathematical Modelling and Control., 2(3):90–99, 2022, `https://www.aimspress.com/article/doi/10.3934/mmc.2022011`.

2. A. Hofman. *An algorithm for solving a control problem for Kolmogorov systems.* Studia Universitatis Babeş-Bolyai Mathematica., 68(2):331–340, 2023, `https://doi.org/10.24193/subbmath.2023.2.09`.

3. A. Hofman, R. Precup. *Vector fixed point approach to control of Kolmogorov differential systems.* Contemporary Mathematics., 5:1968–1981, 2024, `https://doi.org/10.37256/cm.5220242840`.

4. A. Hofman, R. Precup. *Control problems for Kolmogorov type second order*

equations and systems. Journal of Fixed Point Theory and Applications., 27(1):1–18, 2024, https://link.springer.com/article/10.1007/s11784-024-01160-5.

5. A. Hofman. *Fixed point methods with multi-valued operators for control problems.* Discussiones Mathematicae. Differential Inclusions, Control and Optimization., 44(2):153–165, 2024, https://doi.org/10.7151/dmdico.1250.

6. A. Hofman. *Upper and lower solution method for control of second-order Kolmogorov type systems.* Journal of Numerical Analysis and Approximation Theory., 2025, accepted.

Chapter 1

Preliminaries

In the opening chapter of the thesis, we establish the fundamental concepts and results that underlie our research. In this chapter, we present several well-known notions and results, including the general control problem, Kolmogorov type systems, the concept of a matrix convergent to zero, the notion of Pompeiu-Hausdorff metric, Bielecki-type norms and various fixed point theorems, the Arzelà-Ascoli theorem.

The concepts discussed here are well documented in the literature. Some of the notable references include works by C. Avramescu [2], V. Barbu [4], J. M. Coron [7], A. Granas and J. Dugundji [9], R. I. Petru [25], R. Precup [28, 29], L. C. Evans [8].

1.1 General control problem

The control of differential equations is the subject of numerous studies in the literature. Generally speaking, it consists in determining some parameters of the equation or system of equations such that the solution satisfies certain conditions, other than those imposed by the well-posedness of the problems, such as initial or boundary conditions (see V. Barbu [4]).

In the work I. Ş. Haplea, L. G. Parajdi and R. Precup [16] introduced a controllability principle for a general control problem related to operator equations, within the framework of fixed point theory. The general control problem consists in finding the pair (w, λ), a solution of the following system

$$\begin{cases} w = H_0(w, \lambda), \\ w \in W, \ \lambda \in \Lambda, \ (w, \lambda) \in \mathcal{D}, \end{cases} \tag{1.1}$$

associated with the fixed point equation $w = H_0(w, \lambda)$. In this case, w represents the state variable, λ is the control variable, W is the state domain, Λ is the control domain and $\mathcal{D}$ is the controllability domain, usually given by a certain condition or property imposed on w, or both w and λ. Note the very general form of the control problem, in terms of sets, where W, Λ and $\mathcal{D} \subset W \times \Lambda$ are not necessarily structured sets and H_0 is any function from the set $W \times \Lambda$ în W.

In this context, we say that the equation $w = H_0(w, \lambda)$ is controllable in the set $W \times \Lambda$ with respect to $\mathcal{D}$, if problem (1.1) admits solutions. If the solution is unique, we say that the equation is uniquely controllable.

Let Σ be the set of all possible solutions of the fixed point equation and let Σ_1 be the set of those w that are the first components of the solutions of the fixed point equation, i.e.,

$$\begin{aligned} \Sigma &= \{(w, \lambda) \in W \times \Lambda : \ w = H_0(w, \lambda)\}, \\ \Sigma_1 &= \{w \in W : \ \text{there is } \exists\, \lambda \in \Lambda \text{ with } (w, \lambda) \in \Sigma\}. \end{aligned}$$

Clearly, the set of all solutions of the control problem (1.1) is given by $\Sigma \cap \mathcal{D}$.

Consider the set-valued map $F_0 : \Sigma_1 \to \Lambda$ defined as

$$F_0(w) = \{\lambda \in \Lambda : \quad (w, \lambda) \in \Sigma \cap \mathcal{D}\}.$$

Roughly speaking, F_0 gives the 'expression' of the control variable in terms of the state variable.

We have the following general principle for solving the control problem (1.1).

Proposition 1. *If for some extension $F : W \to \Lambda$ of F_0 from Σ_1 to W, there exists a fixed point $w \in W$ of the set-valued map*

$$H(w) := H_0(w, F(w)),$$

i.e.,

$$w = H_0(w, \lambda), \tag{1.2}$$

for some $\lambda \in F(w)$, then the couple (w, λ) is a solution of the control problem (1.1).

Proof. Clearly $(w, \lambda) \in \Sigma$. Hence $w \in \Sigma_1$ and so $F(w) = F_0(w)$. Then $\lambda \in F_0(w)$ and from the definition of F_0, it follows that $(w, \lambda) \in \mathcal{D}$. Therefore (w, λ) solves (1.1). □

The applicability of this general principle was tested in the work of I. Ş. Haplea, L. G. Parajdi and R. Precup [16] on a system that models cellular dynamics in the context of leukemia, as well as in R. Precup [30], where a control problem for the Lotka-Volterra predator-prey system is addressed.

1.2 Kolmogorov type systems

The Kolmogorov system was introduced as a generalization of the mathematical model given by the mathematician Volterra in population dynamics (K. Sigmund [33]). It takes into account general *per capita* rates of two interacting populations and looks as follows:

$$\begin{cases} x' = xf(x, y), \\ y' = yg(x, y). \end{cases} \tag{1.3}$$

Naturally, when studying the interaction between two species, the two rates f and g depend explicitly on a series of parameters. Some of these parameters are specific to the two species and do not undergo changes, others can be influenced, even added, in order to control the evolution and achieve a desired equilibrium.

By changing the variables $x = e^u$ and $y = e^v$, the system (1.3) is transformed into the normal form

$$\begin{cases} u' = f(e^u, e^v) \\ v' = g(e^u, e^v). \end{cases}$$

In this chapter, by a first-order Kolmogorov equation we mean an equation of the form $x' = xf(t, x)$. As before, the change of variable $x = e^u$ leads to $u' = f(t, e^u)$. By analogy, we say that a second-order equation is a second-order Kolmogorov equation if it has the form:

$$\left(\frac{x'}{x}\right)' = f(t, x),$$

equivalently

$$x'' - \frac{1}{x}x'^2 = xf(t, x).$$

In this case, in the language of population dynamics, $f(t, x)$ expresses the change in the *per capita* rate $\frac{x'}{x}$. More generally, we call Kolmogorov equations of order n, the equations of the form

$$\left(\frac{x'}{x}\right)^{(n-1)} = f(t, x).$$

All these equations have the property that by changing the variable $x = e^u$, they become respectively

$$u'' = f(t, e^u)$$

and

$$u^{(n)} = f(t, e^u).$$

1.3 Matrices converging to zero

Matrices converging to zero are important in the study of systems of equations. For example, they take over the role of contraction constants in the vectorial version of Banach's fixed-point theorem, due to Perov.

Definition 2. *A square matrix with non-negative elements $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is said to converge to zero if*

$$M^k \to 0_n \quad as \quad k \to \infty,$$

where O_n represents the zero matrix of order n.

The following statements are equivalent (A. Berman and R. J. Plemmons [5]):

(a) M converges to zero;

(b) $\rho(M) < 1$;

(c) $I - M$ is non-singular and $(I - M)^{-1} = I + M + M^2 + ...$;

(d) $I - M$ is non-singular and its inverse $(I - M)^{-1}$ is also with non-negative values (where I represents the identity matrix of the same dimension).

We mention that a square matrix of order two

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with non-negative values is convergent to zero if and only if

$$\text{tr } M < \min\{2,\ 1 + \det M\}, \tag{1.4}$$

meaning

$$a + d < 2 \quad \text{şi} \quad a + d < 1 + ad - bc. \tag{1.5}$$

Note that if M is convergent to zero, then $a < 1$ and $d < 1$.

1.4 The notion of Pompeiu-Hausdorff metric

If A and B are two subsets of a metric space (X, d) and $a \in A, b \in B$, then define:

$$D(a, B) = \inf_{b\in B} d(a, b), \qquad \rho(A, B) = \sup_{a\in A} D(a, B),$$

$$D(b, A) = \inf_{a\in A} d(a, b), \qquad \rho(B, A) = \sup_{b\in B} D(b, A),$$

and the Pompeiu-Hausdorff metric is defined by

$$H(A,B) = \max\{\rho(A,B), \rho(B,A)\}.$$

We denote by $\mathcal{P}_{b,cl}(X)$ the set of all nonempty, bounded and closed subsets of X and by $\mathcal{P}_{b,cl,cv}(X)$ the set of all nonempty, bounded, closed and convex subsets of X.

1.5 Fixed point theorems

Definition 3. *Let (X, d) be a metric space. A mapping $T : X \to X$ is called a contraction on X, if there exists $q \in [0, 1)$ such that for all $x, y \in X$ we have*

$$d(T(x), T(y)) \leq qd(x, y).$$

Definition 4. *Let $T : X \to X$. A point $x \in X$ is called a fixed point of T if $T(x) = x$*

Theorem 1.1 (Banach)**.** *Let (X, d) be a complete metric space and $T : X \to X$ o contracţie. be a contraction. Then T admits a unique fixed point $x^* \in X$ (i.e. $T(x^*) = x^*$). Moreover, x^* can be obtained by the method of successive approximations starting from an arbitrary element $x_0 \in X$, as the limit of the sequence x_n defined recursively by $x_n = T(x_{n-1})$, i.e. $x_n \to x^*$.*

Theorem 1.2 (Schauder)**.** *Let X be a Banach space, $D \subset X$ a nonempty closed, convex, bounded set and $N : D \to D$ a compact operator (i.e., continuous, with $N(D)$ relatively compact). Then, N has at least one fixed point in D.*

Theorem 1.3 (Krasnoselskii)**.** *Let D be a closed bounded convex subset of a Banach space X, $A : D \to X$ a contraction and $B : D \to X$ a continuous mapping with $B(D)$ relatively compact. If*

$$A(x) + B(y) \in D \quad \text{for every} \;\; x, y \in D,$$

then the mapping $A + B$ has at least one fixed point.

Theorem 1.4 (Avramescu)**.** *Let $(D_1,\ d)$ be a complete metric space, D_2 a closed convex subset of a normed space Y, and let $N_i : D_1 \times D_2 \to D_i$, $i = 1, 2$ be continuous mappings. Assume that the following conditions are satisfied:*

(a) *There is a constant $L \in [0, 1)$ such that*

$$d(N_1(x, y),\ N_1(\overline{x}, y)) \leq Ld(x,\ \overline{x})$$

for all $x, \overline{x} \in D_1$ and $y \in D_2$;

(b) *$N_2(D_1 \times D_2)$ is a relatively compact subset of Y.*

Then there exists $(x, y) \in D_1 \times D_2$ with

$$N_1(x, y) = x, \quad N_2(x, y) = y.$$

Theorem 1.5 (Perov)**.** *Let $(X, \|\cdot\|)$ be a Banach space, D a closed subset of $X \times X$ and $N : D \to D$, $N = (N_1, N_2)$, $N_i : D \to X \ (i = 1, 2)$ be an operator with the following property:*

$$\begin{bmatrix} \|N_1(x) - N_1(y)\| \\ \|N_2(x) - N_2(y)\| \end{bmatrix} \leq M \begin{bmatrix} \|x_1 - y_1\| \\ \|x_2 - y_2\| \end{bmatrix}$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in D$, where M is a convergent to zero matrix of size two. Then N has a unique fixed point in D which is the limit of the sequence $\left(N^k(x)\right)_{k\geq 1}$ of successive approximations starting from any $x \in D$.

We will conclude this section by recalling two fixed point theorems for multivalued mappings and with a multivalued version Krasnoselskii's fixed point theorem for a sum of two operators, a particular case of a more general result obtained by R. I. Petru [25].

Theorem 1.6 (Nadler)**.** *Let (X, d) be a complete metric space, and $N : X \to \mathcal{P}_{b,cl}(X)$ such that*

$$H(N(x), N(y)) \leq Ld(x, y),$$

for all $x, y \in X$, where $L < 1$. Then there is $x^* \in X$ with $x^* \in N(x^*)$.

Theorem 1.7 (Bohnenblust-Karlin)**.** *Let X be a Banach space, D a convex, closed and bounded subset of X and let $N : D \to \mathcal{P}_{b,cl,cv}(X)$ be upper semicontinuous with $N(D)$ relatively compact. Then there is at least one fixed point $x \in D$ for N, i.e., $x \in N(x)$.*

Theorem 1.8. *Let $(X, ||\cdot||)$ be a Banach space and $D \in \mathcal{P}_{b,cl,cv}(X)$. Assume that $N_1 : D \to X$ and $N_2 : D \to \mathcal{P}_{b,cl,cv}(X)$ satisfy :*

(i) N_1 is a contraction.

(ii) N_2 is lower semicontinous with $N_2(D)$ relatively compact.

(iii) $N_1(u) + N_2(\overline{u}) \subset D$ for all $u, \overline{u} \in D$.

Then $N_1 + N_2$ has at least one fixed point in D.

1.6 Bielecki type norms

This section is devoted to a brief presentation of some notions and results that will be used in the next section. It is intended for those less familiar with the theoretical framework in which we place ourselves.

We say that an integral equation is of *Volterra type* if the involved integral is on a variable interval as is the case of an equation of the form

$$x(t) = \varphi(t) + \int_a^t \psi(t, s, x(s))\, ds, \quad t \in [a, b] \tag{1.6}$$

and that it is of *Fredholm type* if the involved integral is given on a fixed interval, as in the equation

$$x(t) = \varphi(t) + \int_a^b \psi(t, s, x(s))\, ds, \quad t \in [a, b].$$

In case that the equation involves both types of integral, we say that it is of *Volterra-Fredholm type*.

When dealing with Volterra type equations it is convenient that instead of the max-norm on the space $C[a,b]$ given by $\|x\| = \max_{t\in[a,b]} |x(t)|$, to consider an equivalent norm defined by

$$\|x\|_\theta = \max_{t\in[a,b]} \left(|x(t)|\, e^{-\theta(t-a)} \right),$$

for some suitable number $\theta > 0$. Such a norm is called a *Bielecki norm* and it is equivalent to the max-norm, as follows from the inequalities

$$e^{-\theta(b-a)} \|x\| \le \|x\|_\theta \le \|x\| \quad (x \in C[a,b]).$$

The utility of Bielecki norms lies in the possibility that, by choosing a sufficiently large value of θ, the Lipschitz or growth constants can be made arbitrarily small.

1.7 Arzelà-Ascoli Theorem

Theorem 1.9. *A subset $M \subset C[a,b]$ is relatively compact if and only if:*

(a): *The set M is uniformly bounded, meaning there exists a constant $C > 0$ such that for any $f \in M$, we have*

$$\sup_{x \in [a,b]} |f(x)| \leq C.$$

(b): *The set M is equicontinuous, meaning for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{f \in M} |f(x) - f(y)| \leq \varepsilon, \quad \text{for any } x, y \in [a,b] \text{ with } |x - y| \leq \delta.$$

Based on the Arzelà-Ascoli theorem and the properties of differentiable functions, the following result holds:

Theorem 1.10. *Let $M \subset C^1[a,b]$ and denote $M' = \{f' : f \in M\}$. Then the following statements are equivalent:*

1. *The set M is relatively compact in $(C[a,b], \|\cdot\|_\infty)$;*
2. *The sets M and M' are uniformly bounded.*

Chapter 2

Control problems for Kolmogorov type systems

This chapter is structured into three parts. **Section 2.1** deals with a first control problem that we address using Banach's fixed-point theorem, which guarantees the existence and uniqueness of the solution. For this, the Lipschitz condition is imposed on the nonlinear terms, and Bielecki type norms are used, which do not require the Lipschitz constants to be subject to restrictions. Furthermore, in **Section 2.2,** the conditions that ensure controllability are weakened, via Schauder's fixed-point theorem, in the case of a control independent of t.

In **Section 2.3,** a third control problem is presented in which the control affects the general growth rate, and in **Section 2.4,** we presented three examples of Kolmogorov type systems that intervene in biology.

The results in this chapter were published in the works of A. Hofman and R. Precup [13].

2.1 First control problem

Let us consider the following control problem for the general Kolmogorov system under initial conditions

$$\begin{cases} x'(t) = x(t)\left(f(x,y) - \lambda(t)\right), \\ y'(t) = y(t)\left(g(x,y) - c\lambda(t)\right), \\ x(0) = x_0,\ y(0) = y_0, \end{cases} \tag{2.1}$$

where $\lambda(t)$ is the control function and c is a positive correction factor, $c \neq 1$. We want to find a positive solution (x, y) so that

$$\frac{y(t)}{x(t)} = r(t), \tag{2.2}$$

where r is a given positive continuous function on some interval $[0, T]$.

Thus the problem consists in finding how to change the *per capita* growth rates for the ratio of the two species to follow a desired evolution giving by $r(t)$ on a fixed time interval $[0, T]$. The correction factor c expresses the fact that the effect of the control intervention on the two rates is manifested differently in the two species.

We have the following result.

Theorem 2.1. *Assume that $f, g \in C^1(\mathbb{R}^2_+)$, $r \in C^1[0, T]$, $r > 0$ on $[0, T]$ and that the functions*

$$x \cdot f_x(x, y), \quad y \cdot f_y(x, y), \quad x \cdot g_x(x, y), \quad y \cdot g_y(x, y) \tag{2.3}$$

are bounded on $\mathbb{R}^2_+$. Then the control problem (2.1)-(2.2) has a unique solution (x, y, λ) with $x, y > 0$.

If the hypothesis on the boundedness of functions (2.3) is removed, we however have the following result.

Theorem 2.2. *Assume that $f, g \in C^1(\mathbb{R}^2_+)$, $r \in C^1[0, T]$, $r > 0$ on $[0, T]$ and that the function*

$$-\frac{c}{1-c} f(x, y) + \frac{1}{1-c} g(x, y) \tag{2.4}$$

is bounded above on $\mathbb{R}^2_+$. Then the control problem (2.1)-(2.2) has a unique solution (x, y, λ) with $x, y > 0$.

2.2 Second control problem

We consider the problem of controllability of the Kolmogorov system

$$\begin{cases} x'(t) = x(t)[f(x, y) - \lambda], \\ y'(t) = y(t) \cdot g(x, y), \\ x(0) = x_0, \ y(0) = y_0, \end{cases} \tag{2.5}$$

where λ is a constant. We want to find a solution so that $x(T) = x_1$

Thus the problem is to change constantly the *per capita* rate of only one of the two populations for it to reach a desired threshold in a given time.

Theorem 2.3. *Let* $f, g \in C\left(\mathbb{R}_+^2\right)$.

(a) *If f and g are bounded on $\mathbb{R}_+^2$, then for every $T > 0$, the control problem has a solution (x, y, λ) with $x, y > 0$.*

(b) *If $x_0, y_0, x_1 \geq 1$, then for each $\rho_0 > \max\{x_0, x_1, y_0\}$, there exists $T_{\rho_0} > 0$ such that for any $T \in (0, T_{\rho_0}]$, the control problem has a solution (x, y, λ) with $0 < x, y \leq \rho_0$.*

2.3 Third control problem

The problem consists in changing the growth rate (not the *per capita* rate) of one of the two populations so that at time T, the total population reaches a desired level γ. More exactly we consider the problem

$$\begin{cases} x'(t) = x(t)f(x(t), y(t)) - \lambda, \\ y'(t) = y(t)g(x(t), y(t)), \\ x(0) = x_0, \ y(0) = y_0, \\ x(T) + y(T) = \gamma. \end{cases} \tag{2.6}$$

Theorem 2.4. *Let $\rho > \max\{|x_0|, |y_0|, |y_0 - \gamma|\}$; $f, g \in C^1\left([-\rho, \rho]^2\right)$; M_ρ a bound of $|xf(x, y)|$, $|yg(x, y)|$ on $[-\rho, \rho]^2$ and $\overline{M}_\rho$ a bound of the absolute value of the partial derivatives of the functions $xf(x, y)$, $yg(x, y)$ on $[-\rho, \rho]^2$. If T is such that*

$$T \leq \frac{\rho - \max\{|x_0|, \ |y_0 - \gamma|\}}{2M_\rho}, \quad T \leq \frac{\rho - |y_0|}{M_\rho}, \quad T < \frac{1}{3\overline{M}_\rho}, \tag{2.7}$$

then the control problem has a unique solution (x, y, λ) with $|x|, |y| \leq \rho$.

2.4 Applications

Example 1

This example refers to Theorem 2.2. Specifically, we consider the following

$$\begin{cases} x' = \left(\dfrac{a}{1 + x^2 + y^2} - \lambda(t)\right) x, \\ y' = \left(\dfrac{b}{1 + x^2 + y^2} - \lambda(t)c\right) y, \end{cases}$$

under the control condition (2.2). Here we have

$$f(x, y) = \frac{a}{1 + x^2 + y^2}, \qquad g(x, y) = \frac{b}{1 + x^2 + y^2}$$

and the functions (2.3) are

$$\begin{aligned} x \cdot f_x(x,y) &= -\frac{2ax^2}{(1+x^2+y^2)^2}, \\ y \cdot f_y(x,y) &= -\frac{2ay^2}{(1+x^2+y^2)^2}, \\ x \cdot g_x(x,y) &= -\frac{2bx^2}{(1+x^2+y^2)^2}, \\ y \cdot g_y(x,y) &= -\frac{2by^2}{(1+x^2+y^2)^2}. \end{aligned}$$

Evidently, their absolute values are bounded on $\mathbb{R}_+^2$ by $2|a|$ and $2|b|$, respectively.

From Theorem 2.2, it follows that the system is uniquely controllable. The expression of the control function $\lambda(t)$ is given by the formula

$$\lambda(t) = \frac{r'(t)}{(1-c)r(t)} + \frac{1}{1-c}\left(f(e^{u(t)}, e^{v(t)}) - g(e^{u(t)}, e^{v(t)})\right) \tag{2.8}$$

in terms of the state variables.

Example 2

The following example illustrates Theorem 2.2. The differential system represents a mathematical model of cellular dynamics in hematology, considered in the work [21].

More specifically, we consider the control problem

$$\begin{cases} x' = \left(a\left(1 - \dfrac{gx+y}{A}\right) - \lambda(t)\right)x, \\ y' = \left(b\left(1 - \dfrac{x+y}{B}\right) - c\lambda(t)\right)y, \end{cases}$$

where $0 < a < b$, $0 < c < 1$, $g \geq 1$ and $A, B > 0$, again under the control condition (2.2) which expresses the desired evolution of the ratio between the density $y(t)$ of leukemic cells and the density $x(t)$ of healthy cells over a period of time. The problem is motivated by the need to develop a treatment scheme for patients with chronic leukemia.

In this case, we have

$$f(x,y) = a\left(1 - \frac{gx+y}{A}\right), \quad g(x,y) = b\left(1 - \frac{x+y}{B}\right),$$

for which, evidently, the boundedness condition of the functions (2.3) does not hold.

Change the function,

$$-cf(x,y)+g(x,y)=b-ac-\left(\frac{b}{B}-\frac{acg}{A}\right)x-\left(\frac{ac}{A}-\frac{b}{B}\right)y,$$

is upper bounded on $\mathbb{R}^2_+$ by $b-ac$ if

$$\frac{acg}{A}\leq\frac{b}{B}. \tag{2.9}$$

Thus, according to Theorem 2.2, if condition (2.9) is met, then the system is controllable. The numerical solution of the problem leads to an approximation of the control function $\lambda(t)$ that can be related to the drug dosage required to achieve the desired patient evolution.

Example 3

We conclude this section of applications with the following example where we consider the well-known SIR epidemiological model

$$\begin{cases} S'=-aSI,\\ I'=aSI-bI,\\ R'=bI.\end{cases}$$

Here $S(t)$, $I\,(t)$ and $R\,(t)$ represent the number of susceptible, infected, and recovered/immunized individuals at time t, respectively, in a closed population of size N. Therefore, $S(t)+I(t)+R(t)=N$, which allows reducing the study to a two-dimensional Kolmogorov system

$$\begin{cases} S'=-aSI,\\ I'=aSI-bI.\end{cases}$$

Let S_0, I_0 and $R_0=N-(S_0+I_0)$ be the initial values of the three functions.

Introducing a constant vaccination rate λ, the system becomes

$$\begin{cases} S'=-aSI-\lambda,\\ I'=aSI-bI.\end{cases}$$

The control problem consists in finding the vaccination rate λ such that at time T, the immunized population $R(T)$ reaches a certain fraction pN of the total population N, with the desired value $p\in(0,1)$. The condition for controllability will be

$$S\,(T)+I\,(T)=(1-p)\,N.$$

The problem is a particular case of the general control problem (2.6). Here $\rho = N$, $\gamma = (1-p)\,N,\ x = S,\ y = I,\ f\,(S, I) = -aI$ and $g\,(S, I) = aS - b$. Through simple calculations, we have $M_N = \overline{M}_N = aN + b$. Thus, Theorem 2.4 guarantees that the system is uniquely controllable in time T if T is sufficiently small in the sense of inequalities (2.7). However, if an upper bound $\overline{\lambda}$ for the vaccination rate λ is imposed, then a lower bound for T is also necessary. Indeed, from

$$\lambda = \frac{1}{T}\,(x_0 + y_0 - \gamma) + \frac{1}{T}\int_0^T (x\,(s)\,f\,(x\,(s)\,, y\,(s)) + y\,(s)\,g\,(x\,(s)\,, y\,(s)))\,ds, \quad (2.10)$$

since $I \leq N$, we have

$$\begin{aligned} \overline{\lambda} \;\geq\; \lambda &= \frac{1}{T}\,(S_0 + I_0 - (1-p)\,N) + \frac{1}{T}\int_0^T (-aSI + (aS - b)\,I)\,ds \\ &= \frac{1}{T}\,(S_0 + I_0 - (1-p)\,N) - \frac{b}{T}\int_0^T I ds \geq \frac{1}{T}\,(S_0 + I_0 - (1-p)\,N) - bN \\ &= \frac{1}{T}\,(pN - R_0) - bN, \end{aligned}$$

from which

$$T \geq \frac{pN - R_0}{bN + \overline{\lambda}}.$$

Chapter 3

Vector fixed point approach to control of Kolmogorov differential systems

In this chapter, we will use the vector method for control problems related to systems of equations. The method is described for the case of Kolmogorov systems that frequently arise in population dynamics. Three types of problems are considered: problems with control of both *per capita* growth rates, problems with control parameters acting on growth rates, and problems that combine the first two types. Controllability is obtained using a vector approach based on Perov's fixed point theorem and matrices that converge to zero.

This chapter is divided into four sections. In **Section 3.1,** we deal with a first control problem where both controls act on *per capita* rates. In **Section 3.2,** the control is applied to growth rates and not to *per capita* rates, while in **Section 3.3,** the control acts on a single equation. Finally, in **Section 3.4,** we provide some examples that use the theorems presented in the three sections, concluding on the existence and uniqueness of solutions.

The results in this chapter were published in the work A. Hofman and R. Precup [15].

In the following we use the following numbers involving the initial and final values:

$$C_1 := |\ln x_0| + \left|\ln \frac{x_0}{x_T}\right|, \quad C_2 := |\ln y_0| + \left|\ln \frac{y_0}{y_T}\right|. \tag{3.1}$$

3.1 First control problem

We consider the control problem

$$\begin{cases} x'(t) = x(t)\left(f(x(t), y(t)) - \lambda\right) \\ y'(t) = y(t)\left(g(x(t), y(t)) - \mu\right). \end{cases} \tag{3.2}$$

With the assumption that the states x, y are bounded, the first result obtained guarantees that the system can be controlled uniquely. Here, the *per capita* rates of both populations are modified.

Theorem 3.1. *Let $\rho > 0$ be such that $\ln \rho > C_1$, $\ln \rho > C_2$, and let $f, g : [0, \rho]^2 \to \mathbb{R}$ be bounded by a constant $C > 0$. Assume that f and g satisfy the Lipschitz conditions*

$$|f(x, y) - f(\bar{x}, \bar{y})| \leq a_{11}\,|x - \bar{x}| + a_{12}|y - \bar{y}|, \tag{3.3}$$

$$|g(x, y) - g(\bar{x}, \bar{y})| \leq a_{21}\,|x - \bar{x}| + a_{22}|y - \bar{y}| \tag{3.4}$$

for all $x, y, \bar{x}, \bar{y} \in [0, \rho]$. Then, for each

$$0 < T \leq \min\left\{\frac{\ln \rho - C_1}{C},\ \frac{\ln \rho - C_2}{C}\right\} \tag{3.5}$$

for which the matrix

$$M := \rho T[a_{ij}]_{1 \leq i,j \leq 2} \tag{3.6}$$

converges to zero, the control problem (3.2) has a unique solution $(x^*, y^*, \lambda^*, \mu^*)$ with x^*, y^* positive and $\|x^*\|_\infty \leq \rho$, $\|y^*\|_\infty \leq \rho$.

For the next result instead of the Lipschitz conditions on f and g, we assume a logarithmic growth.

Theorem 3.2. *Let $f, g : \mathbb{R}_+^2 \to \mathbb{R}$ be continuous and satisfy logarithmic growth conditions*

$$\begin{aligned} |f(x, y)| &\leq a_{11}\,|\ln x| + a_{12}\,|\ln y| + b_1, \\ |g(x, y)| &\leq a_{21}\,|\ln x| + a_{22}\,|\ln y| + b_2, \end{aligned} \tag{3.7}$$

for all $x, y \in (0, \infty)$ and some constants $a_{ij},\ b_i \in \mathbb{R}_+\ (i, j = 1, 2)$. Then for each $T > 0$ for which the matrix

$$M = T[a_{ij}]$$

converges to zero, the control problem (3.2) has at least one solution $(x^*, y^*, \lambda^*, \mu^*)$ with $x^* > 0$ and $y^* > 0$.

3.2 Second control problem

We consider now the control problem

$$\begin{cases} x'(t) = x(t)f(x(t), y(t)) - \lambda \\ y'(t) = y(t)g(x(t), y(t)) - \mu, \end{cases} \tag{3.8}$$

when the control parameters act on the growth rates.

Theorem 3.3. *Assume that the functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ satisfy the following conditions:*

$$\begin{aligned} |xf(x,y) - \overline{x}f(\overline{x},\overline{y})| &\leq a_{11}|x - \overline{x}| + a_{12}|y - \overline{y}|, \\ |yg(x,y) - \overline{y}g(\overline{x},\overline{y})| &\leq a_{21}\,|x - \overline{x}| + a_{22}|y - \overline{y}|, \end{aligned}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}$, and the matrix

$$M = T\,[a_{ij}]_{1 \leq i,j \leq 2}$$

converges to zero. Then the control problem (3.8) has a unique solution.

3.3 Third control problem

For problem

$$\begin{cases} x'(t) = x(t)(f(x(t), y(t)) - \lambda) \\ y'(t) = y(t)g(x(t), y(t)) - \mu, \end{cases} \tag{3.9}$$

we apply again Perov's fixed point theorem by combining the techniques used for the first two problems. Thus we require the Lipschitz continuity of $f(x, y)$ and $yg(x, y)$.

Theorem 3.4. *Let $f, g : [0, \rho] \times \mathbb{R} \to \mathbb{R}$ be such that*

$$\begin{aligned} |f(x,y) - f(\bar{x},\bar{y})| &\leq a_{11}|x - \bar{x}| + a_{12}|y - \bar{y}|, \\ |yg(x,y) - \bar{y}g(\bar{x},\bar{y})| &\leq a_{21}|x - \bar{x}| + a_{22}|y - \bar{y}|, \end{aligned}$$

for all $x, \bar{x} \in [0, \rho]$ and $y, \bar{y} \in \mathbb{R}$. Assume that

$$|f(x,y)| \leq C$$

for $(x, y) \in [0, \rho] \times \mathbb{R}$,

$$C_1 + TC \leq \ln \rho \tag{3.10}$$

and that the matrix

$$M = T \begin{bmatrix} \rho a_{11} & a_{12} \\ \rho a_{21} & a_{22} \end{bmatrix} \tag{3.11}$$

is convergent to zero. Then the control problem has a unique solution $(x^*, y^*, \lambda^*, \mu^*)$ *such that* $x^* > 0$ *and* $\|x^*\|_\infty \leq \rho$ *.*

Remark 3.1. The proofs of the previous theorems show the advantage of the vectorial method compared to the scalar one, namely that it allows us, instead of several conditions imposed on the constants involved in the Lipschitz or growth inequalities, to formulate a single condition imposed cumulatively using a matrix whose elements are these constants.

3.4 Applications

Example 1

The following example illustrates Theorem 3.1. We consider the following self-limiting system

$$\begin{cases} x' = x\left(\frac{10^{-4}}{1+x^2+y^2} - \lambda\right) \\ y' = y\left(\frac{2\cdot 10^{-4}}{1+4x^2+y^2} - \mu\right), \end{cases}$$

where $T = 5,\ \rho = 100,\ x_0 = e,\ y_0 = e^2$ and the final controllability conditions are $x_5 = e^2$ and $y_5 = e$. We have $C = 2 \cdot 10^{-4}$,

$$\begin{array}{llll} \left|\frac{\partial f}{\partial x}\right| = \left|-\frac{2\cdot 10^{-4}x}{(1+x^2+y^2)^2}\right| \leq 10^{-4}, & \left|\frac{\partial f}{\partial y}\right| = \left|-\frac{2\cdot 10^{-4}y}{(1+x^2+y^2)^2}\right| \leq 10^{-4}, \\ \left|\frac{\partial g}{\partial x}\right| = \left|-\frac{8\cdot 2\cdot 10^{-4}x}{(1+4x^2+y^2)^2}\right| \leq 4\cdot 10^{-4}, & \left|\frac{\partial g}{\partial y}\right| = \left|-\frac{2\cdot 2\cdot 10^{-4}y}{(1+4x^2+y^2)^2}\right| \leq 2\cdot 10^{-4}. \end{array}$$

Thus, the Lipschitz conditions (3.3) and (3.4) become

$$\begin{aligned} |f(x,y) - f(\bar{x},\bar{y})| &\leq 10^{-4}|x-\bar{x}| + 10^{-4}|y-\bar{y}|, \\ |g(x,y) - g(\bar{x},\bar{y})| &\leq 4\cdot 10^{-4}|x-\bar{x}| + 2\cdot 10^{-4}|y-\bar{y}|. \end{aligned}$$

Also, in this case, using (3.1), we have $C_1 = 2$ and $C_2 = 3$. For $T = 5$, condition (3.5) is satisfied. In addition, the matrix M given by (3.6) is

$$M = 100 \cdot 5 \cdot \begin{bmatrix} 10^{-4} & 10^{-4} \\ 4\cdot 10^{-4} & 2\cdot 10^{-4} \end{bmatrix} = \begin{bmatrix} 5\cdot 10^{-2} & 5\cdot 10^{-2} \\ 20\cdot 10^{-2} & 10^{-1} \end{bmatrix}.$$

Let us recall the necessary and sufficient condition (1.4) for a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of order two to converge to zero

$$\text{tr } M < \min\{2,\ 1 + \det M\}, \tag{3.12}$$

that is

$$a + d < 2 \quad \text{şi} \quad a + d < 1 + ad - bc. \tag{3.13}$$

Note that if M converges to zero, then $a < 1$ and $d < 1$. In our case, we have

$$\begin{aligned} \text{tr } M &= 5\cdot 10^{-2} + 10^{-1} < 2, \\ \text{tr } M &< 1 + \det M = 1 + (5\cdot 10^{-2}\cdot 10^{-1} - 5\cdot 10^{-2}\cdot 20\cdot 10^{-2}). \end{aligned}$$

Therefore, this condition is satisfied for matrix M. We apply Theorem 3.1 and obtain that the control problem has a unique solution with $\|x^*\|_\infty \leq 100$ and $\|y^*\|_\infty \leq 100$.

Example 2

We apply Theorem 3.2 with the following choice of functions f and g:

$$\begin{aligned} f(x,y) &= \frac{1}{10}\frac{x}{x+y+1}\ln x + 1, \\ g(x,y) &= \frac{1}{10}\frac{y}{x+y+1}\ln y + 1 \quad (x,y>0), \end{aligned}$$

extended by continuity at $x=0$ and $y=0$, i.e., $f(0,y) = g(x,0) = 1 \ (x,y \in \mathbb{R}_+)$. Using the logarithmic growth conditions, we obtain the following relation

$$|f(x,y)| \leq \frac{1}{10}\frac{x}{x+y+1}\ |ln\ x| + 1,$$

since $x,y \in \mathbb{R}_+^2$ we have that $\frac{x}{x+y+1} \ \leq 1$. Therefore the first condition from (3.7) is satisfied with $a_{11} = \frac{1}{10}, a_{12} = 0, b_1 = 1$. Similarly,

$$|g(x,y)| \leq \frac{1}{10}|ln\ y| + 1$$

from which it results that $a_{21} = 0, a_{22} = \frac{1}{10}, b_2 = 1$. We verify if the hypotheses of Theorem 3.2 are fulfilled for $T = 5$ and if the matrix M is convergent to zero.

In this case, we have

$$\begin{aligned} \text{tr } M &= \frac{5}{10} + \frac{5}{10} < 2, \\ \text{tr } M &< 1 + \det M = 1 + \left(\frac{5}{10}\cdot\frac{5}{10}\right) = 3,5. \end{aligned}$$

Thus the matrix M is convergent to zero. The hypotheses for Theorem 3.2 are fulfilled for $T = 5$ and the matrix convergent to zero is

$$M = \begin{bmatrix} 0,5 & 0 \\ 0 & 0,5 \end{bmatrix}.$$

We obtain that the corresponding Kolmogorov system is controllable for any values x and y.

Example 3

The following functions satisfy the hypotheses of Theorem 3.3

$$\begin{aligned} f(x,y) &= \frac{1}{10}(1+\sin y)\frac{\sin x}{x}, \\ g(x,y) &= \frac{1}{10}(1+\sin x)\frac{\sin y}{y}. \end{aligned}$$

Here it is understood that $f(0,y) = \frac{1}{10}(1+\sin y)$ and $g(x,0) = \frac{1}{10}(1+\sin x)$. We have that

$$\left|\frac{\partial(xf(x,y))}{\partial x}\right| = \left|\frac{cos\ x\cdot(1+sin\ y)}{10}\right| \le \frac{2}{10}, \quad \left|\frac{\partial(xf(x,y))}{\partial y}\right| = \left|\frac{sin\ x\cdot cos\ y}{10}\right| \le \frac{1}{10},$$
$$\left|\frac{\partial(yg(x,y))}{\partial x}\right| \le \frac{1}{10}, \quad \left|\frac{\partial(yg(x,y))}{\partial y}\right| \le \frac{2}{10}.$$

Therefore, the Lipschitz conditions become

$$\begin{aligned} |xf(x,y) - \bar{x}f(\bar{x},\bar{y})| &\le \frac{2}{10}|x-\bar{x}| + \frac{1}{10}|y-\bar{y}|, \\ |yg(x,y) - \bar{y}g(\bar{x},\bar{y})| &\le \frac{1}{10}|x-\bar{x}| + \frac{2}{10}|y-\bar{y}|. \end{aligned}$$

We verify if the hypotheses of Theorem 3.3 are satisfied for $T = 3$ and that in this case the matrix M is convergent to zero. Here the matrix M is

$$M = 3\cdot\begin{bmatrix} \frac{2}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{2}{10} \end{bmatrix} = \begin{bmatrix} 0,6 & 0,3 \\ 0,3 & 0,6 \end{bmatrix}.$$

We verify the necessary and sufficient condition for a matrix of order 2 to be convergent to zero, namely

$$\text{tr } M < \min\{2,\ 1+\det M\}.$$

We have that

$$\begin{aligned} \text{tr } M &= \frac{6}{10} + \frac{6}{10} < 2, \\ \text{tr } M &< 1 + \det\ M = 1 + 2,7 = 3,7. \end{aligned}$$

The condition is satisfied and therefore it follows that the matrix M is convergent to zero. We apply Theorem 3.3 and the obtain that the control problem has a unique solution.

Example 4

We illustrate Theorem 3.4 by considering the following functions:

$$\begin{aligned} f(x,y) &= \frac{1}{100\left(1+x^2+y^2\right)}, \\ g(x,y) &= \frac{1}{100}\left(1+\sin x\right)\frac{\sin y}{y}, \end{aligned}$$

for which $a_{11} = a_{12} = a_{21} = \frac{1}{100}$, $a_{22} = \frac{2}{100}$ and $C = \frac{1}{100}$, independent of ρ. Taking $x_0 = 1$ and $x_T = e$, we have $C_1 = 1$. Furthermore, taking $T = 10$ and $\rho = e^2$ all the hypotheses of Theorem 3.4 are satisfied. Here the matrix M is

$$M = \frac{1}{10}\begin{bmatrix} e^2 & 1 \\ e^2 & 2 \end{bmatrix}$$

and it is immediately observed that it converges to zero. According to Theorem 3.4 the control problem has a unique solution such that $x^* > 0$ and $\|x^*\|_\infty \leq \rho$.

Chapter 4

A fixed-point approach to control problems for Kolmogorov type second-order equations and systems

In this chapter, second-order differential equations and systems of the Kolmogorov type are defined. Unlike first-order equations that express the *per capita* rate, second-order equations express the rate of change of the *per capita* rate. Several control problems with fixed finite time T and fixed final state x_T are studied, with additive or multiplicative control. Their controllability is demonstrated using fixed-point methods, the theorems of Banach, Schauder, Krasnoselskii, Avramescu, and Perov.

This chapter is divided into two sections. **Section 4.1** contains two subsections. In **Subsection 4.1.1,** we study an additive control problem using Banach's fixed-point theorem to demonstrate the existence and uniqueness of the solution. We investigate the existence of a solution in the following Theorem 4.2 using logarithmic growth conditions and Schauder's fixed-point theorem. We continue with the next result, which combines the two previously mentioned results, using Krasnoselskii's fixed-point theorem for a sum of two operators. The last result refers to a problem with multiplicative control for which Banach's contraction theorem is used.

We continue with **Section 4.2,** where we use the fixed-point theorems of Perov, Schauder, and Avramescu to demonstrate the controllability of second-order Kolmogorov systems.

The results in this chapter were published in the work of A. Hofman and R. Precup [14].

4.1 Control of second order Kolmogorov equations

4.1.1 Problems with additive control

We consider the following control problem of a second-order Kolmogorov equation

$$\begin{cases} \left(\frac{x'(t)}{x(t)}\right)' = f(t,x(t)) - \lambda \\ x(0) = a, \quad x'(0) = 0 \\ x > 0 \text{ on } [0,T], \quad x(T) = x_T, \end{cases} \tag{4.1}$$

where $a,\ x_T > 0$, and the additive control λ is scalar.

Our first result is an existence and uniqueness theorem of the solution (x,λ) of the control problem with x in a ball of a given radius ρ of the space $C[0,T]$ endowed with the Chebyshev norm $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$. Denote

$$\alpha = \ln a, \quad u_T = \ln x_T, \quad \gamma = 2\frac{\alpha - u_T}{T^2}, \quad R = \ln \rho$$

and for a nonnegative number L, in case that $L = 0$, by $\frac{1}{L}$ let it mean $+\infty$.

Theorem 4.1. *Let $L \in \mathbb{R}_+$ and $\rho > 0$. Assume that*

$$\exp(\max\{|\alpha|, |u_T|\} + 1) \le \rho < \frac{2}{LT^2} \tag{4.2}$$

and the function $f : [0,T] \times [0,\rho] \to \mathbb{R}$ is continuous, $f(\cdot,0) \equiv 0$ and satisfies the Lipschitz condition

$$|f(t,v) - f(t,\overline{v})| \le L|v - \overline{v}|, \tag{4.3}$$

for all $t \in [0,T]$, $v, \overline{v} \in [0,\rho]$. Then the control problem has a unique solution (x^*,λ^*) with $x^* > 0$, $\|x^*\|_\infty \le \rho$, and

$$\lambda^* = \frac{2}{T^2}\left(\alpha - \ln x_T + \int_0^T \int_0^\tau f(s,x^*(s))dsd\tau\right). \tag{4.4}$$

Remark 4.1. *Here again, if we do not require for the solution to satisfy $\|x^*\|_\infty \le \rho$, that is, the radius ρ is not a priori given, but we assume however that $f : [0,T] \times (0,+\infty) \to \mathbb{R}$ is continuous and satisfies the growth condition (5.5) for all $t \in [0,T]$, $v \in (0,+\infty)$ and some constants $l_1, l_2 \in \mathbb{R}_+$ with $l_1 < \frac{2}{T^2}$, then the control problem has at least one solution. This statement is obvious if we use the above result for any $\rho \ge \rho_0$ where*

$$\rho_0 = \exp\left(\frac{2\max\{|\alpha|, |u_T|\} + T^2 l_2}{2 - T^2 l_1}\right).$$

Taking in particular $\rho = \rho_0$, *we find a solution with* $\|x^*\|_\infty \leq \rho_0$.

Using Schauder's fixed point theorem, we do not need f to satisfy a Lipschitz condition. Instead we will assume a logarithmic growth condition.

Theorem 4.2. *Assume that the function* $f : [0,T] \times [0,\rho] \to \mathbb{R}$ *is continuous and satisfies the growth condition*

$$|f(t,v)| \leq l_1 |\ln v| + l_2, \tag{4.5}$$

for all $t \in [0,T]$, $v \in (0,\rho]$ *and some constants* $l_1, l_2 \in \mathbb{R}_+$ *with* $l_1 < \frac{2}{T^2}$. *In addition assume that*

$$\rho \geq \exp\left(\frac{2\max\{|\alpha|, |u_T|\} + T^2 l_2}{2 - T^2 l_1}\right). \tag{4.6}$$

Then the control problem has at least one solution (x^*, λ^*) *with* $x^* > 0$, $\|x^*\|_\infty \leq \rho$, *and* λ^* *given by* (4.4).

Remark 4.2. *Here again, if we do not require for the solution to satisfy* $\|x^*\|_\infty \leq \rho$, *that is, the radius* ρ *is not a priori given, but we assume however that* $f : [0,T] \times (0,+\infty) \to \mathbb{R}$ *is continuous and satisfies the growth condition (4.5) for all* $t \in [0,T]$, $v \in (0,+\infty)$ *and some constants* $l_1, l_2 \in \mathbb{R}_+$ *with* $l_1 < \frac{2}{T^2}$, *then the control problem has at least one solution. This statement is obvious if we use the above result for any* $\rho \geq \rho_0$ *where*

$$\rho_0 = \exp\left(\frac{2\max\{|\alpha|, |u_T|\} + T^2 l_2}{2 - T^2 l_1}\right).$$

Taking in particular $\rho = \rho_0$, *we find a solution with* $\|x^*\|_\infty \leq \rho_0$.

The next result combines the two previous ones assuming that f splits as $f = f_1 + f_2$, where f_1 satisfies a Lipschitz condition while f_2 satisfies a logaritmic growth condition. The result is based on Krasnoselskii's fixed point theorem for a sum of two operators. Here sign $L = 1$ if $L > 0$ and sign $L = 0$ if $L = 0$.

Theorem 4.3. *Assume that the function* $f : [0,T] \times [0,\rho] \to \mathbb{R}$ *is continuous and splits as* $f = f_1 + f_2$, *where* f_1 *is like in Theorem 4.1 and* f_2 *is like in Theorem 4.2. In addition assume that*

$$\rho \geq \exp\left(\frac{2\max\{|\alpha|, |u_T|\} + T^2 l_2 + 2sign\ L}{2 - T^2 l_1}\right). \tag{4.7}$$

Then the control problem has at least one solution (x^*, λ^*) *with* $x^* > 0$, $\|x^*\|_\infty \leq \rho$, *and* λ^* *given by (4.4).*

Since $f = f_1 + f_2$, it can be observed that if $L > 0$ and $f_2 = 0$, then $f = f_1$ and it reduces to Theorem 4.1 (where $l_1 = l_2 = 0$), while Theorem 4.2 reduces to Theorem 4.3 if $f_1 = 0$, when $L = 0$ and $f = f_2$.

4.1.2 Problem with a multiplicative control

We consider the following control problem

$$\begin{cases} \left(\frac{x'(t)}{x(t)}\right)' = \lambda f(t, x(t)) \\ x(0) = a, \quad x'(0) = 0 \\ x > 0 \text{ on } [0, T], \quad x(T) = x_T, \end{cases} \tag{4.8}$$

with the multiplicative control parameter λ.

We have the following result on the unique controllability of the problem under a given bound of the positive solution of the equation.

Theorem 4.4. *Let*

$$\rho \geq \exp\left(|\alpha| + |u_T - \alpha|\right),$$

$\alpha \neq u_T$ *and* $f : [0, T] \times [0, \rho] \to (0, +\infty)$ *a continuous function satisfying the Lipschitz condition*

$$|f(t, x) - f(t, y)| \leq L|x - y|$$

for all $t \in [0, T]$ *and* $x, y \in (0, \rho]$. *If*

$$L < \frac{f_\rho}{\rho T^2 |u_T - \alpha|},$$

where $f_\rho := \int_0^T \int_0^\tau \min_{x \in [\frac{1}{\rho}, \rho]} f(s, x) ds d\tau$, *then there exists a unique solution* (x^*, λ^*) *of the control problem (4.8) with* $x^* > 0$, $\|x^*\|_\infty \leq \rho$, *and*

$$\lambda^* = \frac{u_T - \alpha}{\int_0^T \int_0^\tau f(s, x^*) ds d\tau}.$$

4.2 Control of second order Kolmogorov systems

We consider the following control second-order Kolmogorov system

$$\begin{cases} \left(\frac{x'(t)}{x(t)}\right)' = f(x(t), y(t)) - \lambda \\ \left(\frac{y'(t)}{y(t)}\right)' = g(x(t), y(t)) - \mu \\ x(0) = a, \; x'(0) = 0, \quad y(0) = b, \; y'(0) = 0 \\ x(T) = x_T, \quad y(T) = y_T, \end{cases} \tag{4.9}$$

where $a, b, \; x_T, \; y_T > 0$ and the controls λ and μ are constant.

Denote

$$R = \ln \rho, \quad u_T = \ln x_T, \quad v_T = \ln y_T, \quad \alpha = \ln a, \quad \beta = \ln b,$$
$$\gamma = \frac{2}{T^2}\left(\alpha - u_T\right), \quad \theta = \frac{2}{T^2}\left(\beta - v_T\right).$$

The next theorem guarantees the unique controllability of the system in a given ball.

Theorem 4.5. *Let*

$$\rho \geq \exp\left(1 + \max\left\{|\alpha|, |u_T|, |\beta|, |v_T|\right\}\right) \tag{4.10}$$

and assume that the functions $f, g : [0, \rho]^2 \to \mathbb{R}$ *satisfy* $f(0, \cdot) \equiv 0,\ g(\cdot, 0) \equiv 0$ *and the Lipschitz conditions*

$$\begin{aligned} |f(x,y) - f(\overline{x},\overline{y})| &\leq a_{11}|x - \overline{x}| + a_{12}|y - \overline{y}|, \\ |g(x,y) - g(\overline{x},\overline{y})| &\leq a_{21}|x - \overline{x}| + a_{22}|y - \overline{y}|, \end{aligned}$$

for all $x, y, \overline{x}, \overline{y} \in [0, \rho]$ *and some nonnegative constants* $a_{ij}\ \ (i, j = 1, 2)$, *and that the matrix*

$$M = \frac{\rho T^2}{2}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is convergent to zero. Then the control problem has a unique solution $(x^*, y^*, \lambda^*, \mu^*)$ *with* $x^*, y^* > 0$ *and* $\|x^*\|_\infty,\ \|y^*\|_\infty \leq \rho$.

Here again, if instead of the Lipschitz conditions, f and g only have a logarithmic growth, then one can prove the existence of a least one solution of the control problem.

Theorem 4.6. *Let* $f, g : \mathbb{R}^2_+ \to \mathbb{R}$ *be continuous and satisfy the logarithmic growth conditions*

$$\begin{aligned} |f(x,y)| &\leq a_{11}\left|\ln x\right| + a_{12}\left|\ln y\right| + b_1, \\ |g(x,y)| &\leq a_{21}\left|\ln x\right| + a_{22}\left|\ln y\right| + b_2, \end{aligned} \tag{4.11}$$

for all $x, y \in (0, \infty)$ *and some constants* $a_{ij},\ b_i \in \mathbb{R}_+\ \ (i, j = 1, 2)$. *Then for each* $T > 0$ *for which the matrix*

$$M = \frac{T^2}{2}\left[a_{ij}\right]_{1 \leq i,j \leq 2}$$

converges to zero, the control problem (4.9) *has at least one solution* $(x^*, y^*, \lambda^*, \mu^*)$ *with* $x^* > 0$ *and* $y^* > 0$.

Our last result is an application of Avramescu's fixed point theorem to control problem (4.9) when f satisfies a Lipschitz condition with respect to the first variable only, and g has a logarithmic growth in the last variable.

Theorem 4.7. *Let ρ be such that*

$$\rho \geq \max\left\{\exp\left(1+\max\left\{|\alpha|,|u_T|\right\}\right),\ \exp\frac{2\max\left\{|\beta|,|v_T|\right\}+T^2c}{2-T^2b}\right\} \tag{4.12}$$

and $f,g:[0,\rho]^2\to\mathbb{R}$ be continuous and $f(0,\cdot)\equiv 0$. Assume that

$$\begin{aligned} |f(x,y)-f(\overline{x},y)| &\leq a|x-\overline{x}| \quad \text{for all } x,\overline{x},y\in[0,\rho], \\ |g(x,y)| &\leq b|\ln y|+c \quad \text{for all } x\in[0,\rho],\ y\in(0,\rho], \end{aligned}$$

where $a<\frac{2}{\rho T^2}$ and $b<\frac{2}{T^2}$. Then problem (4.9) has at least one solution $(x^*,y^*,\lambda^*,\mu^*)$ with $x^*,y^*>0$ and $\|x^*\|_\infty,\|y^*\|_\infty\leq\rho$.

Chapter 5

Fixed point methods with multi-valued operators for control problems

In this chapter, we deal with control problems for first-order Kolmogorov equations, where the controllability condition is given by an inclusion. We use fixed-point techniques based on Nadler's, Bohnenblust-Karlin's theorems, and on the multivalued version of Krasnoselskii's theorem for a sum of two operators [9, 18, 25]. For other fixed-point techniques in control theory, we refer to [7, 30].

This chapter comprises three sections. In **Section 5.1,** we present the first result concerning the control of first-order multivalued Kolmogorov equations, where we used Nadler's fixed-point theorem, proving the existence of a solution within a ball of radius ρ in the space $C[0,T]$ endowed with the Chebyshev norm. In **Section 5.2,** we used Bohnenblust-Karlin's fixed-point theorem, considering, instead of the Lipschitz condition on the function f, a more general condition of at most linear growth. Finally, in **Section 5.3,** we presented an application of the multivalued version of Krasnoselskii's fixed-point theorem obtained in [25]. This guarantees the existence of at least one solution to the control problem.

The results in this chapter were published in A. Hofman [12].

In the following we deal with the control problem

$$\begin{cases} x'(t) = x(t)f(t,x(t)) - \lambda x \\ x(0) = x_0 \\ x > 0 \text{ on } \quad [0,T]\,, \quad x_T := x(T) \in [a,b], \end{cases} \tag{5.1}$$

where the controllability condition is of inclusion type, more precisely $x(T) \in [a,b]$. Here $0 < a < b$ and λ is a constant parameter.

5.1 Application of Nadler's fixed point theorem

Our first result is an existence theorem of solution (x, λ) of the control problem with x in a ball of a given radius ρ of the space $C[0,T]$ endowed with the Chebyshev norm $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$. In what follows we make the notations:

$$\alpha = \ln a, \quad \beta = \ln b, \ u_0 = \ln x_0, \ u_T = \ln x_T, \ R = \ln \rho.$$

To prove our result we need the following lemma.

Lemma 5.1. *Let $(X, ||\cdot||)$ be a normed space, x_1 and $x_2 \in X$ and M a bounded subset of X. Then we have*

$$H(x_1 + M, x_2 + M) \leq ||x_1 - x_2||.$$

The lemma above is applied when for an operator N, the value $N(u)$ is a sum of a function and an interval of functions.

Theorem 5.1. *Let $\rho > 0$ and the function $f : [0,T] \times [0,\rho] \to \mathbb{R}$ be continuous, $f(\cdot, 0) \equiv 0$ and the following Lipschitz condition hold*

$$|f(t,v) - f(t,\overline{v})| \leq L|v - \overline{v}|, \tag{5.2}$$

for all $t \in [0,T]$, $v, \overline{v} \in [0,\rho]$, with $0 < L < \frac{1}{\rho T}$. In addition, we assume that

$$\rho \geq \exp\left(|u_0| + \max\{\beta, |\alpha|\} + 1\right). \tag{5.3}$$

Then the control problem (5.1) has solutions (x^*, λ^*) with $x^* > 0$, $\|x^*\|_\infty \leq \rho$ and

$$\lambda^* \in \left[\frac{1}{T}\left(u_0 - \beta + \int_0^T f(s, x^*(s))ds\right), \ \frac{1}{T}\left(u_0 - \alpha + \int_0^T f(s, x^*(s))ds\right)\right]. \tag{5.4}$$

5.2 Application of Bohnenblust-Karlin's fixed point theorem

Using Bohnenblust-Karlin fixed point theorem, we do not need f to satisfy a Lipschitz condition. Instead we will assume a logarithmic growth condition.

Theorem 5.2. *Assume that the function $f : [0,T] \times [0,\rho] \to \mathbb{R}$ is continuous and satisfies the growth condition*

$$|f(t,v)| \leq l_1 |\ln v| + l_2, \tag{5.5}$$

for all $t \in [0, T]$, $v \in (0, \rho]$ *and some constants* $l_1, l_2 \in \mathbb{R}_+$ *cu* $l_1 < \frac{1}{T}$. *with* $l_1 < \frac{1}{T}$. *In addition assume that*

$$\rho \geq \exp\left(\frac{C + l_2 T}{1 - l_1 T}\right), \tag{5.6}$$

where $C := |u_0| + \max\{\beta, |\alpha|\}$. *Then the control problem (5.1) has at least one solution* (x^*, λ^*) *with* $x^* > 0$, $\|x^*\|_\infty \leq \rho$, *and* λ^* *given by (5.4).*

Lemma 5.2. *Let* N *be a multi-valued map from* $D \subset C[0, T]$, *to the subsets of* $C[0, T]$ *defined as*

$$N(u)(t) = \Gamma(u)(t) + \frac{t}{T}[\alpha, \beta],$$

where $\Gamma : D \to C[0, T]$ *is a continuous single-valued map and by* $\frac{t}{T}[\alpha, \beta]$ *one means the set of continuous functions*

$$\left\{ z \in C[0, T] : \; z(t) = \frac{t}{T}\zeta, \quad \zeta \in [\alpha, \beta] \right\}.$$

Then N *is upper semicontinuous.*

5.3 Application of Krasnoselskii's fixed point theorem

Noticing that the expression of $\Gamma(u)$ contains both Volterra and Fredholm integral terms, we are led to use Krasnoselskii's fixed point theorem for a sum of two operators. Thus we have

Theorem 5.3. *Let* $f : [0, T] \times [0, \rho] \to [-C, C]$ *be continuous and*

$$|f(t, v) - f(t, \overline{v})| \leq L|v - \overline{v}|, \tag{5.7}$$

for all $v, \overline{v} \in [0, \rho]$, $t \in [0, T]$ *and some* $C, L > 0$. *In addition assume that*

$$\rho \geq \exp\left(|u_0| + \max\{\beta, |\alpha|\} + 2TC\right). \tag{5.8}$$

Then the control problem (5.1) has solutions (x^*, λ^*) *with* $x^* > 0$, $||x^*||_\infty \leq \rho$ *and* λ^* *as in (5.4).*

Compared with the result given by Theorem 5.1, in this case, there is no restriction on the Lipschitz constant L.

Chapter 6

Algorithms for solving control problems related to Kolmogorov systems

In this chapter, we will use the method of lower and upper solutions to construct an iterative algorithm that allows obtaining the solution to the control problem for Kolmogorov type systems. We will demonstrate that, under the hypothesis of relatively simple conditions, the proposed algorithm is convergent towards the solution of the problem. The result regarding the convergence of the algorithm also represents an existence result for the control problem.

Throughout this chapter, we study two types of Kolmogorov type systems. These systems depend on a real parameter λ, that represents the control variable, and our goal is to find a solution (x, y) such that the following control condition is satisfied:

$$\varphi(x, y) = 0.$$

The function $\varphi : C([0, T]; \mathbb{R}^2) \to \mathbb{R}$ is a general continuous function that satisfies certain specific conditions. A relevant example in practice of choosing φ is the following

$$\varphi(x, y) = \alpha x(T) + \beta y(T) - \gamma, \quad \text{where } \alpha, \beta, \gamma \in \mathbb{R}.$$

The algorithm we propose in what follows requires that the control problem (the system) satisfies two fundamental requirements: for any value of λ, it admits a unique solution and that this solution continuously depends on the parameter λ. These requirements are essential to ensure the stability and convergence of the proposed method.

The algorithm is applied in the presence of a lower solution $(\underline{x}, \underline{y}, 0)$ and upper solution $(\overline{x}, \overline{y}, 1)$ of the control problem.

The triplet $(\underline{x}, \underline{y}, 0)$ is a lower of the control problem if $(\underline{x}, \underline{y})$ is a solution to the Cauchy problem and $\varphi(\underline{x}, \underline{y}) < 0$ and $(\overline{x}, \overline{y}, 1)$ is a upper solution of the control problem if $(\overline{x}, \overline{y})$ is a solution to the Cauchy problem and $\varphi(\overline{x}, \overline{y}) > 0$.

The general form of the algorithm is as follows:

Algorithm 1 (Bisection method)**.**

Step 1. Initialize $\underline{\lambda}_0 := 0$, $\overline{\lambda}_0 := 1$, $\underline{x}_0 := \underline{x}$, $\underline{y}_0 := \underline{y}$, $\overline{x}_0 := \overline{x}$, $\overline{y}_0 := \overline{y}$.

Step 2. At any iteration $k \geq 1$, *define*

$$\lambda_k := \frac{\underline{\lambda}_{k-1} + \overline{\lambda}_{k-1}}{2},$$

and solve the system for $\lambda := \lambda_k$, *obtaining the solution* (x_k, y_k). *If* $\varphi(x_k, y_k) < 0$, *then for the next step choose*

$$\underline{\lambda}_k := \lambda_k,\ \overline{\lambda}_k := \overline{\lambda}_{k-1},\ \underline{x}_k := x_k,\ \underline{y}_k := y_k,\ \overline{x}_k := \overline{x}_{k-1},\ \overline{y}_k := \overline{y}_{k-1},$$

otherwise, if $\varphi(x_k, y_k) > 0$, *consider*

$$\underline{\lambda}_k := \underline{\lambda}_{k-1},\ \overline{\lambda}_k := \lambda_k,\ \underline{x}_k := \underline{x}_{k-1},\ \underline{y}_k := \underline{y}_{k-1},\ \overline{x}_k := x_k,\ \overline{y}_k := y_k,$$

and then repeat Step 2 for $k := k + 1$.

Step 3 (Stopping condition).

The algorithm stops when

$$|\varphi(x_k, y_k)| < \delta,$$

where $\delta > 0$ *represents an acceptable error.*

This chapter is structured into two sections. In **Section 6.1,** we use the bisection algorithm to determine a solution to a first-order Kolmogorov problem such that the control condition is satisfied. We will establish the conditions under which the system admits a unique solution and show that it depends continuously on the control parameter. We will also demonstrate the convergence of the bisection algorithm, thus obtaining a solution to the control problem considered.

The results of this section were published in the work of A. Hofman [11].

For **Section 6.2,** we will proceed in a similar manner to the first section, but for a second-order Kolmogorov system. In this context, using the method of lower and upper solutions, we will adapt the bisection algorithm for the considered problem and demonstrate its convergence. We will also analyze both the algorithm that obtains an exact solution and the approximate algorithm.

The results of this section were published in the work of A. Hofman [10].

6.1 The method of lower and upper solutions for first-order Kolmogorov systems

In what follows, we will present the control problem to which we will apply the bisection algorithm. To begin, we consider a problem with the control of the growth rate of the first equation, namely the problem

$$\begin{cases} x'(t) = x(t)f(x(t), y(t)) - \lambda \\ y'(t) = y(t)g(x(t), y(t)) \\ x(0) = x_0, \ y(0) = y_0. \end{cases} \tag{6.1}$$

Here λ is constant and the controllability condition is

$$\varphi(x, y) = 0,$$

where the function φ is assumed to be continuous.

To be able to use the algorithm above, it is necessary, first of all, to determine the conditions under which problem (6.1) admits a unique solution and, moreover, this solution is continuously dependent on the parameter.

Lemma 6.1. *Assume that $f, g : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz continuous functions and that $|f| \leq C_f$,$|g| \leq C_g$. Then, for any $\lambda \in \mathbb{R}$, the Cauchy problem* (6.1) *admits a unique solution, which depends continuously on the parameter λ.*

Alternatively, we have

Lemma 6.2. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be such that the functions $xf(x, y)$ and $yg(x, y)$ are Lipschitz continuous on the space $\mathbb{R}^2$. Then for any $\lambda \in \mathbb{R}$, the Cauchy problem (6.1) has a unique solution that depends continuously on the parameter λ.*

From Lemmas 6.1 and 6.2, we obtain two convergence results based on Algorithm 1.

Theorem 6.1. *Assume that $f, g : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz continuous functions on $\mathbb{R}^2$ and that $|f| \leq C_f$, $|g| \leq C_g$. In addition, assume that*

$$\varphi(S_1(0), S_2(0)) < 0 \text{ and } \varphi(S_1(1), S_2(1)) > 0. \tag{6.2}$$

Then, Algorithm 1 is convergent to a solution of the control problem.

Similarly, using Lemma 6.2, the following result can be demonstrated.

Theorem 6.2. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that the functions $xf(x,y)$, $yg(x,y)$ are Lipschitz continuous on the space $\mathbb{R}^2$. If (6.2) holds, then the algorithm is convergent to a solution of the control problem.*

6.2 The method of lower and upper solutions for the control of second-order Kolmogorov systems

In this Subsection, using a similar approach to the previous one, we will introduce a method of sub and super solutions for the control of second-order Kolmogorov systems. Two iterative algorithms are defined, one exact and one approximate, and their convergence is studied. The method of work uses the fixed-point technique based on Perov's theorem, matrices converging to zero, and Bielecki-type norms.

Thus, we deal with control problems of the type

$$\begin{cases} \left(\frac{x'(t)}{x(t)}\right)' = f(x(t), y(t), \lambda) \\ \left(\frac{y'(t)}{y(t)}\right)' = g(x(t), y(t), \lambda), \end{cases} \tag{6.3}$$

for $t \in [0, T]$, with the initial values

$$x(0) = a, \quad x'(0) = 0, \quad y(0) = b, \; y'(0) = 0, \tag{6.4}$$

where $a, b > 0$. Here λ is a vector from $\mathbb{R}^m$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$. Such kind of problems have applications to various domains, particularly in biomathematics. The controllability condition is

$$\Psi(x(T), y(T)) = 0,$$

where $\Psi : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. For example, we can take

$$\Psi(s, \tau) = s - k\tau \quad \text{or} \quad \Psi(s, \tau) = s - k,$$

with k a given constant.

First, in the present set-framework, we introduce the notions of lower and upper solutions the control problem. (see [23], [11]).

Definition 6.1. *We call a lower solution of the control problem a triple $(\underline{x}, \underline{y}, \underline{\lambda})$ where $(\underline{x}, \underline{y})$ is a solution of the Cauchy problem with $\lambda = \underline{\lambda}$ and*

$$\Psi(\underline{x}(T), \underline{y}(T)) < 0.$$

Definition 6.2. *A triple $(\overline{x}, \overline{y}, \overline{\lambda})$ is an upper solution of the control problem if $(\overline{x}, \overline{y})$ is the solution of the Cauchy problem with $\lambda = \overline{\lambda}$ and*

$$\Psi(\overline{x}(T), \overline{y}(T)) > 0.$$

Lower and upper solutions can be obtained with the aid of the computer by repeated trials giving various control variable values.

For the convergence of the algorithm, we must guarantee that the Cauchy problem (6.3)-(6.4) has a unique solution for each λ and that it depends continuously on the parameter λ.

Theorem 6.3. *Let $\alpha = ln\ a,\ \beta = ln\ b$ and*

$$\rho \geq \exp\left(1 + \max\left\{|\alpha|, |\beta|\right\}\right). \tag{6.5}$$

Assume that $f, g : \mathbb{R}^2 \times \mathbb{R}^m \to \mathbb{R}$ with $f(0, y, \lambda) \equiv 0,\ g(x, 0, \lambda) \equiv 0$, for any $x, y \in \mathbb{R}$, $\lambda \in \mathbb{R}^m$ satisfy the Lipschitz conditions

$$\begin{aligned} |f(x,y,\lambda) - f(\overline{x},\overline{y},\mu)| &\leq a_{11}|x-\overline{x}| + a_{12}|y-\overline{y}| + a_{13}|\lambda-\mu|, \\ |g(x,y,\lambda) - g(\overline{x},\overline{y},\mu)| &\leq a_{21}|x-\overline{x}| + a_{22}|y-\overline{y}| + a_{23}|\lambda-\mu|, \end{aligned}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$, and $\lambda, \mu \in \mathbb{R}^m$. In addition assume that matrix

$$M = \frac{\rho T^2}{2} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{6.6}$$

is convergent to zero. Then for any $\lambda \in \mathbb{R}^m$ the Cauchy problem (6.3)-(6.4) has a unique solution (x, y) satisfying $\|x\|_\infty \leq \rho$ and $\|y\|_\infty \leq \rho$, that depends continuously on the parameter λ.

In what follows, we present an algorithm that, in the limit, converges towards the solution of the control problem. We call it an exact algorithm because, by repeating the process a number of times (possibly infinite), the solution to the problem is reached.

6.2.1 Exact algorithm.

Let $(\underline{x}, \underline{y}, \underline{\lambda})$ and $(\overline{x}, \overline{y}, \overline{\lambda})$ be lower and upper solutions of the control problem with $\underline{\lambda} < \overline{\lambda}$.

Step 1. Initialize $\underline{\lambda}_0 := \underline{\lambda}$, $\overline{\lambda}_0 := \overline{\lambda}$, $\underline{x}_0 := \underline{x}$, $\underline{y}_0 := \underline{y}$, $\overline{x}_0 := \overline{x}$, $\overline{y}_0 := \overline{y}$.

Step 2. At any iteration $k \geq 1$, define $\lambda_k := \dfrac{\underline{\lambda}_{k-1} + \overline{\lambda}_{k-1}}{2}$ and solve problem

(6.3)-(6.4) for $\lambda = \lambda_k$. Obtain the solution

$$(x_k, y_k) = (e^{S_1(\lambda_k)}, e^{S_2(\lambda_k)}).$$

If $\Psi(x_k(T), y_k(T)) < 0$, then put

$$\underline{\lambda}_k := \lambda_k,\ \overline{\lambda}_k := \overline{\lambda}_{k-1},\ \underline{x}_k := x_k,\ \underline{y}_k := y_k,\ \overline{x}_k := \overline{x}_{k-1},\ \overline{y}_k := \overline{y}_{k-1},$$

otherwise, for $\Psi(x_k(T), y_k(T)) > 0$, take

$$\underline{\lambda}_k := \underline{\lambda}_{k-1},\ \overline{\lambda}_k := \lambda_k,\ \underline{x}_k := \underline{x}_{k-1},\ \underline{y}_k := \underline{y}_{k-1},\ \overline{x}_k := x_k,\ \overline{y}_k := y_k$$

and we repeat Step 2 for $k := k + 1$. Obviously, if $\Psi(x_k(T), y_k(T)) = 0$, then we have the solution and we are finished.

Step 3 (Stopping condition).

The algorithm stops if

$$|\Psi(x_k, y_k)| < \delta,$$

for a given error $\delta > 0$.

Using the Theorem 6.3 we can prove obtain the convergence result of the above algorithm.

Theorem 6.4. *Under the assumptions of Theorem 6.3, the algorithm is convergent to a solution of the control problem.*

We assume that the Cauchy problem can be approximately solved with a desired error ε. In this situation, the algorithm changes as follows.

6.2.2 Approximation algorithm.

Let $\varepsilon > 0$ be an admissible error, $(\tilde{\underline{x}}, \tilde{\underline{y}}, \tilde{\underline{\lambda}})$ and $(\tilde{\overline{x}}, \tilde{\overline{y}}, \tilde{\overline{\lambda}})$ be approximate lower and upper solutions of the Cauchy problem.

Step 1. Initialize $\underline{\lambda}_0 := \widetilde{\underline{\lambda}}$, $\overline{\lambda}_0 := \widetilde{\overline{\lambda}}$, $\widetilde{\underline{x}}_0 := \widetilde{\underline{x}}$, $\tilde{\underline{y}}_0 := \widetilde{\underline{y}}$, $\widetilde{\overline{x}}_0 := \widetilde{\overline{x}}$, $\widetilde{\overline{y}}_0 := \widetilde{\overline{y}}$.

Step 2. At any iteration $k \geq 1$, define $\lambda_k := \dfrac{\underline{\lambda}_{k-1} + \overline{\lambda}_{k-1}}{2}$ solve approximatively the Cauchy problem and find the approximate solution $(\tilde{x}_k, \tilde{y}_k)$.

If $\Psi(\widetilde{\underline{x}}_k(T), \widetilde{\underline{y}}_k(T)) < 0$, then put

$$\underline{\lambda}_k := \lambda_k,\ \overline{\lambda}_k := \overline{\lambda}_{k-1},\ \widetilde{\underline{x}}_k := \widetilde{x}_k,\ \widetilde{\underline{y}}_k := \widetilde{y}_k,\ \widetilde{\overline{x}}_k := \widetilde{\overline{x}}_{k-1},\ \widetilde{\overline{y}}_k := \widetilde{\overline{y}}_{k-1},$$

otherwise, for $\Psi(\widetilde{\overline{x}}_k(T), \widetilde{\overline{y}}_k(T)) > 0$, take

$$\underline{\lambda}_k := \underline{\lambda}_{k-1},\ \overline{\lambda}_k := \lambda_k,\ \underline{\tilde{x}}_k := \underline{\tilde{x}}_{k-1},\ \underline{\tilde{y}}_k := \underline{\tilde{y}}_{k-1},\ \overline{\tilde{x}}_k := \tilde{x}_k,\ \overline{\tilde{y}}_k := \tilde{y}_k,$$

and we repeat Step 2 for $k = k + 1$.

Step 3 (Stopping condition).

The algorithm stops if

$$|\Psi(\widetilde{x}_k, \widetilde{y}_k)| < \delta,$$

for a given error $\delta > 0$.

Theorem 6.5. *Under the assumptions of Theorem 6.3, if in addition Ψ satisfies*

$$\left|\Psi(t, s) - \Psi(\overline{t}, \overline{s})\right| \leq L(|t - \overline{t}| + |s - \overline{s}|),$$

for all $t, \overline{t}, s, \overline{s} \in \mathbb{R}$, then the approximate algorithm provides the triple (x^*, y^*, λ^*), where $\lambda^* = \lim_{k\to\infty} \underline{\lambda}_k = \lim_{k\to\infty} \overline{\lambda}_k$, and the pair (x^*, y^*) is the exact solution of the Cauchy problem for $\lambda = \lambda^*$ and

$$\Psi(x^*(T), y^*(T)) \in [-2\varepsilon L, 2\varepsilon L]\,. \tag{6.7}$$

Remark 6.1. **(a)** *The estimate (6.7) shows that controllability condition is satisfied with the error $2\varepsilon L$.*

(b) *If $(\tilde{x}^*, \tilde{y}^*)$ is an approximate solution corresponded to $\lambda = \lambda^*$, with the error ε, then one has*

$$\Psi(\tilde{x}^*(T), \tilde{y}^*(T)) \in [-4\varepsilon L, 4\varepsilon L]\,. \tag{6.8}$$

Indeed, we have that

$$|\Psi(x^*(T), y^*(T)) - \Psi(\tilde{x}^*(T), \tilde{y}^*(T))| \leq L(|x^*(T) - \tilde{x}^*(T)| + |y^*(T) - \tilde{y}^*(T)|) \leq 2\varepsilon L.$$

We get that

$$-4\varepsilon L \leq \Psi(x^*(T), y^*(T)) - 2\varepsilon L \leq \Psi(\tilde{x}^*(T), \tilde{y}^*(T)) \leq \Psi(x^*(T), y^*(T)) + 2\varepsilon L \leq 4\varepsilon L,$$

hence the conclusion (6.8).

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