BABEŞ-BOLYAI UNIVERSITY
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INVESTIGATIONS ON DISCRETE MORSE THEORY AND APPLICATIONS

PhD thesis abstract

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CLUJ-NAPOCA
2013

Keywords: critical cell, critical simplex, discrete Morse functions, real Morse-Smale characteristic, Morse-Smale complex, the estimation number of critical simplices

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Introduction

The discrete Morse theory was formulated by Robin Forman in some articles from period 1995-2003, the reference article being [31]. A presentation with many details with respect of this subject can be found in [30], [35]. The articles [33], [36] contain extensions of this theory, and in the articles [37], [39], [61] appears some applications.

The discrete Morse theory can be formulated very clearly in the language of CW-complexes. In this paper we will present notions and fundamental results with respect to CW-complexes. This represents a type of topological spaces introduced by J.H.C. Whitehead in [92], [93], with the purpose to satisfy some natural requirements in homotopy theory.

This paper is naturally divided into four chapters which will be described in the lines to come.

First chapter contains four sections and has a monographic character. The main purpose of this chapter is to introduce the basic notions and results that will be used in the paper. We follow here the excellent books of R. Forman [31], [32], G. Cicortaş [20], D. Andrica [2].

Chapter 2 is divided in four sections and contains original results of the author obtained in the study of Morse-Smale characteristic for discrete Morse functions defined on the unit circle S^1, the 2-dimensional sphere S^2, the real projective plane $P^2(\mathbb{R})$, the Klein bottle $K = P^2(\mathbb{R})\#P^2(\mathbb{R})$, the Möbius band M, the torus $T^2 = S^1 \times S^1$, the torus with two holes $T_2 = T^2\#T^2$, the Dunce hat DH, results which are mentioned in the papers of I.C. Lazăr and V. Revnic [57], V. Revnic [81]. This chapter is based on the following papers of D. Andrica [2], R. Forman [31], T. Lewiner [59], [60], I.C. Lazăr şi V. Revnic [57], V. Revnic [81].

Chapter 3 is divided in four sections and has a monographic character. There are presented notions from discrete Morse theory, definitions and properties for perfect discrete Morse functions defined on 2-dimensional complexes. This chapter is based on the following papers of M. Armstrong [10], J. Hopcroft and R.E. Tarjan [53], H. Lopes, J. Rossignac, A. Safonova, A. Szymczak and G. Tavares [65], T. Lewiner [59], [60], [61], J.R. Munkeres [79], R. Ayala, D. Fernandez-Ternero and J.A. Vilches [11].

Chapter 4 is divided in four sections and has a monographic character with respect to optimality which we can obtain in discrete Morse theory when we use chain complexes. This chapter is based on the following papers of H. Molina-Abril and P. Real [73], [74], [75], A. Hatcker [50], R. Ayala, D. Fernandez-Ternero and J.A. Vilches [11], M.M. Cohen [21], M. Desbrun, E. Konso and Y. Tong [24], S. Eilengerg and S. Mac Lane [29], R. Forman [30], [31], [38], E.E. Moise [77], F. Sergeraert [85], T. Lewiner, H. Lopes, G. Tavares and L. Matmidia [63].

I am grateful to my scientific advisor, prof. dr. Dorin Andrica for his valuable advice, for significant support during the preparation of this paper, for guidance, for patience. I am indebted to associate professors Cornel Pintea, dr. Paul Blaga, dr. Liana Ţopan for constant support and for discussions which were helpful in writing this paper. I would also like to thank to all members of Department of Geometry of Babeş-Bolyai University from Cluj-Napoca, for their useful remarks and helpful suggestions.

Chapter 1

Morse theory for regular CW-complexes

1.1 CW-complexes, regular CW-complexes

We will consider only finite CW-complexes. CW-complexes are formed by cells. We consider the next sets:
$B^n = \{x \in E^n : |x| \leq 1\}$, the closed unit ball in n-dimensional Euclidian space
$S^{(n-1)} = \{x \in E^n : |x| = 1\}$, the boundary of B^n (is the unit $(n-1)$-sphere).

Definition 1.1.1 A n-**cell** is a topological space which is homeomorphic to $intB^n$.
A closed n-cell is a topological space which is homeomorphic to B^n.

If σ is a closed n-cell, then we denote by $\dot{\sigma}$ the subset of σ corresponding to $S^{(n-1)} \subset B^n$ under any homeomorphism between B^n and σ.

Definition 1.1.2 A cell is a topological space which is a n-cell for some n.
The basic operation of CW-complexes is the notion of attaching a cell.

Definition 1.1.3 Let X be a topological space, σ a n-cell and $f : \dot{\sigma} \to X$ a continuous function. We let $X \bigcup_f \sigma$ denote the disjoint union of X and σ quotiented out by the equivalence relation that each point $s \in \dot{\sigma}$ is identified with $f(s) \in X$.
We refer to this operation by saying that $X \bigcup_f \sigma$ is the results of attaching the cell σ to X and the map f is called the attaching map. The attaching map must be defined on all of $\dot{\sigma}$, that is, the entire boundary of σ must be glued to X.

The type of homotopy of $X \bigcup_f \sigma$ depends only on the type of homotopy of X and the homotopy class of f.

Example 1.1.4 Let X be a circle, then Figure 1.1(i) shows one possible result of attaching a 1-cell to X. Attaching a 1-cell to circle cannot lead to the space illustrated in Figure 1.1.(ii) since the entire boundary of the 1-cell has not been glued to the circle.

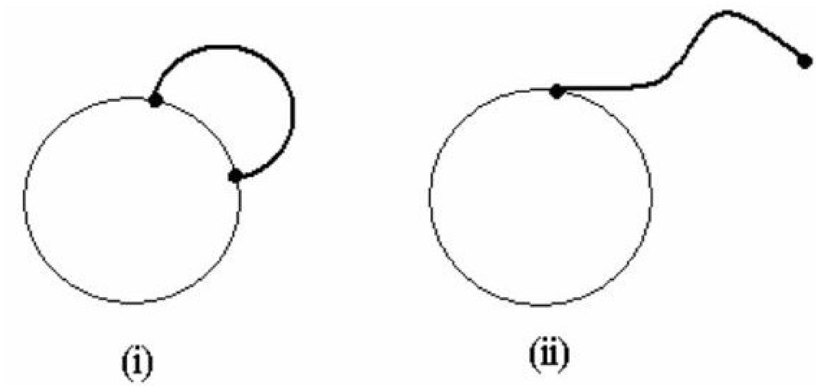


Figure 1.1. (i) A 1-cell attached to a circle; (ii) A 1-cell unattached to a circle.

Definition 1.1.5 A space X is a CW-complex if he is constructed by attaching cells in the following way:

1. We start with a discrete set X_0, which is formed by 0-cells and we call this vertices.

2. Inductively, we obtain the n-skeleton X_n, when we attaching to the space X_{n-1} the n-cells e_α^n with the attaching maps $\varphi_\alpha : S^{n-1} \to X_{n-1}$. We denote $X_n = X_{n-1} \bigcup_\alpha e_\alpha^n$.

3. We can stop this algorithm by putting $X = X_n$ for $n < \infty$ or we can continue taking $X = \bigcup_n X_n$.

A set $A \subset X$ is open (closed) if and only if $A \cap X_n$ is open (closed) in X, for each n.

If $X = X_n$ we say that X is a finite dimensional CW-complex, and n is the dimension of X. Otherwise X has ∞ dimension.

The sequence $X_0, X_1, X_2, \ldots, X_{n-1}, X_n, \ldots$ is called a CW-decomposition for the space X.

Example 1.1.6 1) A celular complex with dimension 1, $X = X_1$ is called a graph. He is obtained by a finite number of vertices, which are the 0-cells, and to this vertices we attach the edges which are the 1-cells of the complex X.

Definition 1.1.7 A finite simplicial complex is obtained from a set of vertices V and a family K of subsets of V, with the following properties:

(i) $V \subseteq K$

(ii) if $\alpha \in K$ and $\beta \subseteq \alpha$, then $\beta \in K$.

We denote the simplicial complex by K. The elements of K are called simplices. If $\alpha \in K$ contains $d+1$ linear independent vertices, then we say that the dimension of α is d or α is a d-simplex and we denote with $\alpha^{(d)}$. If $\beta \subset \alpha$, we say that β is a face of α and denote $\beta \prec \alpha$ or $\alpha \succ \beta$. We observe that a d-simplex is a d-cell.

2) A finite simplicial complex it will be a CW-complex, in which the CW-decomposition is formed by closed simplices.

In a CW-complex X any cell has a characteristic map $\Phi_\alpha : \overline{e_\alpha^n} \to X$, which extend the attaching map φ_α, and $\Phi_\alpha|_{e_\alpha^n}$ is a homeomorphism.

A subcomplex of a CW-complex X is a closed subset A of X which is an union of cells from X. Any subcomplex A is an CW-complex.

For example any skeleton X_n is a subcomplex for the CW-complex X.

If A is a subcomplex of X then (X, A) denote a CW-pair. A CW-complex is finite if has a finite number of cells. Any of these CW-complex is compact, because when we attach cells the compactness is kept. Conversely, any compact subspace of a CW-complex is contained in a finite subcomplex.

We recall some properties for CW-complexes:

Theorem 1.1.8 *Any CW-complex space is a normal topological space and in particular Hausdorff.*

Theorem 1.1.9 *Any point in CW-complex admits contractible open neighborhoods, so the CW-complexes are locally contractible spaces. In particular, the CW-complexes are locally linear conex spaces.*

Theorem 1.1.10 *Any finite CW-complex X is an Euclidean retract, that means for a n exist an embedding $i : X \to \mathbb{R}^n$ such that $i(X)$ is a retract of a neighborhood in $\mathbb{R}^n$.*

Theorem 1.1.11 *A compact manifold is homotopic equivalent with a finite CW-complex.*

Let M be a finite CW-complex, K the set of all cells from M and K_p the set of all cells from M with the dimension p.

We will use the following notations.

(i) $\sigma^{(p)}$ is the σ cell of dimension p.

(ii) $\sigma \prec \tau$, σ is a face of τ if $\sigma \neq \tau$ and $\sigma \subset \overline{\tau}$, where $\overline{\tau}$ is the closure of τ.

(iii) $\sigma \preceq \tau$ if $\sigma \prec \tau$ or $\sigma = \tau$.

Suppose $\sigma^{(p)}$ is a face of $\tau^{(p+1)}$.

Let B be the closed ball of dimension $p+1$ and $h : B \to M$ the characteristic function for τ, that means h is a continuous map which goes inside B homeomorphically in τ.

Definition 1.1.12 We say that $\sigma^{(p)}$ is a regular face for $\tau^{(p+1)}$ if:

(i) the restriction $h : h^{-1}(\sigma) \to \sigma$, $x \mapsto h(x)$ is a homeomorphism;

(ii) $\overline{h^{-1}(\sigma)}$ is a closed p-ball.

Otherwise we say that $\sigma^{(p)}$ is the irregular face of $\tau^{(p+1)}$.

Definition 1.1.13 M is a regular CW-complex if has only regular faces.

In particular, any simplicial complex is a regular CW-complex.

Suppose $\sigma^{(p)}$ is a regular face for $\tau^{(p+1)}$. We choose an orientation for each cell from M and consider σ, τ elements in the groups $C_p(M, \mathbb{Z})$ and $C_{p+1}(M, \mathbb{Z})$. The we have:

$$\langle \partial\tau, \sigma \rangle = \pm 1,$$

where $\langle \partial\tau, \sigma \rangle$ is the incidence number for τ and σ.

Theorem 1.1.14 *Suppose that $\tau^{(p+1)} \succ \sigma^p \succ \nu^{(p-1)}$, then one of the following relations hold:*

(i) σ is an irregular face of τ;

(ii) ν is an irregular face of σ;

(iii) there exist a p-cell $\widetilde{\sigma} \neq \sigma$ such that $\tau \succ \widetilde{\sigma} \succ \nu$.

Theorem 1.1.15 *Let M be a regular CW-complex. Suppose that for two numbers $p, r \geq 1$, we have $\tau^{(p+r)} \succ \nu^{(p-1)}$. Then there exists $p+r-1$ dimensional cells $\sigma^{(p)}$ and $\widetilde{\sigma}^{(p)}$ which satisfies the following relations:*

$$\widetilde{\sigma} \neq \sigma, \ \tau \succ \sigma \succ \nu, \ \tau \succ \widetilde{\sigma} \succ \nu.$$

Definition 1.1.16 Two CW-complexes M, N are isomorphic if there exists a homeomorphism $h : N \to M$ which attach to each cell of N an unique cell from M.

We say that $\widetilde{M}$ is a subdivision of M if there exists a homeomorphism $h : \widetilde{M} \to M$ which attach to each cell of $\widetilde{M}$ an unique cell from M. We say that two complexes M and N are equivalent if there exists a finite subdivision $\widetilde{M}$ of M and a finite subdivision $\widetilde{N}$ of N such that $\widetilde{M}$ and $\widetilde{N}$ are isomorphic.

The next notion is very important in what follows.

Definition 1.1.17 Let M be a CW-complex and $\sigma^p \prec \tau^{(p+1)}$ two cells of M with the following properties:

(i) σ is a regular face of τ;

(ii) σ is not a face for another cell.

Consider $N = M \setminus (\sigma \cup \tau)$. We say that M collapses to N.

More generally, we say that M collapses to N and we write $M \searrow N$, if N is obtained from M by a finite operations as above.

1.2 Discrete Morse functions on CW-complexes

Let M be a finite CW-complex. Let K be the set of all cells from M and K_p the set of all cells with dimension p. In [31] R. Forman define the discrete Morse function and the critical point of such function.

Definition 1.2.1 A discrete Morse function on M is a function $f : K \to \mathbb{R}$ which for every $\sigma \in K_p$ verifies:

(i) If $\sigma^{(p)}$ is the irregular face of $\tau^{(p+1)}$ then $f(\tau) > f(\sigma)$.

Furthermore, we have

$$\#\{\tau^{(p+1)} \succ \sigma^{(p)} \mid f(\tau) \leq f(\sigma)\} \leq 1.$$

(ii) If $\nu^{(p-1)}$ is the irregular face of $\sigma^{(p)}$ then $f(\nu) < f(\sigma)$.

Furthermore, we have

$$\#\{\nu^{(p-1)} \prec \sigma^{(p)} \mid f(\nu) \geq f(\sigma)\} \leq 1.$$

With other words, there is at most one $(p+1)$-cell and at most one $(p-1)$-cell such that $f(\sigma)$ be balanced function values on those cells.

Note that a discrete Morse function is not a continuous function on K.

Definition 1.2.2 Let f be a discrete Morse function on M. We say that $\sigma \in K_p$ is a critical point of index p if we have:

(i) $\#\{\tau^{(p+1)} \succ \sigma^{(p)} \mid f(\tau) \leq f(\sigma)\} = 0$;

(ii) $\#\{\nu^{(p-1)} \prec \sigma^{(p)} \mid f(\nu) \geq f(\sigma)\} = 0$.

The real number $f(\sigma)$ is called critical value of f.

Example 1.2.3 If M is regular then from Definitions 1.2.1 and 1.2.2 we have the minimum of f is a vertex, which is a critical point of index 0. This thing we can obtain from the fact that if $p \geq 1$, then each p-cell has at least two faces of dimension $p-1$.

Example 1.2.4 If M is a n-dimensional manifold without boundary then the maximum of f is an n-face, which must be a critical point of index n.

This property yields from the fact that if $p \leq n-1$, then each p-cell is a face of at least two cells with dimension $p+1$.

From Definition 1.2.2 we have σ is not a critical p-cell if and only if one of the following relations hold:

(i) there exists $\tau^{(p+1)} \succ \sigma^{(p)}$ such that $f(\tau) \leq f(\sigma)$;

(ii) there exists $\nu^{(p-1)} \prec \sigma^{(p)}$ such that $f(\nu) \geq f(\sigma)$.

Lemma 1.2.5 *The above conditions (i), (ii) cannot both be true.*

Any CW-complex M admits a discrete Morse function, for example

$$f : K \to \mathbb{R}, \ f(\sigma) = \dim(\sigma).$$

In this case, any cell is a critical point of f.

A discrete Morse function defined on a CW-complex can induce a discrete Morse function on the subcomplex. Mutual a discrete Morse function defined on a subcomplex can be extended to all the complex.

Proposition 1.2.6 *Let M be a CW-complex and N a subcomplex of M. The restriction of any discrete Morse function on M is a discrete Morse function on N too. If $\sigma \subseteq N$ is a critical point of f, then σ is a critical point of $f|_N$.*

The proof results directly from Definitions 1.2.1 and 1.2.2.

Proposition 1.2.7 *Let M be a CW-complex and N a subcomplex of M. Any discrete Morse function defined on N can be extended at a discrete Morse function on M.*

With other words, if f is a discrete Morse function on N, then exists a discrete Morse function g on M such that $g(\sigma) = f(\sigma)$, for each $\sigma \subseteq N$.

The discrete Morse function g defined as above has the disadvantage that any face of $M \setminus N$ is critical point. In applications is important to find functions with few number of critical points.

Proposition 1.2.8 *Let M be a CW-complex and N a subcomplex of M such that $M \searrow N$. Let f be a discrete Morse function on N and $c = \max\limits_{\sigma \subseteq N} f(\sigma)$.*

The function f can be extended to a discrete Morse function on M such that $N = M_c$ and f don't have critical points in $M \setminus N$.

Proposition 1.2.8 prove the standard n-simplex Δ_n admits a discrete Morse function with only one critical point, because Δ_n collapses in a vertex.

1.3 The Morse theorems for regular CW-complexes

Discrete Morse theory for regular CW-complexes was introduced by R. Forman in [31]. In the same paper he extend the results for general CW-complexes.

Let M be a regular CW-complex and f a discrete Morse function defined on M. For $c \in \mathbb{R}$, we define the level subcomplex:

Definition 1.3.1 For $c \in \mathbb{R}$ we define:

$$M_c(f) = M_c = \bigcup_{f(\sigma) \leq c} \bigcup_{\tau \preceq \sigma} \tau, \ \sigma \in K.$$

M_c is the set of all cells with f take values $\leq c$, with all of their faces.

In particular, M_c is a subcomplex of M.

To prove that a cell $\sigma \in K$ with the property $f(\sigma) < c$ is contained in M_c, we must to show that there exists $\tau \in K$ such that $\sigma \prec \tau$ and $f(\tau) \leq c$.

We observe that it is sufficient to consider only τ with the property:

$$\dim(\tau) = \dim(\sigma) + 1.$$

This follows from the next lemma.

Lemma 1.3.2 [31] *Let σ be a p-cell of M and suppose $\tau \succ \sigma$. Then there exists a $(p+1)$-cell $\widetilde{\tau}$ with the properties $\sigma \prec \widetilde{\tau} \preceq \tau$, $f(\widetilde{\tau}) \leq f(\tau)$.*

Theorem 1.3.3 [31] *If $a, b \in \mathbb{R}$ such that $[a, b]$ contains no critical values of f, then M_b collapses to M_a, denoted by $M_b \searrow M_a$.*

Theorem 1.3.4 *If $\sigma^{(p)}$ is a critical cell with dimension p with $f(\sigma) \in [a, b]$ and $[a, b]$ contains no critical values, then M_b is homotopic equivalent with $M_a \bigcup_{\dot{e}^p} e^p$ (where e^p is a cell of dimension p and she is attached by M_a.*

Let $M_p(f)$ be the number of critical simplices of index p with respect f.
The numbers M_p are called the Morse numbers of f.
From Theorems 1.3.3 and 1.3.4 we have the following property:

Corollary 1.3.5 *M is homotopic equivalent with a CW-complex which has exactly $M_p(f)$ cells of dimension p.*

Example 1.3.6

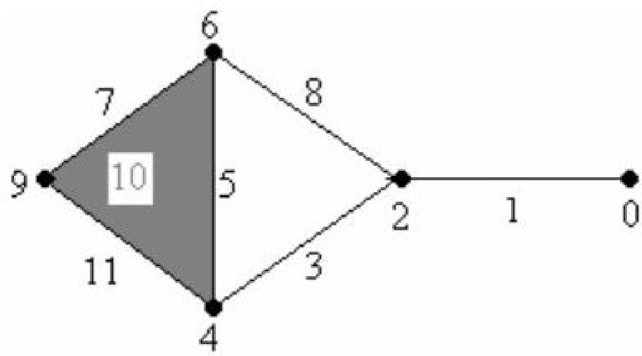


Figure 1.2

In Figure 1.2, $f^{-1}(0)$ is a critical simplex of index 0, $f^{-1}(8)$ is a critical simplex of index 1. Other critical simplices doesn't exists, so M is homotopic equivalent with a circle, this is obvious because we can see this thing from figure.

Example 1.3.7

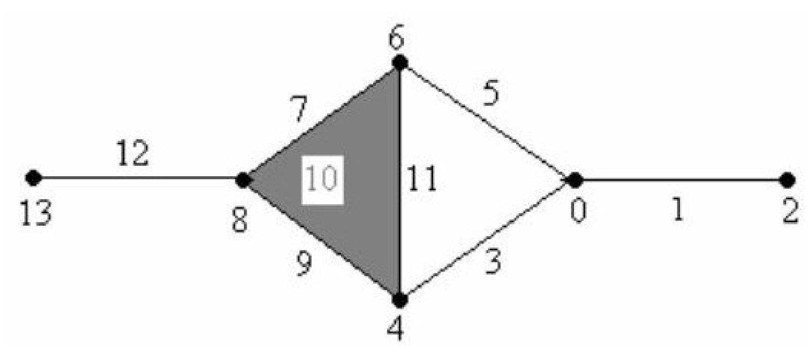


Figure 1.3

In Figure 1.3 we have a simplicial complex M with a discrete Morse function defined on M. We have two critical simplices $f^{-1}(0)$ and $f^{-1}(9)$, and other critical simplices doesn't exist. The simplicial complex M can be constructed following the next steps:

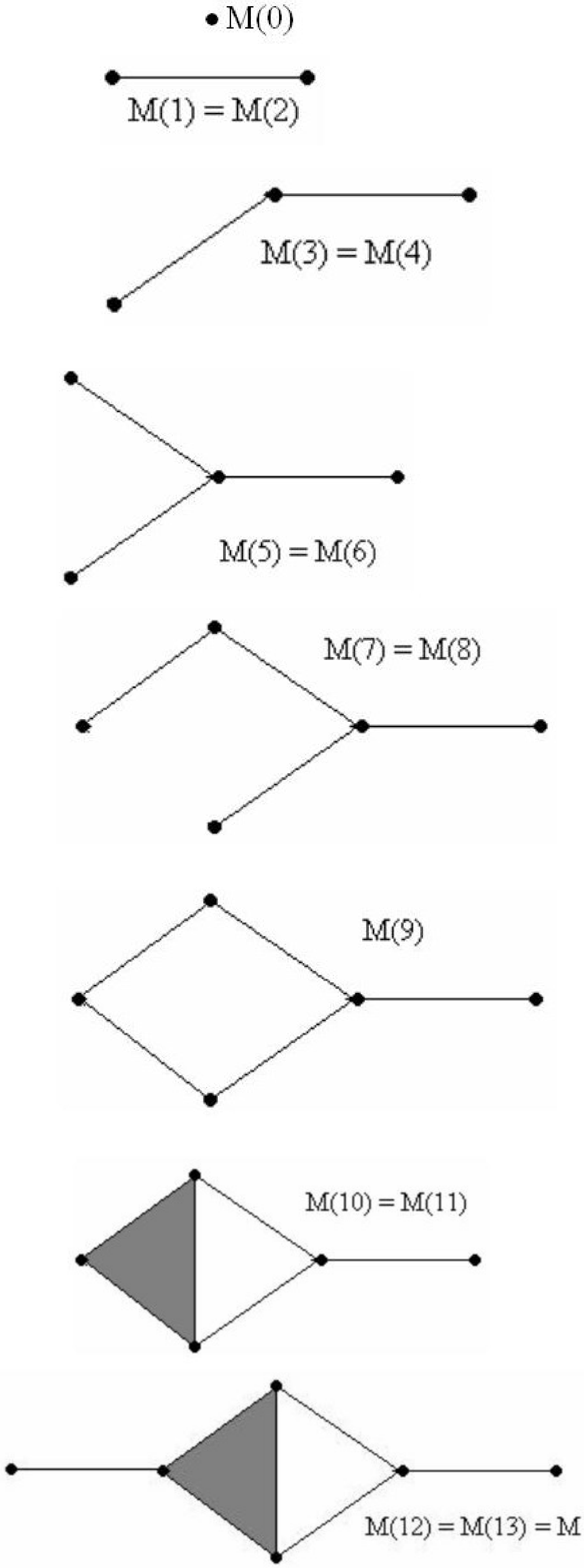


Figure 1.4

1.4 The homological version of Morse inequalities

1.4.1 The homological algebra of Morse inequalities. The polynomial order

Let $p \in \mathbb{R}[T]$, be a polynomial in T variable, T, $p = p_0 + p_1T + p_2T^2 + \ldots + p_nT^n$.

Definition 1.4.1 Let $q \in \mathbb{R}[T]$, be another polynomial in T, $q = q_0 + q_1T + q_2t^2 + \ldots + q_mT^m$.
We say that p dominate q and we write $p \succeq q$ dacă if the formal sequence

$$\frac{p(T) - q(T)}{T + 1}$$

is a polynomial with positive coefficients.

The relation $\succeq$ has the following properties:
1) "$\succeq$" is a partial order relation on $\mathbb{R}[T]$;
2) $p \succeq q$ implies $p(-1) = q(-1)$;
3) "$\succeq$" is an additive order relation, that means

$$p_1 \succeq q_1,\ p_2 \succeq q_2 \text{ implies } p_1 + p_2 \succeq q_1 + q_2.$$

4) Suppose that $p \succeq q$ and let $r \in \mathbb{R}[T]$ a polynomial with positive coefficients. Then $pr \succeq qr$.
The relation $p \succeq 0$ implies that the rational function

$$\begin{aligned}(p_0 + p_1T + p_2T^2 + \ldots + p_nT^n)(1 + T)^{-1} \\ = (p_0 + p_1T + p_2T^2 + \ldots + p_nT^n)(1 - T + T^2 - T^3 + \ldots) \\ = p_0 + (p_1 - p_0)T + (p_2 - p_1 + p_0)T^2 + \ldots\end{aligned}$$

is a polynomial with positive coefficients, that means the following inequalities holds:

$$\begin{aligned}&p_0 \geq 0 \\ &p_1 - p_0 \geq 0 \\ &p_2 - p_1 + p_0 \geq 0 \\ &\ldots\ldots\ldots\ldots \\ &p_n - p_{n-1} + p_{n-2} - \ldots + (-1)^n p_0 \geq 0.\end{aligned} \tag{1.4.1}$$

Remark 1.4.2 In the last inequality we have equality because p is divisible with $T + 1$, so we have $p(-1) = 0$.

It is obvious that we have $p \succeq 0$ if and only if the relations (1.4.1) hold.

Remark 1.4.3 If $p \succeq 0$, then $p_j \geq 0,\ j = 0, 1, \ldots, n$.

In general, these inequalities are more weaker than $p \succeq 0$, how is shown in the next example

Example 1.4.4 The polynomial $p = 1 + T^3$ has positive coefficients, but $p \not\succeq 0$.

The relation $p \succeq q$ is equivalent with $p - q \succeq 0$, so with relations:

$$\begin{aligned}
&p_0 \geq q_0 \\
&p_1 - p_0 \geq q_1 - q_0 \\
&p_2 - p_1 + p_2 \geq q_2 - q_1 + q_0 \\
&\ldots\ldots\ldots\ldots \\
&p_{n-1} - p_{n-2} + p_{n-3} - \ldots + (-1)^{n-1}p_0 \geq q_{n-1} - q_{n-2} + q_{n-3} - \ldots + (-1)^{n-1}q_0 \\
&p_n - p_{n-1} + p_{n-2} - \ldots + (-1)^n p_0 = q_n - q_{n-1} + q_{n-2} - \ldots + (-1)^n q_0
\end{aligned} \tag{1.4.2}$$

From $p \succeq q$ we have the inequalities $p_j \geq q_j$, $j = 0, 1, \ldots$

Lemma 1.4.5 *If $p \succeq q$ and for an index j we have $p_j \geq q_j$, then $p_{j-1} \geq q_{j-1}$ or $p_{j+1} \geq q_{j+1}$.*

Lemma 1.4.6 (Morse principle) *Let $p, q \in \mathbb{R}[T]$. Suppose that p contains only even powers and q has positive coefficients. Then:*

$$p \succeq q \Rightarrow p = q.$$

1.4.2 The Poincaré polynomial

Let K be a field, C_q be the vector space over K, $\dim_K C_q < \infty$ and $\dim_K C_q = 0$ for q big enough. We denote:

$$C_* = \bigoplus_{q \geq 0} C_q.$$

Definition 1.4.7 The Poincaré polynomial associated to C_* is:

$$P_{C_*} = \sum_{q \geq 0} (\dim_K C_q) T^q.$$

Lemma 1.4.8 (Euler-Poincaré-Morse) *Let C_* be a chain complex with a finite dimension over K and $H_* = H_*(C)$ the homology of C_*. Then*

$$P_{C_*} \succeq P_{H_*}.$$

1.4.3 Morse inequalities

Let K be a field, $\beta_i = \dim H_i(M, K)$ the Betti numbers with coefficients in K and χ the characteristic of the regular CW-complex M.

Theorem 1.4.9 (The strong Morse inequalities) *For each $j \geq 0$ we have:*

$$m_j - m_{j-1} + \ldots \pm m_0 \geq \beta_j - \beta_{j-1} + \ldots \pm \beta_0.$$

From Theorem 1.4.9 we obtain

Theorem 1.4.10 (The weak Morse inequalities)
(i) For each $j \geq 0$ we have

$$m_j \geq \beta_j.$$

(ii) $\chi(M) = \beta_0 - \beta_1 + \beta_2 - \ldots \pm \beta_m = m_0 - m_1 + m_2 - \ldots \pm m_m$, where m is the dimension of M.

1.4.4 Gradient vector fields

After all, how is one to go about assigning numbers to each of the simplices of a simplicial complex so that they satisfy the axioms of a discrete Morse function? Fortunately, in practice one need not actually find a discrete Morse function. Finding the gradient vector field of the Morse function is sufficient. This requires a bit of explanation.

Let us now return to the example in Figure 1.5(ii). Noncritical simplices occur in pairs. For example, the edge $f^{-1}(1)$ is not critical because it has a "lower dimensional neighbor" which is assigned a higher value, i.e., the vertex $f^{-1}(2)$. Similarly, the vertex $f^{-1}(2)$ is not critical because it has a "higher dimensional neighbor" which is assigned a lower value, i.e., the edge $f^{-1}(1)$. We indicate this pairing by drawing an arrow from the vertex $f^{-1}(2)$, pointing into the edge $f^{-1}(1)$. Similarly, we draw an arrow from the vertex $f^{-1}(4)$ pointing into the edge $f^{-1}(3)$ (See Figure 1.6.)

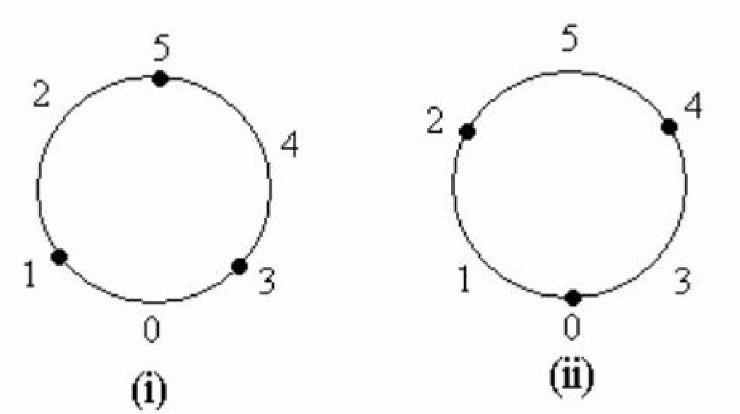


Figure 1.5

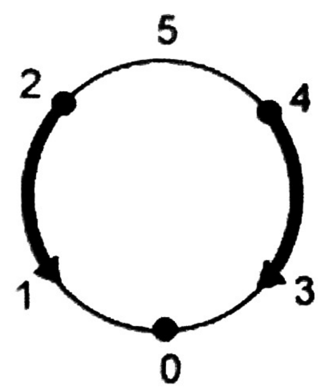


Figure 1.6. The gradient vector field of the Morse function shown in Figure 1.5

We can apply this process to any simplicial complex with a discrete Morse function. The arrows are drawn as follows. Suppose $\alpha^{(p)}$ is a non-critical simplex with $\beta^{(p+1)} > \alpha^{(p)}$ satisfying $f(\beta) \leq f(\alpha)$. We then draw an arrow from α to β. Figure 1.7 illustrates a more complicated example. Note that the discrete Morse function drawn in this figure has one critical vertex, $f^{-1}(0)$, and one critical edge, $f^{-1}(11)$. This simplicial complex is homotopy equivalent to a CW complex with exactly one 0-cell and one 1-cell, i.e., a circle.

Every simplex α satisfies exactly one of the following:
(i) α is the tail of exactly one arrow.
(ii) α is the head of exactly one arrow.
(iii) α is neither the head nor the tail of an arrow.

Note that a simplex is critical if and only if it is neither the tail nor the head of any arrow.

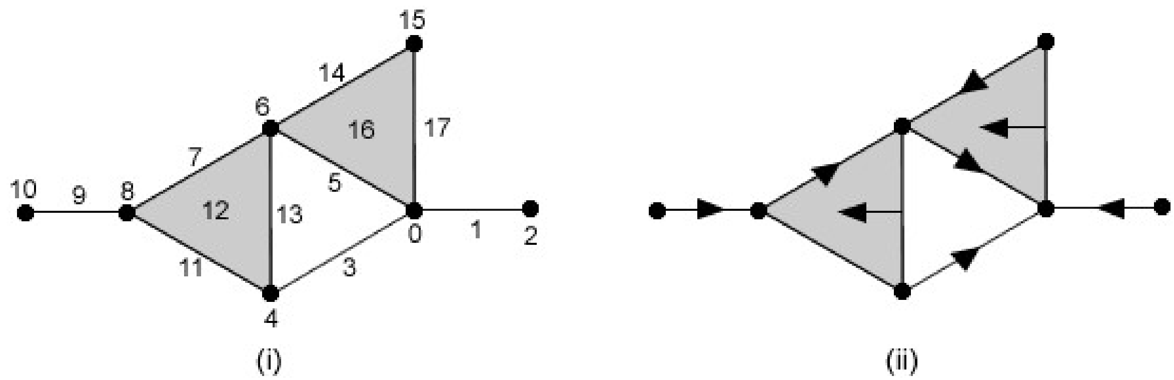


Figure 1.7. Another example of a gradient vector field

These arrows are much easier to work with than the original discrete Morse function. In fact, this gradient vector field contains all of the information that we will need to know about the function for most applications. If one is given a simplicial complex and one wishes to apply the theory of the previous section, one need not find a discrete Morse function. One "only" needs to find a gradient vector field.

Suppose we attach arrows to the simplices so that each simplex satisfies exactly one of properties (i), (ii), (iii) above. Then how do we know if that set of arrows is the gradient vector field of a discrete Morse function?

Let K be a simplicial complex with a discrete Morse function f. Then rather than thinking about the discrete gradient vector field V of f as a collection of arrows, we may equivalently describe V as a collection of pairs $\{\alpha^{(p)} < \beta^{(p+1)}\}$ of simplices of K, where $\{\alpha^{(p)} < \beta^{(p+1)}\}$ is in V if and only if $f(\beta) \leq f(\alpha)$. In other words, $\{\alpha^{(p)} < \beta^{(p+1)}\}$ is in V if and only if we have drawn an arrow that has α as its tail, and β as its head. The properties of a discrete Morse function imply that each simplex is in at most one pair of V. This leads us to the following definition.

Definition 1.4.11 A discrete vector field V on K is a collection of pairs $\{\alpha^{(p)} < \beta^{(p+1)}\}$ of simplices of K such that each simplex is in at most one pair of V.

If one has a smooth vector field on a smooth manifold, it is quite natural to study the dynamical system induced by flowing along the vector field. One can begin the same sort of study for any discrete vector field.

Given a discrete vector field V on a simplicial complex K, a V-path is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}$$

such that for each $i = 0, \ldots, r$, $\{\alpha < \beta\} \in V$ and $\beta_i > \alpha_{i+1} \neq \alpha_i$. We say such a path is a non-trivial closed path if $r \geq 0$ and $\alpha_0 = \alpha_{r+1}$. If V is the gradient vector field of a discrete Morse function f, then we sometimes refer to a V-path as a gradient path of f.

One idea behind this definition is the following result.

Theorem 1.4.12 *Suppose V is the gradient vector field of a discrete Morse function f. Then a sequence of simplices as in*

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \ldots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}$$

is a V-path if and only if $\alpha_i < \beta + i > \alpha_{i+1}$ for each $i = 0, 1, \ldots, r$, and

$$\alpha_i \prec \beta_i \succ \alpha_{i+1} \ \textit{pentru oricare}\ i = 0, 1, \ldots, r \ \textit{şi}$$

$$f(\alpha_0) \geq f(\beta_0) > f(\alpha_1) \geq f(\beta_1) > \ldots \geq f(\beta_r) > f(\alpha_{r+1}).$$

That is, the gradient paths of f are precisely those "continuous" sequences of simplices along which f is decreasing. In particular, this theorem implies that if V is a gradient vector field, then there are no nontrivial closed V-paths. In fact, the main result of this section is that the converse is true.

Theorem 1.4.13 *A discrete vector field V is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed V-paths.*

We can see the similarity with the following standard theorem from the subject of directed graphs.

Theorem 1.4.14 *Let G be a directed graph. Then there is a real-valued function of the vertices that is strictly decreasing along each directed path if and only if there are no directed loops.*

The importance of Theorem 1.4.13 we will see in the next example. We will construct a discrete vector field for the real projective plane and we use this theorem to verify that this vector field is a gradient vector field.

Example 1.4.15 Figure 1.8 (i) shows a triangulation of the real projective plane $\mathbb{P}^2$. Note that the vertices along the boundary with the same labels are to be identified, as are the edges whose endpoints have the same labels. In Figure 1.8(ii) we illustrate a discrete vector field V on this simplicial complex. One can easily see that there are no closed V-paths (since all V-paths go to the boundary of the figure and there are no closed V-paths on the boundary), and hence is a gradient vector field. The only simplices which are neither the head nor the tail of an arrow are the vertex labeled 1, the edge e, and the triangle t. Thus, the projective plane is homotopy equivalent to a CW complex with exactly one 0-cell, one 1-cell and one 2-cell.

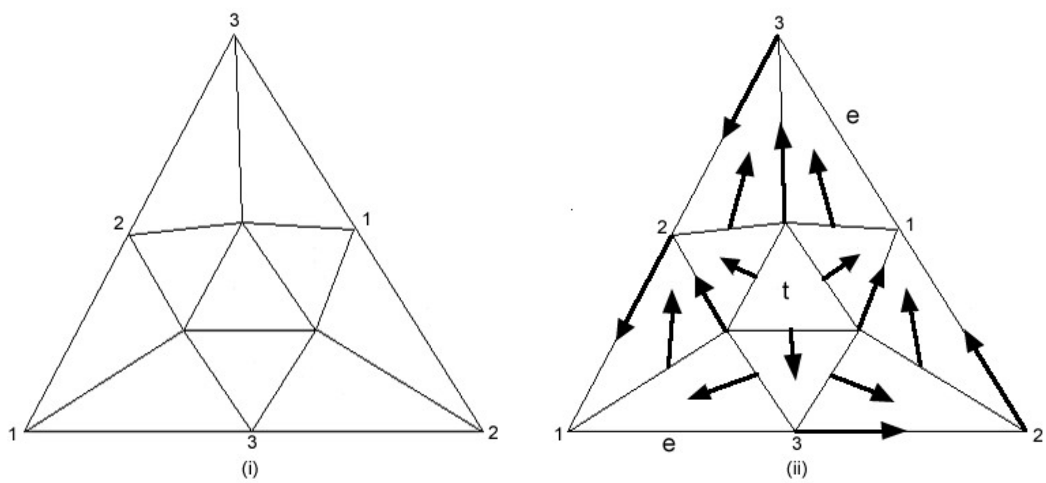


Figure 1.8.
(i) A triangulation of the real projective plane
(ii) A discrete gradient vector field on $\mathbb{P}^2$

1.4.5 The combinatorial description of gradient vector fiels

The notion of a gradient vector field has a very nice purely combinatorial description using which we can recast the Morse Theory in an appealing form. We begin with the Hasse diagram of K, that is, the partially ordered set of simplices of K ordered by the face relation. Consider the Hasse diagram as a directed graph. The vertices of the graph are in 1-1 correspondence with the simplices of K, and there is a directed edge from β to α if and only if α is a codimension-one face of β (See

Figure 1.9(i)). Now let V be a combinatorial vector field. We modify the directed graph as follows. If $\{\alpha < \beta\} \in V$ then reverse the orientation of the edge between α and β, so that it now goes from α to β (See Figure 1.10). A V-path can be thought of as a directed path in this modified graph. There are some directed paths in this modified Hasse diagram which are not V-paths as we have defined them.

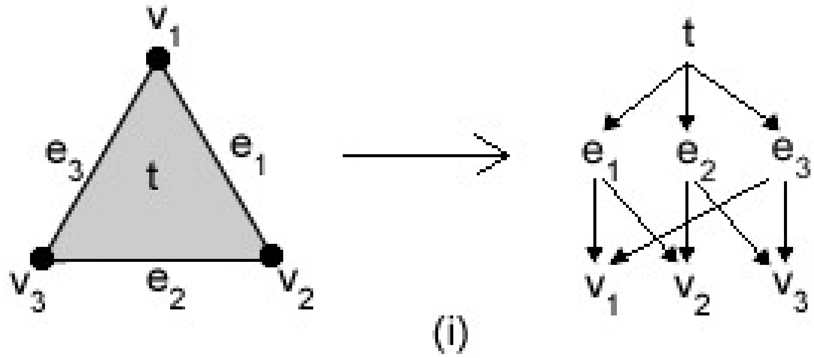


Figure 1.9. From a discrete vector field to a directed Hasse diagram

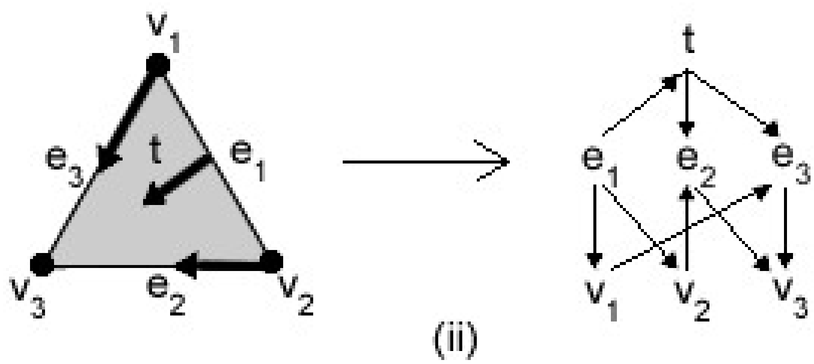


Figure 1.10. From a discrete vector field to a directed Hasse diagram

Theorem 1.4.16 *There are no nontrivial closed V-paths if and only if there are no nontrivial closed directed paths in the corresponding directed Hasse diagram.*

Thus, in this combinatorial language, a discrete vector field is a partial matching of the Hasse diagram, and a discrete vector field is a gradient vector field if the partial matching is acyclic in the above sense.

We can now restate some of our earlier theorems in this language. There is a very minor complication in that one usually includes the empty set as an element of the Hasse diagram (considered as a simplex of dimension -1) while we have not considered the empty set previously.

Theorem 1.4.17 *Let V be an acyclic partial matching of the Hasse diagram of K (of the sort described above – assume that the empty set is not paired with another simplex). Let u_p denote the number of unpaired p-simplices. Then K is homotopy equivalent to a CW-complex with exactly u_p cells of dimension p, for each $p \geq 0$.*

An important special case is when V is a complete matching, that is, every simplex (this time including the empty simplex) is paired with another simplex. In this case, we have the following result.

Theorem 1.4.18 *Let V be a complete acyclic matching of the Hasse diagram of K, then K collapses onto a vertex, so that, in particular, K is contractible.*

Chapter 2

The discrete Morse-Smale characteristic

2.1 Morse functions on simplicial complexes

Let K be a finite simplicial complex.

Definition 2.1.1 A function $f : K \to \mathbb{R}$ is a discrete Morse function if for every $\sigma^{(p)} \in K$, we have the following relations:
(1) $\#\{\beta^{(p+1)} \succ \alpha^{(p)} \mid f(\beta) \leq f(\alpha)\} \leq 1$ and
(2) $\#\{\gamma^{(p-1)} \prec \alpha^{(p)} \mid f(\gamma) \geq f(\alpha)\} \leq 1$.

Example 2.1.2 We consider two simplicial complexes. Here we indicate functions by writing next to each simplex the value of the function on that simplex.

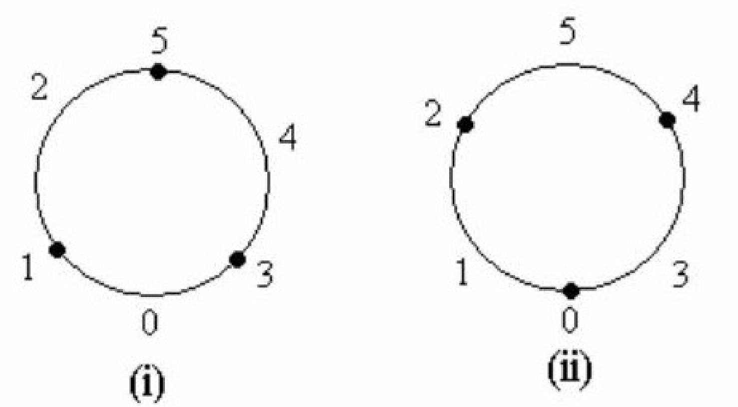


Figure 2.1

The function (i) from Figure 2.1 Is not a discrete Morse function because the edge $f^{-1}(0)$ violates rule (2) from definition, since it has 2 lower dimensional neighbors on which f takes on higher values. Moreover the vertex $f^{-1}(5)$ violates rule (1), because it has 2 higher dimensional neighbors on which f takes on lower values. The function (ii) from Figure 2.1 is a discrete Morse function. Note that a discrete Morse function is not a continuous function on K because we didn't consider any topology on K. The function associate to each simplex an unique value.

Recall the fact that a p-simplex $\alpha^{(p)}$ is critical of index p if the following relation holds:
(1) $\#\{\beta^{(p+1)} \succ \alpha^{(p)} \mid f(\beta) \leq f(\alpha)\} = 0$ and
(2) $\#\{\gamma^{(p-1)} \prec \alpha^{(p)} \mid f(\gamma) \geq f(\alpha)\} = 0$.

Example 2.1.3 In Figure 2.1 (ii): the vertex $f^{-1}(0)$ and the edge $f^{-1}(5)$ are critical simplices and there are no other.

If K is a simplicial complex of dimension m with a discrete Morse function defined on him, then we denote with μ_j the number of critical simplices of index j.

For any field F, we denote with $\beta_j = \dim H_j(K, F)$ the Betti numbers with respect to F, $j = 0, 1, \ldots, m$.

Then the following weak Morse inequalities hold:

(i) for each $j = 0, 1, \ldots, m$ (where m is the dimension of K), we have: $\mu_j \geq \beta_j$.

(ii) $\mu_0 - \mu_1 + \mu_2 - \ldots + (-1)^m \mu_m = \beta_0 - \beta_1 + \beta_2 - \ldots + (-1)^m \beta_m = \chi(K)$ (the Euler relation).

But the strong Morse inequalities hold to:

For each $j = 0, 1, \ldots, m$,

$$\mu_j - \mu_{j-1} + \ldots + (-1)^j \mu_0 \leq \beta_j - \beta_{j-1} + \ldots + (-1)^j \beta_0.$$

Consider K a simplicial complex with exactly c_j simplices of dimension j, for each $j = 0, 1, \ldots, m$, where $m = \dim K$. Let $C_j(K, \mathbb{Z})$ be the space $\mathbb{Z}^{c_j}$.

More precisely $C_j(K, \mathbb{Z})$ is the free abelian group generated by the j-simplices of K, each simple has an orientation.

Then for each j there exists boundary maps $\partial_j : C_j(K, \mathbb{Z}) \to C_{j-1}(K, \mathbb{Z})$, which satisfy the relation $\partial_{j-1} \circ \partial_j = 0$.

Then the chain complex

$$0 \longrightarrow C_m(K, \mathbb{Z}) \xrightarrow{\partial_m} C_{m-1}(K, \mathbb{Z}) \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_1} C_0(K, \mathbb{Z}) \longrightarrow 0$$

calculates the homology of K. For each $j = 0, 1, \ldots, m$, we define: $H_j(C, \partial) = \mathrm{Ker}\,(\partial_j)/\mathrm{Im}\,(\partial_{j+1})$.

Then for each j the following isomorphism holds: $H_j(C, \partial) \cong H_j(K, \mathbb{Z})$, where $H_j(K, \mathbb{Z})$ is the singular homology of K.

2.2 Operation with discrete Morse functions

Let f and g be discrete Morse functions defined respectively on two cell complexes K and L, and V and W their corresponding discrete gradient vector fields.

If L is a subcomplex of K, then $g = f|_L$ is a valid discrete Morse function.

If L is a subdivision of K, one can construct g out of f in order to get the same number of critical cells. This can be done by refining locally the discrete gradient vector field on each subdivided cell, as in Figure 2.2.

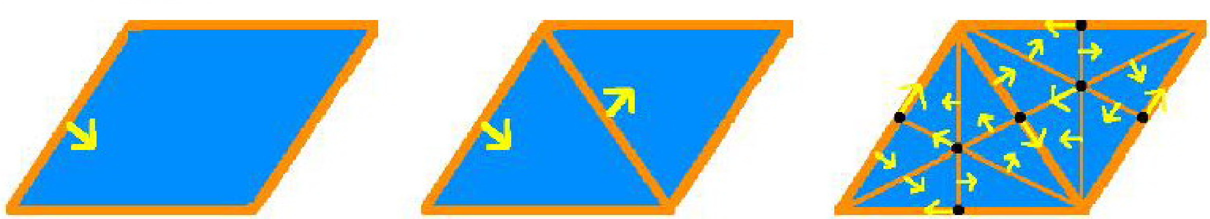


Figure 2.2. Refining a discrete gradient vector field

A complete proof can be found in [31, secţiunea 12].

The cartesian product $K \times L$ of two celular complexes is a cell complex with the following relations:

$$(\alpha_K, \alpha_L) \prec (\beta_K, \beta_L)$$

if $\alpha_K = \beta_k$ and $\alpha_L \prec \beta_L$, or $\alpha_K \prec \beta_K$ and $\alpha_L = \beta_L$.

A discrete gradient vector field $V \times W$ can be defined on $K \times L$ in order to have $\sum_q m_q(f) \cdot m_{p-q}(g)$ critical cells of index p:

$$\begin{cases} \text{dacă } \alpha_K \text{ is not critical, } V(\alpha_K) = 0, & (V \times W)(\{\alpha_K, \alpha_L\}) = 0 \\ \text{dacă } \alpha_K \text{ is not critical, } V(\alpha_K) = \beta_K, & (V \times W)(\{\alpha_K, \alpha_L\}) = \{\beta_K, \alpha_L\} \\ \text{dacă } \alpha_K \text{ is critical, } V(\alpha_L) = 0, & (V \times W)(\{\alpha_K, \alpha_L\}) = 0 \\ \text{dacă } \alpha_K \text{ is critical, } V(\alpha_L) = \beta_L, & (V \times W)(\{\alpha_K, \alpha_L\}) = \{\alpha_K, \beta_L\} \end{cases}$$

This corresponds to discrete Morse function $f \times g : K \times L \to \mathbb{R}$, where

$$(f \times g)(x, y) = f(x)g(y).$$

Figures 2.3 and 2.4 represents an example of cartesian product between a segment and a triangle.

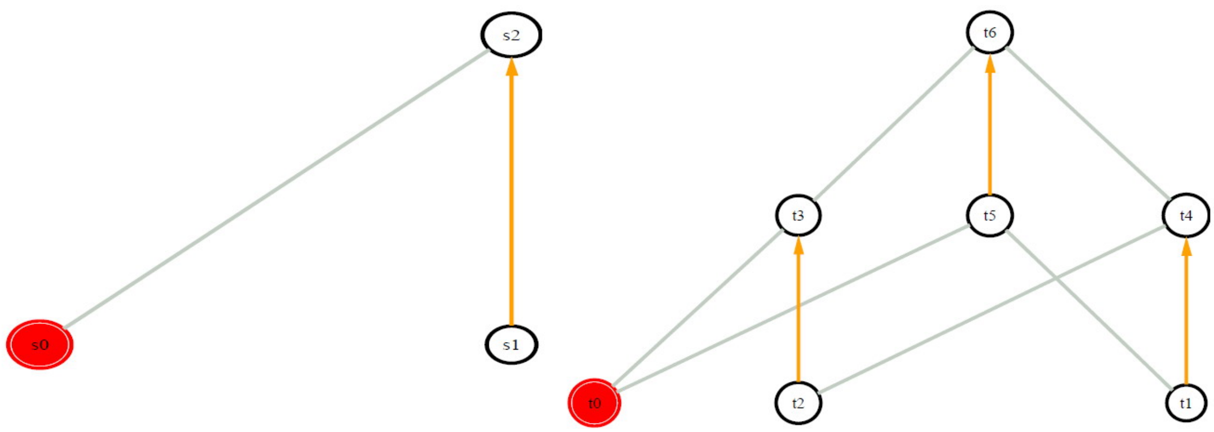


Figure 2.3. The Hasse diagram of a discrete gradient vector field on a segment and on a triangle

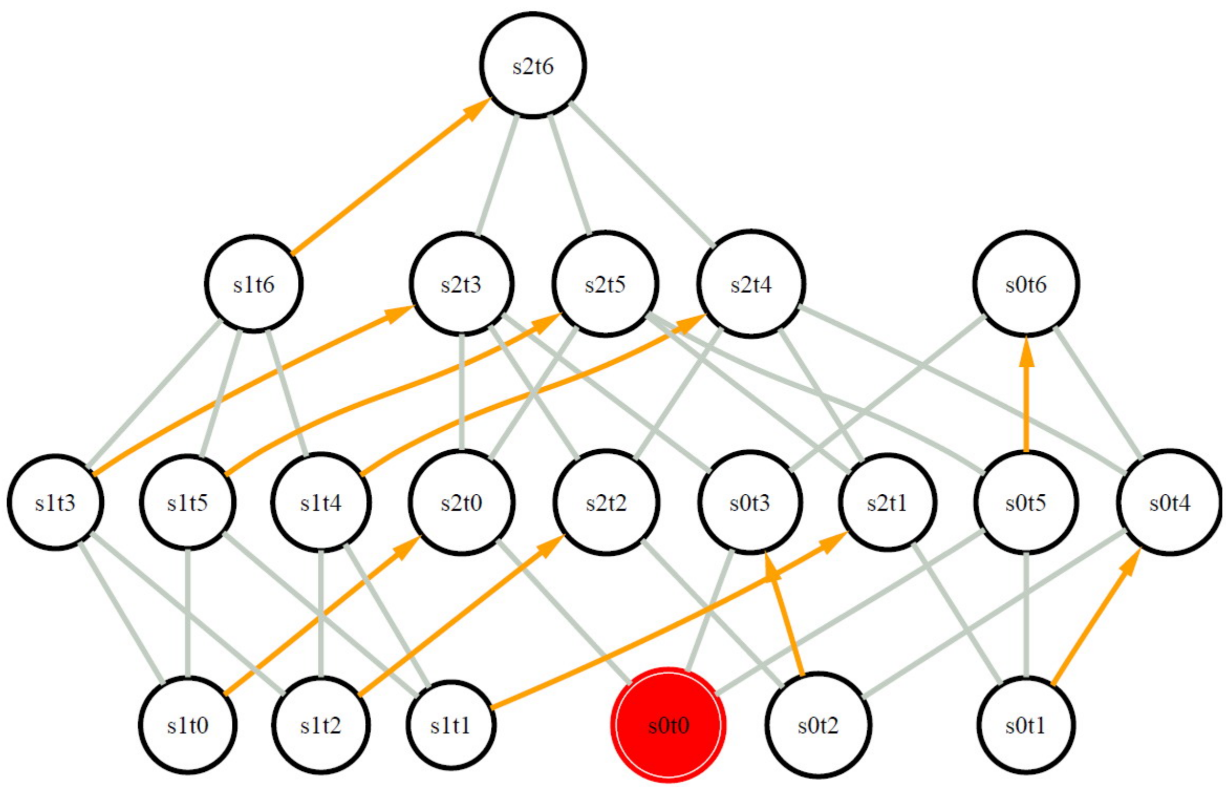


Figure 2.4. The Hasse diagram of the cartesian product of the discrete gradient vector field of Figure 2.3

2.3 The discrete Morse-Smale characteristic of simplicial complexes

Let K^m be a finite m-dimensional simplicial complex. Lete $\Omega(K)$ denote the set containing all discrete Morse functions defined on K. It is clear that $\Omega(K) \neq \emptyset$, because we have the trivial example of discrete Morse function $f(\sigma) = \dim \sigma$, $\sigma \in K$.

For $f \in \Omega(K)$, let $\mu_j(f)$ denote the number of critical simplices of index j of K, for function f, $j = 0, 1, \ldots, m$.

Let $\mu(f)$ be the number denoted by

$$\mu(f) = \sum_{j=0}^{m} \mu_j(f),$$

that means $\mu(f)$ is the total number of critical simplices of K, for a function f.

Definition 2.3.1 The number

$$\gamma(K) = \min\{\mu(f) : \ f \in \Omega(K)\},$$

is called the discrete Morse-Smale characteristic of K.

So the discrete Morse-Smale characteristic represents the minimal number of critical simplices of all discrete Morse functions defined on K.

In similar way we define the numbers $\gamma_j(k)$, for $j = 0, 1, \ldots, m$.

Deci caracteristica Morse-Smale discretă reprezintă numărul minim de simplexe critice pentru toate funcţiile Morse discrete definite pe K.

In similar way we define the numbers $\gamma_j(K)$, for $j = 0, 1, \ldots, m$:

$$\gamma_j(K) = \min\{\mu_j(f) : \ f \in \Omega(K)\},$$

which represents the numbers of critical simplices of dimension j of all discrete Morse functions defined on K.

A complicated problem in combinatorial topology is to calculate effectively this numbers associated to a finite dimensional complex. We don't know yet a finite algorithm to calculate this numbers for simplicial complex.

Let L^m be a finite simplicial complex and let $\psi : L \to K$ be a simplicial isomorphism.

We consider the discrete Morse functions $f : K \to \mathbb{R}$ and $g : L \to \mathbb{R}$, defined on simplicial complexes K and L such that the following diagram commutes:

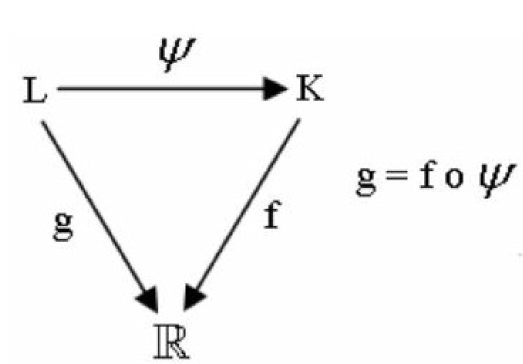


Figure 2.5

We define the critical sets of functions f and g:

$$C(f) = \{\alpha \mid \alpha \text{ este simplex critic a lui } f\}$$

$$C(g) = \{\beta \mid \beta \text{ este simplex critic a lui } g\}$$

Because ψ is a simplicial isomormorphism then the following relation holds

$$C(f) = \psi(C(g)).$$

So we have the inequality $\#C(f) \geq \gamma(L)$, for every discrete Morse function defined on simplicial complex K. This imply that $\gamma(K) \geq \gamma(L)$. In a similar way we prove this inequality $\gamma(K) \leq \gamma(L)$ and so we obtain $\gamma(K) = \gamma(L)$.

In conclusion, for isomorphism simplicial complexes K and L we have:

$$\gamma(K) = \gamma(L) \text{ and } \gamma_j(K) = \gamma_j(L),\ j = 0, 1, \dots, m.$$

So the numbers $\gamma(K)$ and $\gamma_j(K)$, $j = 0, 1, \dots, m$, are invariants of the simplicial complex K.

2.4 Exact discrete Morse functions. F-perfect discrete Morse functions

Let K^m be a finite m-dimensional simplicial complex. It is clear that $\Omega(K) \neq \emptyset$, because we have the trivial example of discrete Morse function $f(\sigma) = \dim \sigma$, for each simplex $\sigma \in K$.

Let $H_j(K, F)$, $j = 0, 1, \dots, m$, be the singular homology groups with coefficients in F and

$$\beta_j(K, F) = \text{rank} H_j(K, F) = \dim_F H_j(K, F),\ j = 0, 1, \dots, m,$$

the Betti numbers of K with respect to F.

For each $f \in \Omega(K)$, $j = 0, 1, \dots, m$, the weak Morse inequalities holds:

$$\mu_j(f) \geq \beta_j(K, F).$$

Definition 2.4.1 A discrete Morse function $f \in \Omega(K)$ is called exact (or minimal) if:

$$\mu_j(f) = \gamma_j(K),\ \text{pentru } j = 0, 1, \dots, m.$$

So the number of critical simplices of any dimension of an exact discrete Morse function defined on a finite simplicial complex is always minimal.

Definition 2.4.2 A discrete Morse function $f \in \Omega(K)$ is called F-perfect if:

$$\mu_j(f) = \beta_j(K, F),\ \text{for each } j = 0, 1, \dots, m.$$

Considering the weak Morse inequalities and the definition for Morse-Smale characteristic we obtain the following inequalities:

$$\mu_j(f) \geq \min\{\mu_j(f) :\ f \in \Omega(K)\} = \gamma_j(K) \geq \beta_j(K, F).$$

Theorem 2.4.3 *The simplicial complex K has F-perfect discrete Morse functions if and only if*

$$\gamma(K) = \beta(K, F),$$

where $\beta(K, F) = \sum_{j=0}^{m} \beta_j(K, F)$ is the total Betti number of K with respect to F.

Let K^m be a finite m-dimensional simplicial complex. We known that $C_j(K, \mathbb{Z})$, $j = 0, 1, \ldots, m$, are the free finite abelian groups with j generators in K.

Because the factorization subgroups of a finitely generated group are finite generated then the singular homology group $H_j(K, \mathbb{Z})$ is finite generated.

From fundamental theorem of free finite generated groups we have

$$H_j(K, \mathbb{Z}) \simeq A_j \oplus B_j,$$

where A_j is a free group and B_j is the torsion subgroup of $H_j(K, \mathbb{Z})$.

So the singular homology groups $H_j(K, \mathbb{Z})$, $j = 0, 1, \ldots, m$, are finite generated and we have

$$H_j(K, \mathbb{Z}) \simeq (\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}) \oplus (\mathbb{Z}_{n_{j_1}} \oplus \ldots \oplus \mathbb{Z}_{n_{\beta(j)}})$$

where $\mathbb{Z}$ is taken by β_j times in the free group and $\beta_j = \beta_j(K, \mathbb{Z})$, $j = 0, 1, \ldots, m$, represents the Betti numbers of K with respect to the group $(\mathbb{Z}, +)$, i.e.

$$\beta_j(K, \mathbb{Z}) = \text{rank} H_j(K, \mathbb{Z}).$$

Example 2.4.4 We consider the circle S^1. The singular homology of S^1 is:

$$H_j(S^1, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = 0 \\ \mathbb{Z}, & j = 1 \end{cases}$$

So we have:

$$\beta_0(S^1, \mathbb{Z}) = 1, \ \beta_1(S^1, \mathbb{Z}) = 1.$$

Then the total Betti number is:

$$\beta(S^1, \mathbb{Z}) = \beta_0(S^1, \mathbb{Z}) + \beta_1(S^1, \mathbb{Z}) = 1 + 1 = 2.$$

Using Theorem 2.4.3 we obtain

$$\gamma(S^1) = \beta(S^1, \mathbb{Z}) = 2.$$

So this implies that one can define on S^1 a discrete Morse function which has exactly two critical simplices. This function is $\mathbb{Z}$-exact and is defined in Figure 2.6. The critical simplices from figure are encircled.

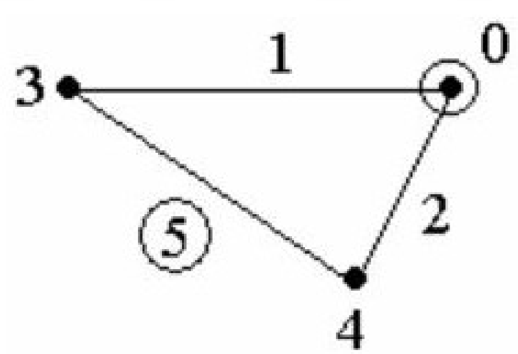


Figure 2.6. A discrete Morse function with two critical simplices defined on unit circle S^1

Example 2.4.5 We consider the 2-dimensional sphere S^2 with his triangulation in Figure 2.7.

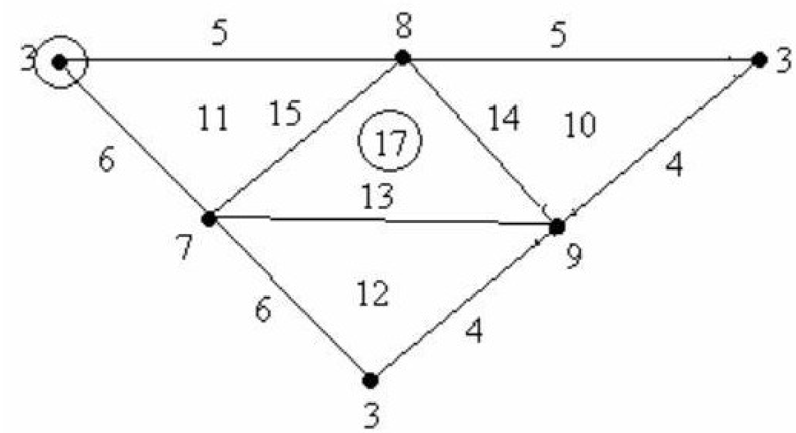


Figure 2.7. A discrete Morse function with two critical simplices defined on 2-dimensional sphere S^2

The singular homology of S^2 is:

$$H_j(S^2, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = 0 \\ 0, & j = 1 \\ \mathbb{Z}, & j = 2 \end{cases}$$

So we have:

$$\beta_0(S^2, \mathbb{Z}) = 1,\ \beta_1(S^2, \mathbb{Z}) = 0,\ \beta_2(S^2, \mathbb{Z}) = 1.$$

Then the total Betti number is

$$\beta(S^2, \mathbb{Z}) = \sum_{j=0}^{2} \beta_j(S^2, \mathbb{Z}) = 1 + 0 + 1 = 2.$$

Using Theorem 2.4.3 we obtain

$$\gamma(S^2) = \beta(S^2, \mathbb{Z}) = 2.$$

So this means that we can define on sphere S^2 a discrete Morse functions with exactly two critical simplices. This functions are $\mathbb{Z}$-exact and is defined in Figure 2.7, the critical simplices are encircled.

Example 2.4.6 We consider the real projective plane $P^2(\mathbb{R})$ with his triangulation in Figure 2.8.
The singular homology of $P^2(\mathbb{R})$ with respect to $\mathbb{Z}_2$ is:

$$H_j(P^2(\mathbb{R}), \mathbb{Z}_2) = \mathbb{Z}_2,\ j = 0, 1, 2.$$

So we have

$$\beta_0(P^2(\mathbb{R}), \mathbb{Z}_2) = 1,\ \beta_1(P^2(\mathbb{R}), \mathbb{Z}_2) = 1,\ \beta_2(P^2(\mathbb{R}), \mathbb{Z}_2) = 1.$$

Then the total Betti number is

$$\beta(P^2(\mathbb{R}), \mathbb{Z}_2) = \beta_0(P^2(\mathbb{R}), \mathbb{Z}_2) + \beta_1(P^2(\mathbb{R}), \mathbb{Z}_2) + \beta_2(P^2(\mathbb{R}), \mathbb{Z}_2) = 1 + 1 + 1 = 3.$$

Using Theorem 2.4.3 we obtain

$$\gamma(P^2(\mathbb{R})) = \beta(P^2(\mathbb{R}), \mathbb{Z}_2) = 3.$$

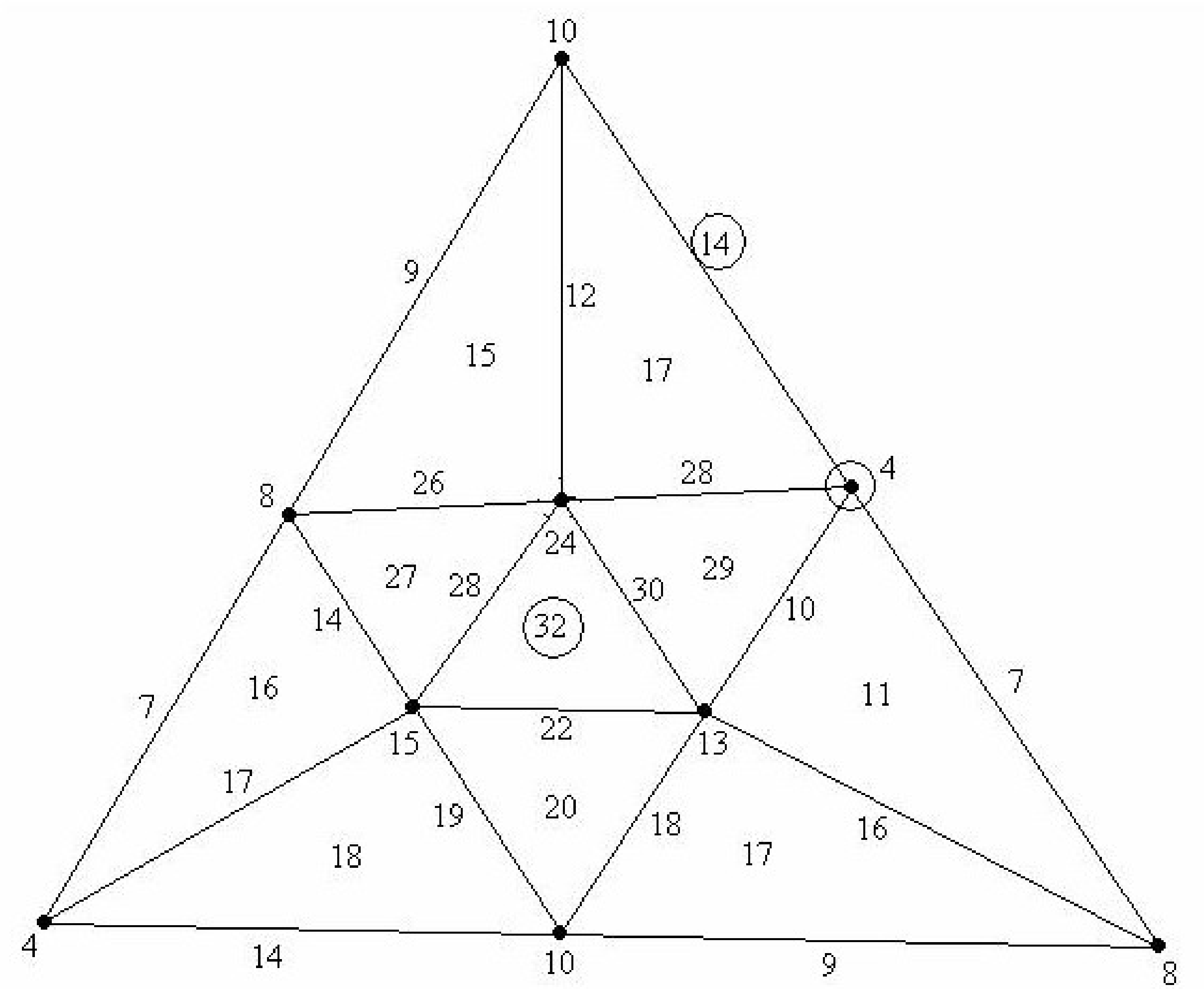


Figure 2.8. A discrete Morse function with three critical simplices defined on the real projective plane $P^2(\mathbb{R})$

So this means that we can define on real projective plane $P^2(\mathbb{R})$ a discrete Morse function with exactly three critical simplices. This function is $\mathbb{Z}_2$-exact and is defined in Figure 2.8. The critical simplices from figure are encircled.

Example 2.4.7 We consider the Möbius band M. The singular homology of M with respect to $\mathbb{Z}$ is

$$H_j(M,\mathbb{Z}) = \mathbb{Z},\ \text{pentru } j = 0,1 \text{ and } H_2(M,\mathbb{Z}) = 0.$$

From the universal coefficients formula for homology (see [2, pag. 118]) it follows:

$$H_k(M;\mathbb{Z}) \simeq (H_k(M;\mathbb{Z}) \otimes \mathbb{Z}_2) \oplus \mathrm{Tor}(\mathbb{Z}_2; H_{k-1}(M;\mathbb{Z})),\ k \in \mathbb{Z},$$

where $\mathrm{Tor}(\mathbb{Z}_2; H_{k-1}(M;\mathbb{Z}))$ is the torsion product of the groups $(\mathbb{Z}_2,+)$ and $H_{k-1}(M;\mathbb{Z})$.

Then we have

$$H_0(M;\mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_1(M;\mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_2(M;\mathbb{Z}_2) \simeq 0,$$

and this implies that

$$\beta_0(M;\mathbb{Z}_2) = 1,\ \beta_1(M;\mathbb{Z}_2) = 1,\ \beta_2(M;\mathbb{Z}_2) = 0.$$

Then the total Betti number is

$$\beta(M,\mathbb{Z}_2) = 1 + 1 + 0 = 2.$$

On the other hand, using Theorem 2.4.3, we obtain

$$\gamma(M) = \beta(M, \mathbb{Z}_2) = 2.$$

So this means that we can define on Möbius band M a discrete Morse function with exactly two critical simplices. This function is $\mathbb{Z}_2$-exact and is defined in Figure 2.9, where the critical simplices are encircled.

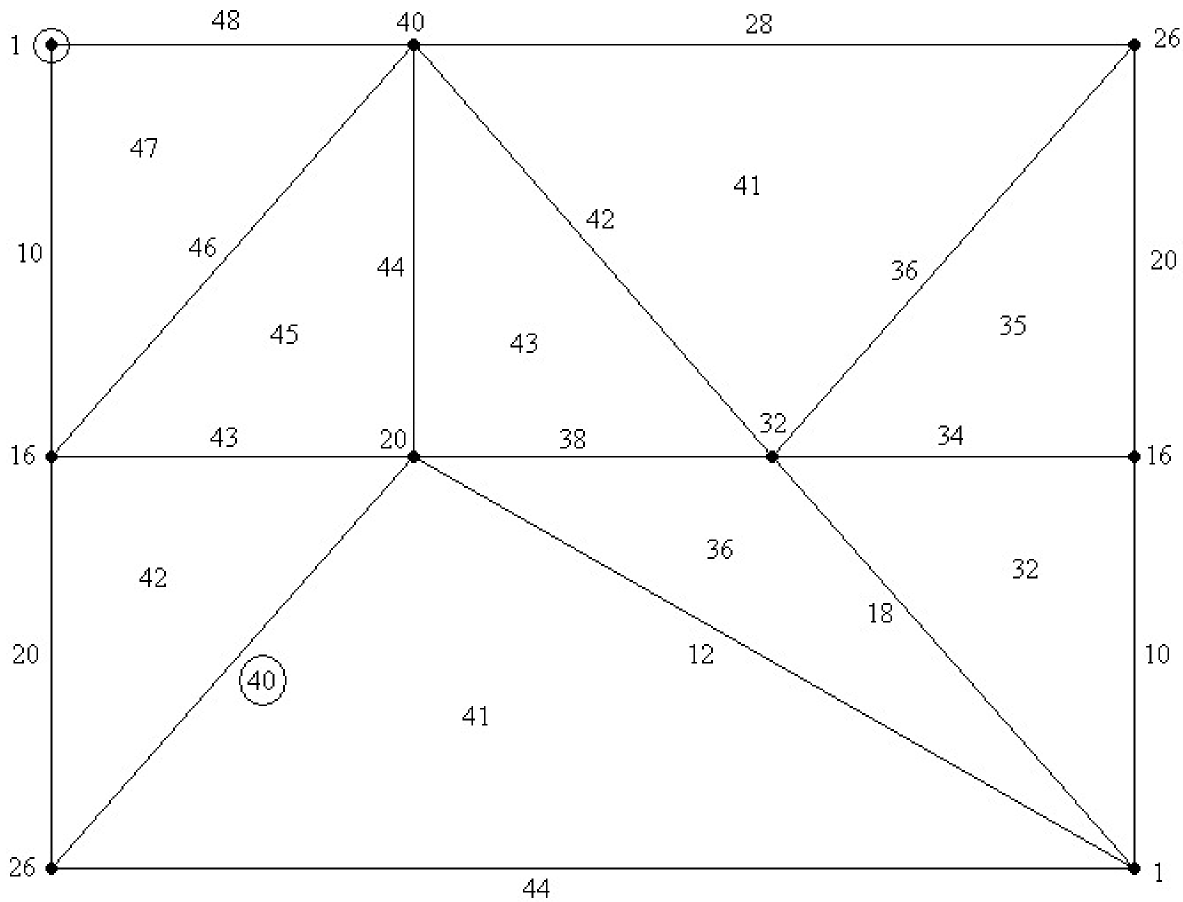


Figure 2.9. A discrete Morse function with two critical simplices defined on the Möbius band M

Example 2.4.8 We consider the Klein bottle $K = P^2(\mathbb{R}) \# P^2(\mathbb{R})$ with his triangulation in Figure 2.10, where # represents the conex sum.

The singular homology of K with respect to $\mathbb{Z}$ is

$$H_0(K, \mathbb{Z}) = \mathbb{Z},\ H_1(K, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z},\ H_2(K, \mathbb{Z}) = 0.$$

Using the universal coefficients formula for homology we obtain

$$H_0(K, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_1(K, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2,\ H_2(K, \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

This implies that $\beta_0(K, \mathbb{Z}_2) = 1,\ \beta_1(K, \mathbb{Z}_2) = 2,\ \beta_2(M, \mathbb{Z}_2) = 1$, then the total Betti number is

$$\beta(K, \mathbb{Z}_2) = 1 + 2 + 1 = 4.$$

Using Theorem 2.4.3, we obtain

$$\gamma(K) = \beta(K, \mathbb{Z}_2) = 4.$$

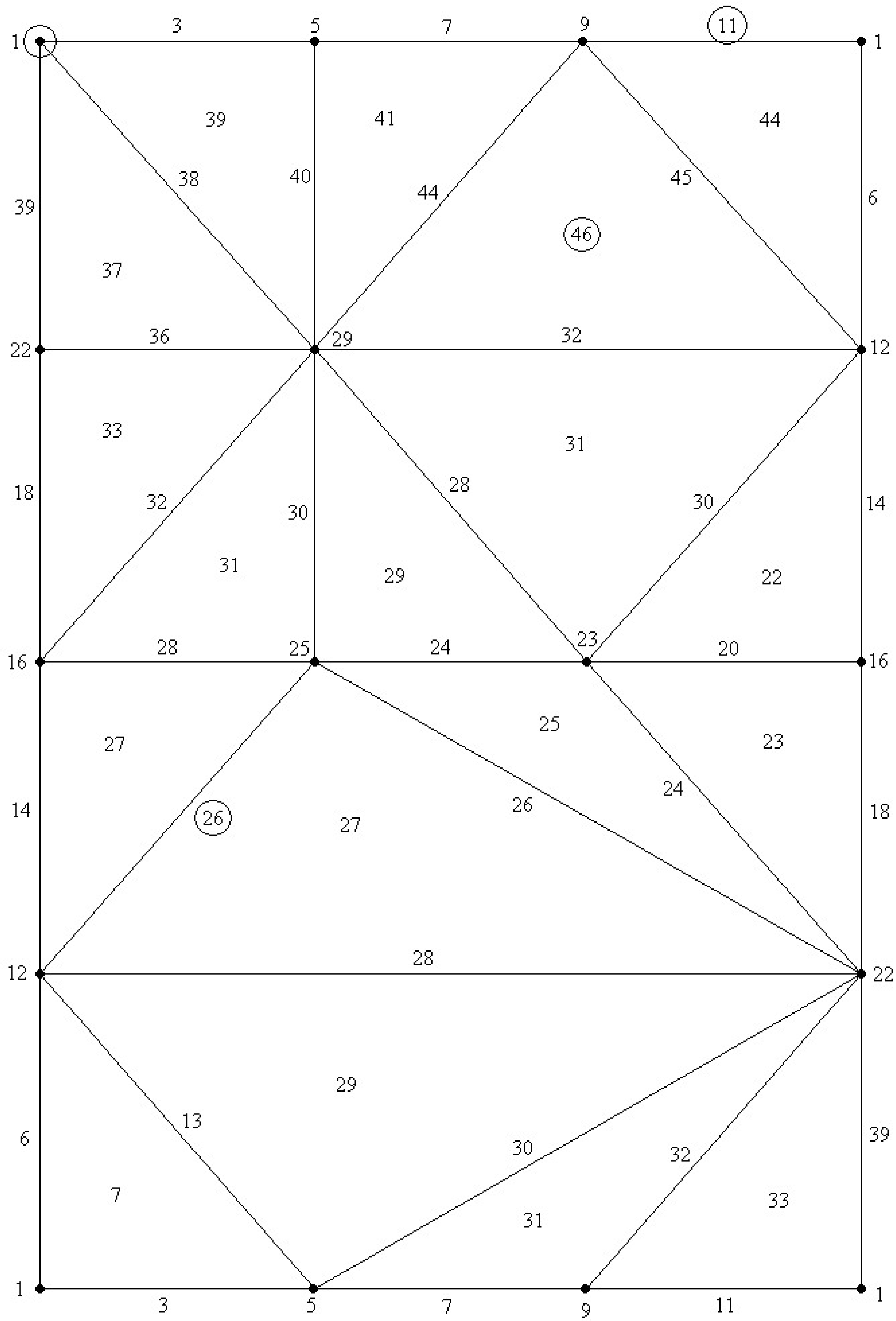


Figure 2.10. A discrete Morse function with four critical simplices defined on the Klein bottle K

So this means that we can define on Klein bottle K a discrete Morse function with exactly four critical simplices. This function is $\mathbb{Z}_2$-exact and is defined in Figure 2.10. The critical simplices from figure are encircled.

Example 2.4.9 We consider the torus $T^2 = S^1 \times S^1$, with his triangularion in Figure 2.11.

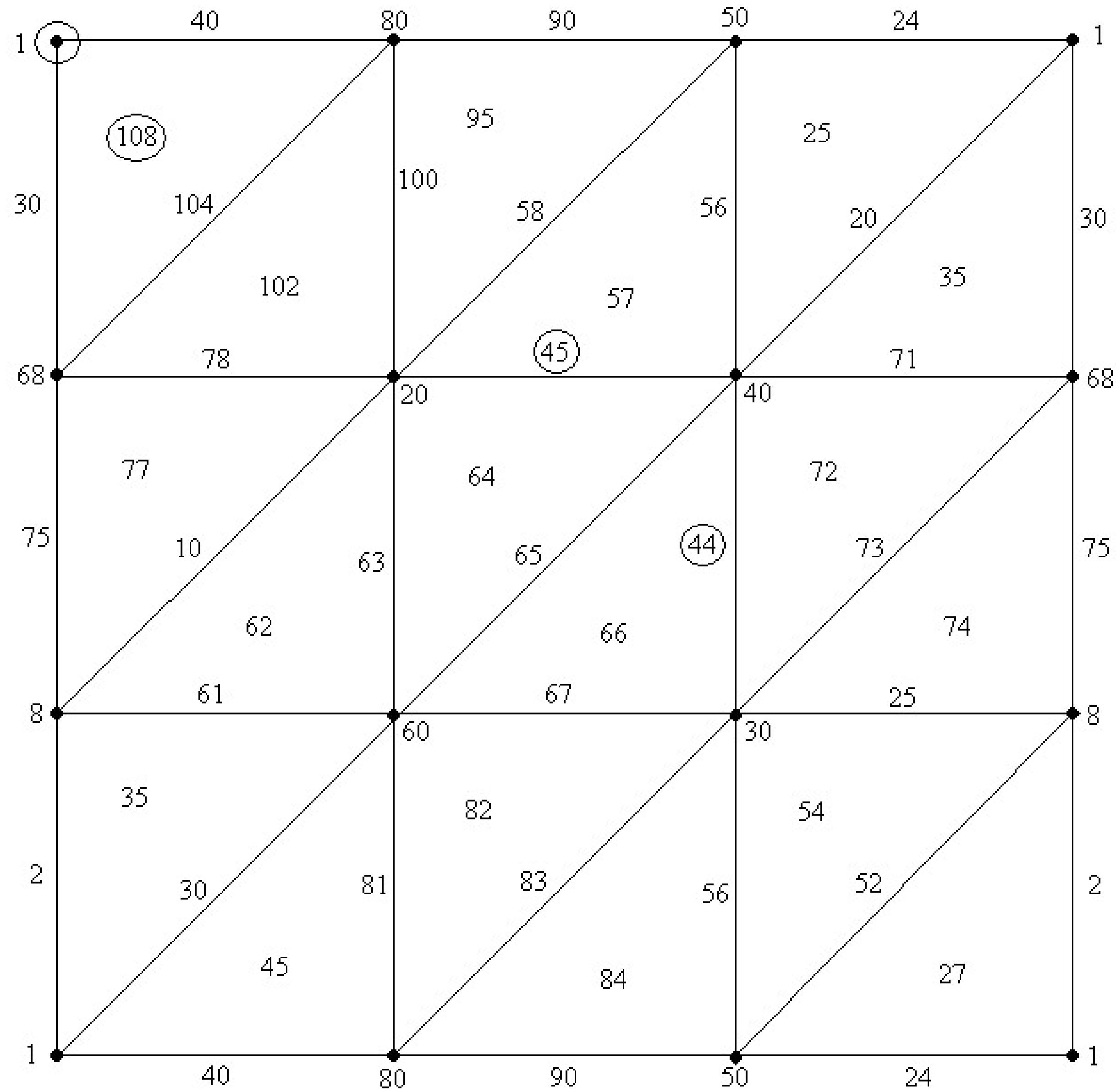


Figure 2.11. A discrete Morse function with four critical simplices defined on the torus $S^1 \times S^1$

The singular homology of the torus $T^2 = S^1 \times S^1$ peste $\mathbb{Z}$ is

$$H_0(T^2) = \mathbb{Z},\ H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z},\ H_2(T^2) = \mathbb{Z}.$$

Then, using the universal coefficients formula we obtain

$$H_0(T^2, \mathbb{Z}_2) \simeq \mathbb{Z}_2,\ H_1(T^2, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2,\ H_2(T^2, \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

This implies that

$$\beta_0(T^2, \mathbb{Z}_2) = 1,\ \beta_1(T^2, \mathbb{Z}_2) = 2,\ \beta_2(T^2, \mathbb{Z}_2) = 1,$$

then the total Betti number is

$$\beta(T^2, \mathbb{Z}_2) = \sum_{j=0}^{2} \beta_j(T^2, \mathbb{Z}_2) = 1 + 2 + 1 = 4.$$

Using Theorem 2.4.3, we obtain

$$\gamma(S^1 \times S^1) = \beta(S^1 \times S^1, \mathbb{Z}_2) = 4.$$

This implies that we can define on torus T^2 a discrete Morse function with exactly four critical simplices. This function is $\mathbb{Z}_2$-exact and is defined in Figure 2.11. The critical simplices from figure are encircled.

Example 2.4.10 We consider the triangulation of the torus with two holes obtained from a conex sum of two torus T^2, which is represented in Figure 2.12.

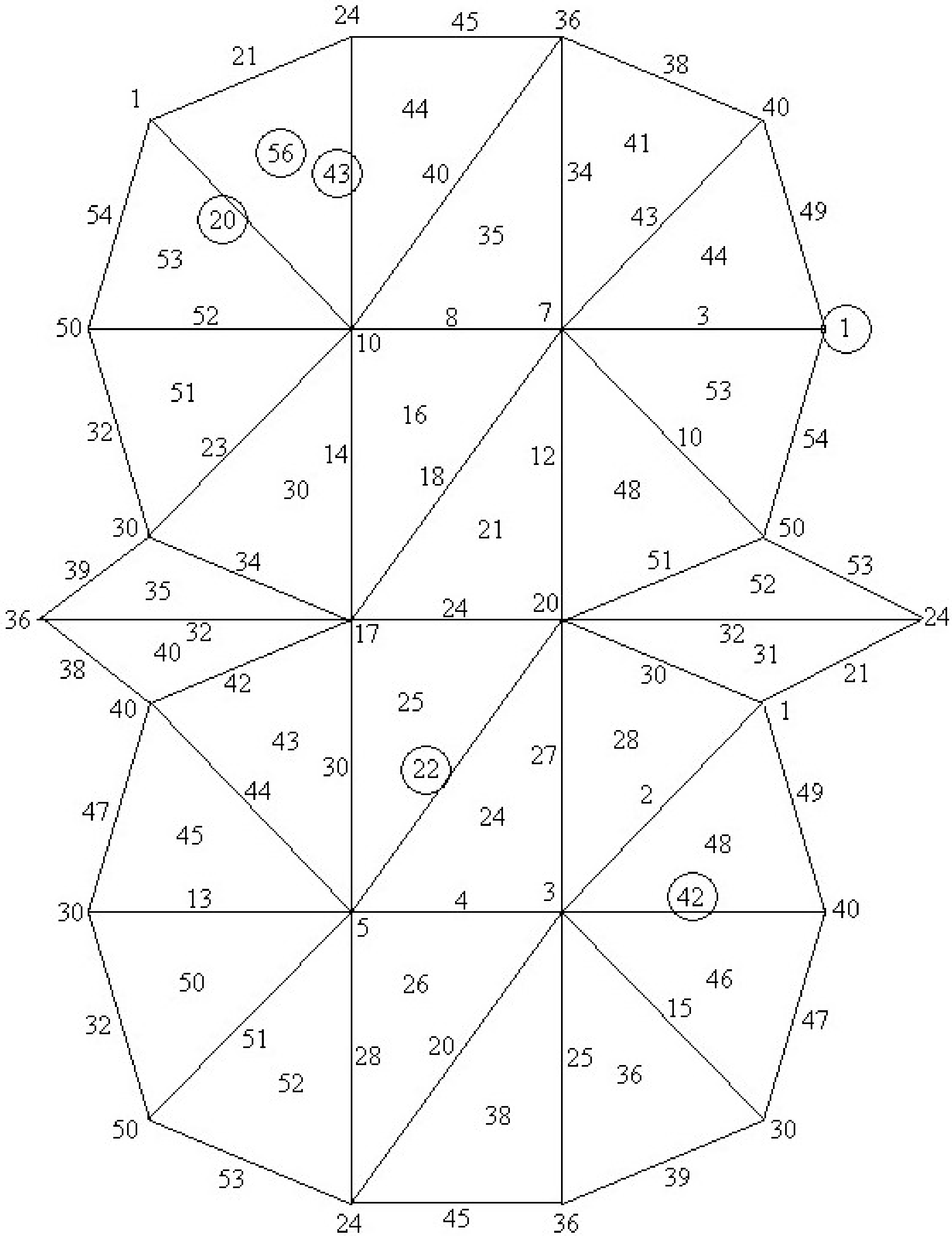


Figure 2.12. A discrete Morse function with six critical simplices defined on the conex sum of two torus

The singular homology of a torus with two holes $T_2 = T^2 \# T^2$ with respect to $\mathbb{Z}$ is

$$H_0(T_2) = \mathbb{Z}, \quad H_1(T_2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad H_2(T_2) = \mathbb{Z}.$$

Then, using the universal coefficients formula we obtain

$$H_0(T_2, \mathbb{Z}_2) \simeq Z_2, \quad H_1(T_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H_2(T_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

This implies that

$$\beta_0(T_2, \mathbb{Z}_2) = 1, \quad \beta_1(T_2, \mathbb{Z}_2) = 4, \quad \beta_2(T_2, \mathbb{Z}_2) = 1,$$

so the total Betti number is

$$\beta(T_2, \mathbb{Z}_2) = \sum_{j=0}^{2} \beta_j(T_2, \mathbb{Z}_2) = 1 + 4 + 1 = 6.$$

Using Theorem 2.4.3, we obtain $\gamma(T_2) = \beta(T_2, \mathbb{Z}_2) = 6$, this implies that we can define on torus with two holes T_2, a discrete Morse function with exactly six critical simplices. This function is $\mathbb{Z}_2$-exact and is defined in Figure 2.12. The critical simplices from figure are encircled.

Example 2.4.11 We consider the triangulation of Dunce hat which is shown in Figure 2.13.

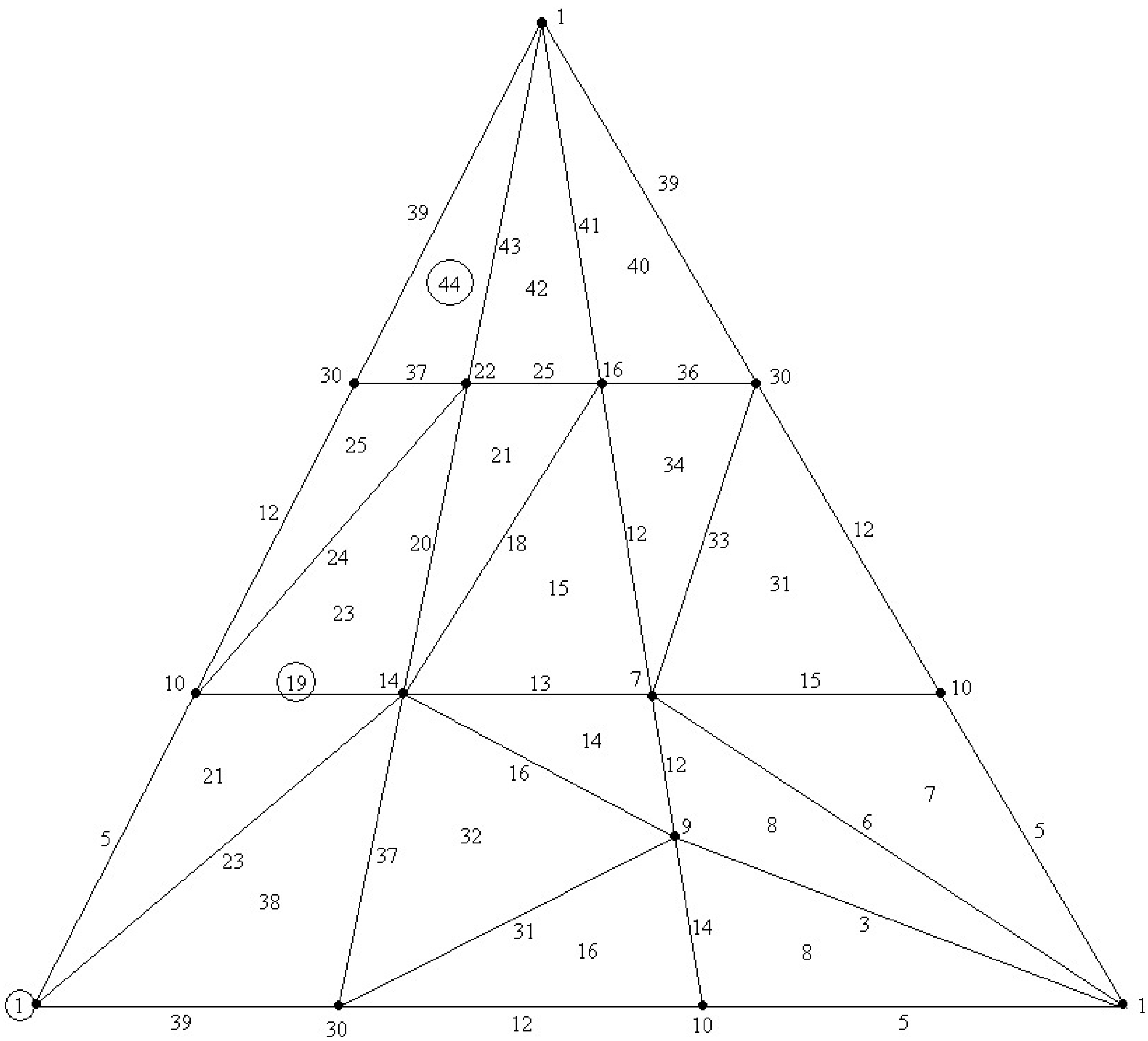


Figure 2.13. A discrete Morse function with three critical simplices defined on the Dunce hat

The singular homology of the Dunce hat DH with respect to $\mathbb{Z}$ is

$$H_0(DH) = \mathbb{Z}, \quad H_1(DH) = \mathbb{Z}, \quad H_2(DH) = \mathbb{Z}.$$

Then using the universal coefficients formula we obtain

$$H_0(DH, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_1(DH, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_2(DH, \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

This implies that

$$\beta_0(DH, \mathbb{Z}_2) = 1, \quad \beta_1(DH, \mathbb{Z}_2) = 1, \quad \beta_2(DH, \mathbb{Z}_2) = 1,$$

then the total Betti number is:

$$\beta(DH, \mathbb{Z}_2) = \sum_{j=0}^{2} \beta_j(DH, \mathbb{Z}_2) = 1 + 1 + 1 = 3.$$

Using Theorem 2.4.3, we obtain $\gamma(DH) = \beta(DH, \mathbb{Z}_2) = 3$, this implies that we can define on the Dunce hat DH a discrete Morse function with exactly three critical simplices. This function is $\mathbb{Z}_2$-exact and is defined in Figure 2.13. The critical simplices from figure are encircled.

Chapter 3

Perfect discrete Morse functions

In this chapter we study the perfect discrete Morse functions on a 2-simplicial complex. These are those discrete Morse functions such that the number of critical i-simplices coincides with the i-th Betti number of the complex. We establish conditions under which a 2-complex admits a perfect discrete Morse function and conversely, we get topological properties of a 2-complex admitting such kind of functions. These results can be considered as a first step in the study of perfect discrete Morse functions on 3-dimensional simplicial complexes.

3.1 The classification theorem of surfaces

Theorem 3.1.1 (The classification of surfaces without boundary) *Any compact connected surface without boundary is homeomorphic to exactly one of the following surfaces: a sphere S^2, a connected sum of $g > 0$ tori T_g, or connected sum of $g > 0$ projective planes M_g.*

A proof for this theorem can be found in [10].

The sphere (Figure 3.1(a)) and connected sums of tori (Figure 3.1(b)) are orientable surfaces, and the Möbius strip is not orientable.

The number g is called the genus of the surface. The Betti number (with coefficient in $\mathbb{Z}_2$) and the orientability completely characterize the topology of a surface, as we can explicitly calculate the homology groups of the standard surfaces [10].

Figure 3.1. Examples of surfaces without boundary:
(a) 2-sphere; (b) Sum of 2 tori;
(c) Klein bottle: connected sum of 2 projective planes

Proposition 3.1.2 (Homology groups of the standard surfaces)

$$\begin{array}{lll} H_0(S^2) = \mathbb{Z}_2, & H_1(S^2) = 0, & H_2(S^2) = \mathbb{Z}_2, \\ H_0(T_g) = \mathbb{Z}_2, & H_1(T_g) = 2g \cdot \mathbb{Z}_2, & H_2(T_g) = \mathbb{Z}_2, \\ H_0(M_g) = \mathbb{Z}_2, & H_1(M_g) = g \cdot \mathbb{Z}_2, & H_2(M_g) = \mathbb{Z}_2. \end{array}$$

Proposition 3.1.3 (The classification of surfaces with boundary) *Any compact connected surface with a non-empty boundary is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum of tori; or finite connected sum of projective planes, in each case with some finite number of open disks removed.*

A proof of this extension of the theorem for closed surfaces can also be found in [10].

The homology group H_0 remains unchanged, and H_2 vanishes for surfaces with boundary.

For example, identifying the boundaries of two disks with the boundary of a cylinder (Figure 3.2(a)) creates a space homeomorphic to the sphere. Identifying the boundary of a disk with the boundary of a Möbius strip (Figure 3.2(b)) creates a space homeopmorphic to the projective plane.

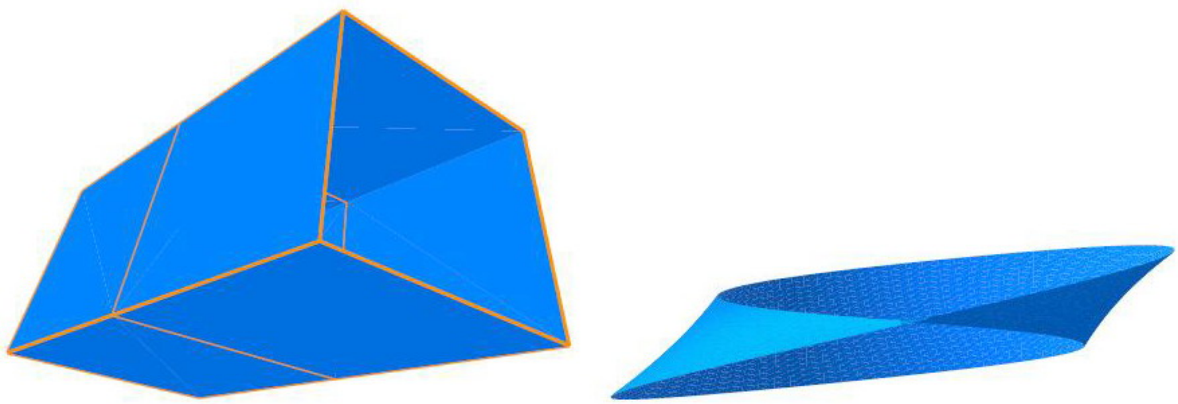


Figure 3.2. Examples for surfaces with a non-empty boundary:
(a) Cylinder; (b) Möbius strip

3.2 Sufficient conditions for exactness of discrete Morse functions

In this section we will enounce sufficient conditions to guarantee that a discrete Morse function is exact or optimal.

Proposition 3.2.1 (Surfaces without boundary) *Let f be a discrete Morse function defined on a compact connected surface without boundary. If f has exactly one critical vertex and one critical face, and possibly many critical edges, then it is optimal.*

Proposition 3.2.2 (Surfaces with boundary) *Let f be a discrete Morse function defined on a compact connected surface with a non-empty boundary. If f has one critical vertex, no critical face, and possibly many critical edges, then it is optimal.*

The construction algorithm for an exact discrete Morse function is described in the following.

Given a finite cell complex K that has the topology of a 2-manifold, the algorithm proceeds on each connected component in 4 steps:

1. Construct a spanning tree T on the dual pseudograph of K.

2. If K has a boundary, add one loop to T.
3. Define a discrete Morse function on T.
4. Define a discrete Morse function on the complement of T.

First step. (Construction of a face-spanning tree). The spanning tree T can be constructed out of the dual pseudograph by any of the standard algorithms [53]. In particular, we can use some mesh compression's strategies. For example, the compression algorithm which is used in [65] (Figure 3.3), and which in the case of the torus conduct to the trees from Figure 3.4.

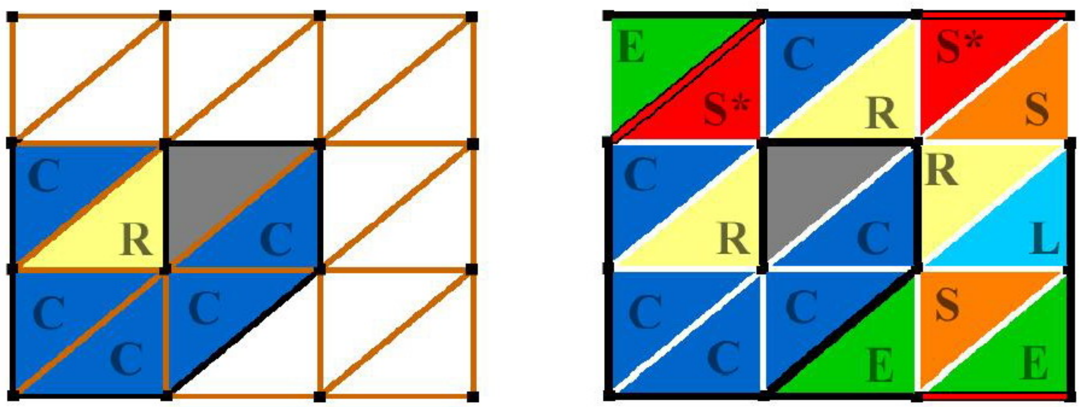


Figure 3.3. The mesh compression codes on a triangulated torus

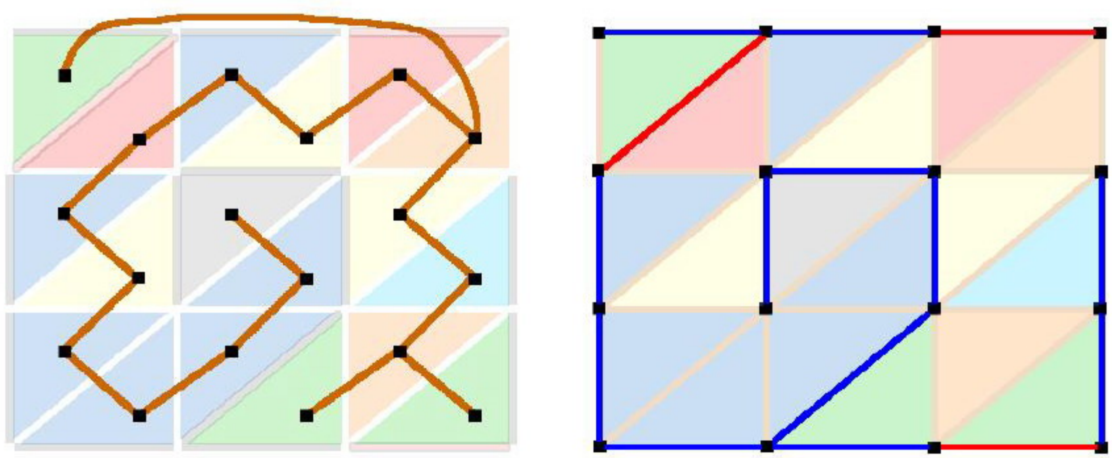


Figure 3.4. The resulting face-spanning tree T and its complement graph G

Second step. (Addition of one loop). We test if the surface has boundary. If we found a loop, we add it to T, so T becomes a pseudograph. For example on Figure 3.5, the loop (with value 21) has been added to T.

Third step. (Definition of the function on T) We select a root of T, and we assign to every node of T (i.e. 2-cells of K) its height in the tree plus a constant c. We assign to every link of T (i.e. 1-cells of K) the minimum value of its two end nodes (cf. Figure 3.5). The result of this process on the example of Figure 3.3 is shown in Figure 3.6.

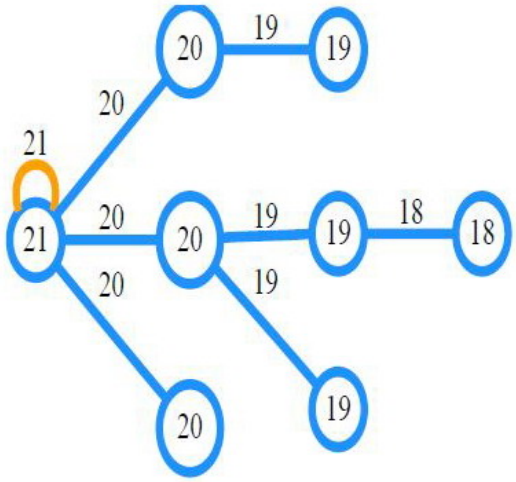


Figure 3.5. Loop added
at step 2 to a face-spanning
tree T (without critical cells)

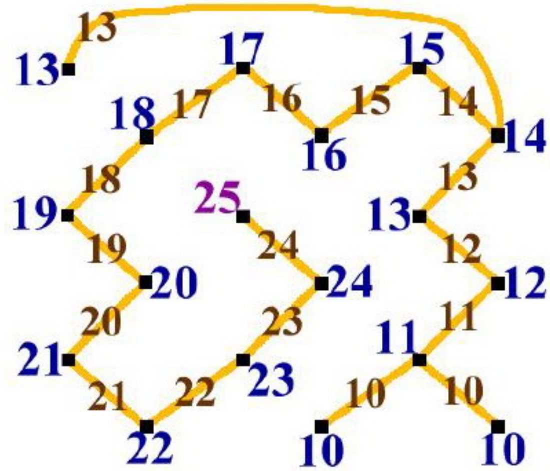


Figure 3.6. The discrete Morse function
on the spanning tree T
of Figure 3.3 (1 critical face)

In our construction, the initial value of c must be at least the number of vertices of K plus 1.
Fourth step. (Definition of the function on the complement of T) We will now consider G, the complement of T. G is a graph with no loop whose nodes are the vertices of K, and whose links are the edges of K that are not represented in T. We build another spanning tree U on G. We assign to every node of G its edge distance to a selected root of U and to every link of U the maximum value of its two end nodes. We finally assign the value $(c-1)$ to each link of $G \setminus U$ (cf. the critical edge of Figure 3.7 with value 12). The result of this process on the example of Figure 3.3 is shown in Figure 3.8.

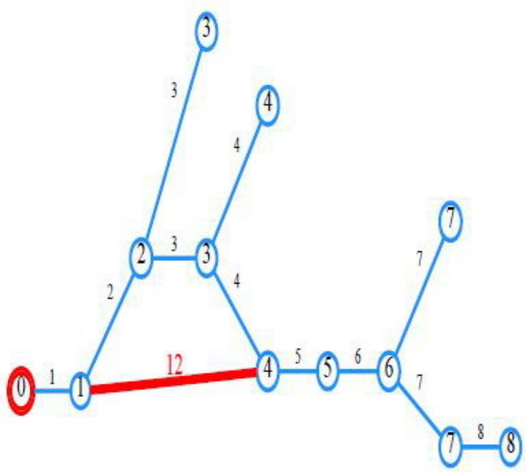


Figure 3.7. The complement graph G
of the cylinder model and its discrete
Morse function
(1 critical vertex and 1 critical edge)

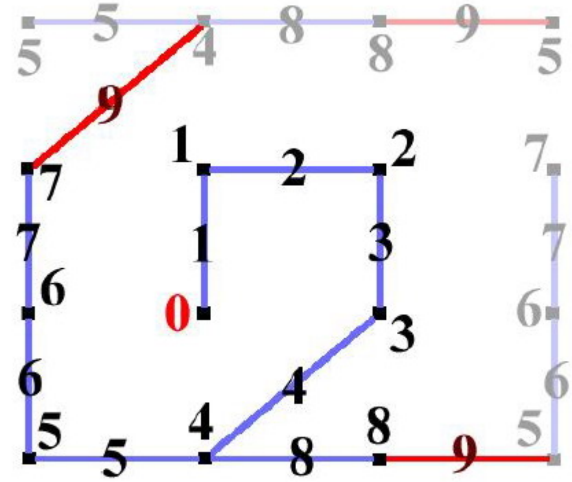


Figure 3.8. The discrete Morse function
on the complement graph G
of Figure 3.3
(1 critical vertex and 2 critical edges)

3.3 $\mathbb{Z}$-perfect discrete Morse functions on 2-complexes

We will start the study of the existence of perfect discrete Morse functions on 2-complexes by considering the case of homology with integer coefficients. Next result proves that every 1-dimensional complex admits $\mathbb{Z}$-perfect discrete Morse function.

Proposition 3.3.1 *Every connected graph admits 1 $\mathbb{Z}$-perfect discrete Morse function.*

Since the spanning tree is built in linear time and in the proof of Proposition 3.3.1 every simplex is visited at most once, it is possible to define an algorithm which constructs a $\mathbb{Z}$-perfect discrete Morse function on G whose time complexity is linear.

Proposition 3.3.1 can be extended to any 2-complex which collapses to a graph in the following way:

Corollary 3.3.2 *Let K be a 2-complex which collapses to a graph G contained in K. Then K admits a $\mathbb{Z}$-perfect discrete Morse function.*

it is interesting to point out that since every surface with boundary collapses to a graph then such kind of surfaces admit a $\mathbb{Z}$-perfect discrete Morse function.

Next result show us how collapsibility of a general complex can be put in terms of its Morse-Smale characteristic.

Lemma 3.3.3 *Let K be a complex. Then K is collapsible if and only if $\gamma(K) = 1$.*

Now we are going to give several results on the links between the existence of $\mathbb{Z}$-perfect discrete Morse functions on a given 2-complex with some trivial homology groups and its simple homotopy type.

Proposition 3.3.4 *Let K be a compact connected 2-complex admitting a $\mathbb{Z}$-perfect discrete Morse function. The following results hold:*

1. If K is acyclic, then K is collapsible.

2. If $H_0(K) = 0$ and $H_1(K) = 0$, then K has the same simple homotopy type as a wedge of spheres S^2.

3. If $H_0(K) = 0$ and $H_2(K) = 0$, then K has the same simple homotopy type as a graph.

Corollary 3.3.5 *Let K be a compact connected 2-complex. If K is acyclic and no-collapsible then K does not admit $\mathbb{Z}$-perfect discrete Morse functions.*

Let $co(K)$ be the minimal number of 2-simplices $f_1, \ldots, f_{co(K)}$, that need to be removed from K so that $K - \{f_1, \ldots, f_{co(K)}\}$ collapses to a graph. Next theorem establishes how $co(K)$ and the existence of $\mathbb{Z}$-perfect discrete Morse functions are related.

Theorem 3.3.6 *Let K be a compact connected 2-complex. Then K admits a $\mathbb{Z}$-perfect discrete Morse function if and only if $co(K) = \beta_2(K;\mathbb{Z})$.*

Corollary 3.3.7 *Let K be a compact connected surface without boundary. Then K admits a $\mathbb{Z}$-perfect discrete Morse function if and only if K is orientable.*

Remark 3.3.8 Corollary 3.3.7 can be extended in a straightforward way to 2-pseudomanifolds, in the following way. A 2-pseudomanifold K admits a $\mathbb{Z}$-perfect discrete Morse function if and only if K is orientable.

3.4 F-perfect discrete Morse functions on 2-complexes

In this section we are going to introduce the general problem of the existence of F-perfect discrete Morse functions on 2-complexes where F is any field. This problem arises in a natural way when we study why a given 2-complex does not admit $\mathbb{Z}$-perfect discrete Morse functions. It is essentially due to two main influences: the nature of the first fundamental group of the complex or to the existence of torsion elements in the first homology group of the complex.

In the first case we find examples as Dunce hat, Bing's house and more generally any acyclic and non-collapsible 2-complex. All of these complexes have in common that they are homologically but not homotopically trivial. In fact we are going to prove that these kind of complexes do not admit any kind of F-perfect discrete Morse functions, for any field F.

Theorem 3.4.1 *If K is an acyclic and non-collapsible connected 2-complex then K does not admit any F-perfect discrete Morse function for all field F.*

Notice that in the second case, although these kind of complexes do not admit $\mathbb{Z}$-perfect discrete Morse functions, they admit F-perfect discrete Morse functions for a suitable field F. In particular, Corollary 3.3.7 characterizes those surfaces admitting $\mathbb{Z}$-perfect discrete Morse functions as orientable surfaces. However, next result shows the influence of a change of coefficients in the existence of perfect discrete Morse functions.

Proposition 3.4.2 *Any non-orientable compact connected surface without boundary admits a $\mathbb{Z}_2$-perfect discrete Morse function.*

Corollary 3.4.3 *Any compact connected surface admits a $\mathbb{Z}_2$-perfect discrete Morse function.*

Remark 3.4.4 Moreover, any pseudo-projective space, that is, the space obtained by gluing a 2-ball to S^1 be means of a map of degree p, admits $\mathbb{Z}_p$-perfect discrete Morse functions. It can be proved by repeating the argument of the proof of Proposition 3.4.2.

The existing results in the literature on optimal discrete Morse functions are mainly restricted to triangulated surfaces. This chapter extends this study to the case of general simplicial 2-complexes. The notions of optimal (exact) and perfect discrete Morse functions are not equivalent. Then a new problem arises: how to determine an optimal discrete Morse function on a given 2-complex acyclic non-contractible one. On the other hand, the results from Chapter 3 can be regarded as a first step in the study of perfect discrete Morse functions on triangulated 3-manifolds. It can be carried out by collapsing the considered 3-manifold to a 2-complex, the so called spine of the 3-manifold, and then we reduce the problem to get a perfect function on a 2-complex.

Chapter 4

Homological optimality by discrete Morse theory

4.1 The homology of chain complexes

Let Λ be a commutative field and $\{x_1, x_2, \ldots, x_n\}$ a finite set of symbols. Let l be a positive integer. Consider $B^l = \{x \in E^l \mid \ |x| \leq 1\}$ be the closed unit ball in the l-dimensional Euclidean space E^l.

We denote by $\Lambda[x_1, x_2, \ldots, x_n]$ the formal linear combinations $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n$, with $\lambda_i \in \Lambda$.

Definition 4.1.1 A differential operator for a cell complex K with coefficients in Λ is a linear map $d : \Lambda[K] \to \Lambda[K]$, such that the image of a p-cell σ is a linear combination of some $(p-1)$-cells of the boundary $\partial(\sigma)$ and satisfying $d \circ d = 0$.

Considering a cell complex K embedded in E^l, its geometric realization $|K|$ is a regular triangulable cell complex and therefore a differential operator ∂ with coefficients in the field Λ can always be defined. This operator, called boundary operator, completely determines the singular homology of $|K|$ $|K|$ ([50]).

The chain complex canonically associated to the cell complex K is the graded differential vector space $(C_*(K), \partial)$, where $C_p(K) = \Lambda[K_p]$, $p = 0, 1, \ldots, r$, and $\partial : C_*(K) \to C_{*-1}(K)$ is the previous boundary operator for the cell complex K. For instance, to find a boundary operator ∂ for a simplicial complex is straightforward, but it is not, in general, an easy task for others cell complexes. The following is one of the fundamental results in the theory of cell complexes ([50]).

Theorem 4.1.2 *Let K a finite cell complex. There are algebraic boundary maps $\partial_p : C_p(K, \Lambda) \to C_{p-1}(K, \Lambda)$, for each p, so that $\partial_{p-1} \circ \partial_p = 0$ and such that the resulting differential complex $\{C_p(K, \Lambda), \partial_p\}_{p=0}^r$ calculates the homology of $|K|$. That is, if we define $H_p(C, \partial) = \mathrm{Ker}\,(\partial_p)/\partial_{p+1}(C)$ we have $H_p(C, \partial) \cong H_p(|K|, \Lambda)$.*

4.2 The algebrization of discrete Morse theory

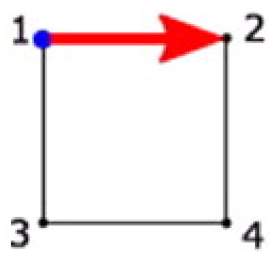
The aim of Discrete Morse Theory is to find simplicial collapses that transform a complex K to a simpler one. This can be done using an essentially algebraic framework in which discrete Morse functions are convenient tools to keep track of the collapses, and the order in which they are done.

Now, we recover all the algebraic machinery underlying in Discrete Morse Theory, establishing a new framework for dealing with special chain complexes associated to finite cell complexes and we show that trees are a convenient combinatorial tool for solving the homological computation problem.

Definition 4.2.1 An integral chain complex (C, d, ϕ) is a graded module $C = \{C_p\}_{p=0}^{n}$ endowed with two linear maps: a differential operator $d : C_* \to C_{*-1}$ and an integral operator $\phi : C_* \to C_{*+1}$, satisfying the global nilpotency properties $d \circ d = 0$, $\phi \circ \phi = 0$.

This integral operator, can also be called chain homotopy operator ([29]).

We will represent an integral operator by an arrow from the cell of lower dimension to the cell of higher dimension (Figure 4.1).



σ	d	ϕ
$\langle 1\rangle$	0	$\langle 1,2\rangle$
$\langle 2\rangle$	0	0
$\langle 3\rangle$	0	0
$\langle 4\rangle$	0	0
$\langle 1,2\rangle$	$\langle 1\rangle + \langle 2\rangle$	0
$\langle 1,3\rangle$	$\langle 1\rangle + \langle 3\rangle$	0
$\langle 2,4\rangle$	$\langle 2\rangle + \langle 4\rangle$	0
$\langle 3,4\rangle$	$\langle 3\rangle + \langle 4\rangle$	0

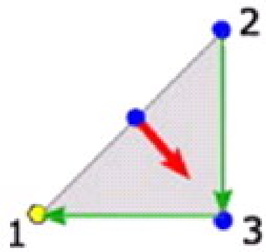
Figure 4.1. An integral chain complex and an integral operator (represented by an arrow)

Definition 4.2.2 An integral chaim complex (C, d, ϕ) is d-pure if the condition $d = d \circ \phi \circ d$ (called homology condition) is satisfied.

An integral chain complex (C, d, ϕ) is ϕ-pure if the condition $\phi = \phi \circ d \circ \phi$ (called Strong Deformation Retract condition) is satisfied.

An integral chain complex that is both, d-pure and ϕ-pure, is called homology integral chain complex. In this case, d (resp. ϕ) is a homology differential (resp. integral) operator.

For instance, the integral chain complex in Figure 4.2, is a homology integral chain complex (the conditions $d = d \circ \phi \circ d$ and $\phi = \phi \circ d \circ \phi$ are satisfied for every cell of the complex).



σ	d	ϕ
$\langle 1\rangle$	0	0
$\langle 2\rangle$	0	$\langle 1,2\rangle + \langle 1,3\rangle$
$\langle 3\rangle$	0	$\langle 1,2\rangle$
$\langle 1,2\rangle$	$\langle 1\rangle + \langle 2\rangle$	$\langle 1,2,3\rangle$
$\langle 1,3\rangle$	$\langle 1\rangle + \langle 3\rangle$	0
$\langle 2,3\rangle$	$\langle 2\rangle + \langle 3\rangle$	0
$\langle 1,2,3\rangle$	$\langle 1,2\rangle + \langle 1,3\rangle + \langle 2,3\rangle$	0

Figure 4.2. A homology integral chain complex. The homology integral operator is represented by arrows

Given two integral chain complexes (C, d, ϕ) and (C, d', ϕ') a map of integral chain complexes $(f, g) : (C, d, \phi) \to (C', d', \phi')$ is a couple of linear maps $f : C \to C'$, $g : C' \to C$ such that

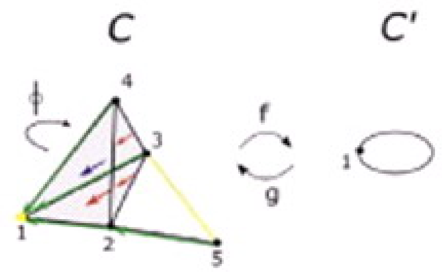
$$f \circ d = d' \circ f, \ \ g \circ d' = d \circ g, \ \ f \circ \phi = \phi' \circ f, \ \ g \circ \phi' = \phi \circ g.$$

Definition 4.2.3 Given two integral chain complexes (C, d, ϕ) and (C', d', ϕ'), we say that they are integral chain equivalent if there exists a map of integral chain complexes (f, g) such that

$$f \circ g = id_{C'} - d' \circ \phi' - \phi' \circ d' \text{ and } g \circ f = id_C - d \circ \phi - \phi \circ d.$$

Two integral chain equivalent complexes are shown in Figure 4.3.

(a)



(b)

σ	$d(\sigma)$	$\phi(\sigma)$
$\langle 2\rangle$	$\langle 1,2\rangle$	
$\langle 3\rangle$	$\langle 1,3\rangle$	
$\langle 4\rangle$	$\langle 1,4\rangle$	
$\langle 5\rangle$	$\langle 1,2\rangle + \langle 2,5\rangle$	
$\langle 2,3\rangle$	$\langle 1,2,3\rangle$	
$\langle 2,4\rangle$	$\langle 1,2,4\rangle$	
$\langle 3,4\rangle$	$\langle 1,3,4\rangle$	
$\langle 2,3,4\rangle$	$\langle 2,3\rangle + \langle 2,4\rangle + \langle 3,4\rangle$	$\langle 1,2,3,4\rangle$

Figure 4.3. (a) Two integral chain equivalent complexes (C, d, ϕ) and (C', d', ϕ').
(b) The d and ϕ values of the complex (C, d, ϕ)

The homology $H_*(C, d, \phi)$ of an integral chain complex (C, d, ϕ) is the graded abelian group $H_*(C)$, such that $(H_*(C), 0, 0)$ is integral chain equivalent to (C, d, ϕ). The differential (resp. integral) holomology of an integral chain complex (C, d, ϕ) is the homology of $(C, d, 0)$, (resp. the homology of $(C, 0, \phi)$).

If (C, d, ϕ) is a homology integral chain complex, then

$$H_*(C, d, \phi) \simeq H_*(C, d, 0) \simeq H_*(C, 0, \phi).$$

The notion of pure integral chain complex is underlying in the work of Sergeraert ([85]), Forman ([31], [30]) and that of theory of discrete differential forms ([24]). The integral chain equivalence relation can be seen as the natural extension of the classical chain homotopy equivalence between chain complexes to the integral case ([29]).

The computation of the homology of a chain complex (C, d) can be directly obtained from an integral operator $\phi : C_* \to C_{*+1}$, satisfying the strong deformation retract (SDR, for short) and homology conditions with regards to the differential operator d ([46], [47]).

Proposition 4.2.4 *Let (C, d, ϕ) be an integral chain complex. Let $\pi : C_* \to C_*$ be the linear map (called the flow of (C, d, ϕ)) defined by*

$$\pi = id_C - d \circ \phi - \phi \circ d$$

and let $\Delta : C_* \to C_*$ be the linear map (called Laplacian of (C, d, ϕ)) defined by

$$\Delta = d \circ \phi + \phi \circ d.$$

Then the following properties hold:
(a) $d \circ \pi = d - d \circ \phi \circ d = \pi \circ d$ and $\phi \circ \pi = \phi - \phi \circ d \circ \phi = \pi \circ \phi$.
In the case of a homology integral chain complex,

$$d \circ \pi = 0 = \pi \circ d \text{ and } \phi \circ \pi = 0 = \pi \circ \phi.$$

(b) $d \circ \Delta = d \circ \phi \circ d = \Delta \circ d$ *and* $\phi \circ \Delta = \phi \circ d \circ \phi = \Delta \circ \phi$.
In the case of a homology integral chain complex,

$$d \circ \Delta = d = \Delta \circ d \text{ and } \phi \circ \Delta = \phi = \Delta \circ \phi.$$

(c) Given a p-chain a, we have the following equality

$$a = \pi(a) + \Delta(a).$$

(d) $\pi^2 = \pi - \phi \circ (d - d \circ \phi \circ d) - (d - d \circ \phi \circ d) \circ \phi = \pi - d \circ (\phi - \phi \circ d \circ \phi) - (\phi - \phi \circ d \circ \phi) \circ d$.
(e) $\Delta^2 = (d + \phi)\Delta(d + \phi)$.
(f) $\pi \circ \Delta = (d - d \circ \phi \circ d) \circ \phi + \phi \circ (d - d \circ \phi \circ d) = d \circ (\phi - \phi \circ d \circ \phi) + (\phi - \phi \circ d \circ \phi) \circ d = \Delta \circ \pi$.

Definition 4.2.5 The integral chain complex $\pi(C, d, \phi) = (\pi(C), d|_{\pi(C)}, \phi|_{\pi(C)})$ is the harmonic complex associated to (C, d, ϕ). If (C, d, ϕ) is a d-pure or a ϕ-pure integral chain complex, then

$$\pi^2 = \pi \circ \pi = \pi \text{ and } \pi(C) = \{x \in C \mid x = \pi(x)\}.$$

In other words, the harmonic complex $(\pi(C), d|_{\pi(C)}, \circ)$ associated to a pure integral chain complex (C, d, ϕ) is formed by the π-equivariant chains of C. If (C, d, ϕ) is a homology integral chain complex, its harmonic complex is of the kind $(\pi(C), 0, 0)$ and given any p-chain the chain map π describes a representative cycle of the homology class associated to this $p$chain.

In Figure 4.3 two integral chain equivalent complexes (C, d, ϕ) and (C, d', ϕ') are shown. The complex (C, d, ϕ) on the left is a ϕ-pure integral chain complex. The complex (C, d', ϕ') on the right is the harmonic complex of the complex (C, d, ϕ). (C, d', ϕ') is a homology integral chain complex ($d'(\sigma) = 0$, $\phi'(\sigma) = 0$, $\forall\ \sigma \in C'$).

Definition 4.2.6 The integral chain complex $\Delta(C, d, \phi) = (\Delta(C), d|_{\Delta(C)}, \phi|_{\Delta(C)})$ is the Laplacian complex associated to (C, d, ϕ). If (C, d, ϕ) is a d-pure or ϕ-pure integral chain complex, then

$$\Delta^2 = \Delta \circ \Delta = \Delta \text{ and } \Delta(C) = \{x \in C \mid x = \Delta(x)\}.$$

In other words, the Laplacian complex $\Delta(C, d, \phi)$ associated to a pure integral chain complex (C, d, ϕ) is formed by all the Δ-equivariant chains.

Proposition 4.2.7 *If (C, d, ϕ) is a (differential or integral) pure integral-chain complex, we can derive the following properties:*
(1) $\pi \circ \Delta = 0 = \Delta \circ \pi$.
(2) $(C, d, \phi) = \pi(C, d, \phi) \oplus \Delta(C, d, \phi)$ *as integral-chain complexes.*
In particular, $\operatorname{Ker} \Delta = \pi(C)$ *and* $\Delta(C) = \operatorname{Ker} \pi$.
(3) $\Delta(C) = \phi(C) \oplus (d \circ \phi)(C)$ *as graded modules.*

In order to emphasize the dependency of π and Δ with regards to d and ϕ, we will denote these maps by $\pi_{(d,\phi)}$ and $\Delta_{(d,\phi)}$, respectively.

The following proposition will be fundamental in developing an integral-chain framework for Discrete Morse Theory. In fact, it shows that to use pure integral operators as chain homotopies decomposing finitely generated chain complexes is a key point.

Proposition 4.2.8 *If (C, d, ϕ) is a (differential or integral) pure integral-chain complex, we have that*

$$\operatorname{Ker} \phi \cong \pi(C) \oplus \phi(C) \cong \operatorname{Ker} \Delta(C) \oplus \phi(C)$$

as graded modules.

In particular, a map of integral-chain complexes (f, g) satisfies that

$$f \circ \pi_{(d,\phi)} = \pi_{(d',\phi')} \circ f, \;\; g \circ \pi_{(d',\phi')} = \pi_{(d,\phi)} \circ g,$$

$$f \circ \Delta_{(d,\phi)} = \Delta_{(d',\phi')} \circ f, \;\; g \circ \Delta_{(d',\phi')} = \Delta_{(d,\phi)} \circ g.$$

That is, f and g are compatible with regards to the respective flows and Laplacians.

In spite of its simplicity, the following result is essential for developing our homological theory of integral-chain complexes.

Lemma 4.2.9 (Integral-chain Lemma) *An integral chain complex (C, d, ϕ) is integral-chain equivalent to its harmonic complex $\pi(C, d, \phi)$. This last harmonic complex $\pi(C, d, \phi)$ is of the form $(\pi(C), d_\pi, \phi_\pi)$, where*

$$d_\pi(\pi(x)) = d - (d \circ \phi \circ d)(x) \text{ and } \phi_\pi(\pi(x)) = \phi - (\phi \circ d \circ \phi)(x).$$

Corollary 4.2.10 *The harmonic complex $\pi(C, d, \phi)$ associated to a d-pure (resp. ϕ-pure) integral-chain complex (C, d, ϕ) is of the form $(\pi(C), 0, \phi_\pi)$ (resp. $(\pi(C), d_\pi, 0)$) and we have $\phi_\pi(\pi(x)) = \phi - (\phi \circ d \circ \phi)(x)$ (resp. $d_\pi(\pi(x)) = d - (d \circ \phi \circ d)(x)$).*

An example of Lemma 4.2.9 can be seen in Figure 4.3, where (C, d, ϕ) is a ϕ-pure integral-chain complex, and (C', d', ϕ') is its harmonic complex.

Now, we give some definitions related to integral-chain perturbation of complexes.

Definition 4.2.11 An integral chain complex (C, d, ϕ) is called d-pointwise nilpotent (resp. ϕ-pointwise nilpotent) if for any $u(a) \in \mathbb{C}$ there is some $n(a) \in \mathbb{N}$ with

$$d \circ (id_C - d \circ \phi - \phi \circ d)^{n(a)} = 0 \text{ (resp. } \phi \circ (id_C - d \circ \phi - \phi \circ d)^{n(a)} = 0).$$

The smallest value for $n(a)$ is called the degree of differential (resp. integral) nilpotency of a.

Proposition 4.2.12 *Given an ϕ-pointwise (resp. d-pointwise) nilpotent chain complex (C, d, ϕ), it is integral chain equivalent to a ϕ-pure (resp. d-pure) integral chain complex $(C, d, \widetilde{\phi})$ (resp. $(C, \widetilde{d}, \phi)$).*

From now on, all the integral chain complexes considered in the paper will be ϕ-pointwise nilpotents. Analogous results can be determined for d-pointwise nilpotent chain complexes.

4.3 Discrete Morse Theory and optimality

In this section, we will show that all the results above can be used as a tool for reaching interesting combinatorial results in DMT.

Definition 4.3.1 Let (K, ∂) a finite cell complex. An operator $h : C_*(K) \to C_{*\pm r}(K)$ is said to be combinatorial if for all p-cell $\sigma^{(p)}$, $h(\sigma^{(p)}) = \lambda \beta^{(p\pm r)}$, where $\lambda \in \mathbb{Z}$ and β is a $(p \pm r)$-cell.

Now, we give some basic notions of DMT with some slight modifications and without using, in principle, discrete Morse functions.

Definition 4.3.2 A combinatorial vector field $\mathcal{V}$ defined on a connected cell complex K is a collection of disjoint pairs of cells $\{\alpha^{(p)} < \beta^{(p+1)}\}$.

Definition 4.3.3 A $\mathcal{V}$-path or gradient path γ is an alternating sequence of cells $a_0^{(p)}, b_0^{(p\pm1)}, a_1^{(p)}, b_1^{(p\pm1)}, a_2^{(p)}, \ldots$ such that for each pair of consecutive cells, one is a facet of the other, and the following condition is satisfied: either $\{a_i^{(p)} < b_i^{(p\pm1)}\}$ or $\{b_i^{(p\pm1)} < a_{i+1}^{(p)}\}$ belongs to $\mathcal{V}$, $\forall\, i \geq 0$.

If the final cell in the gradient path γ above is $\alpha_r^{(p)}$, then we say that γ has length r ([38]). If it ends at $\beta_r^{(p\pm1)}$ then we say that γ has length $r\frac{1}{2}$. If the cells b_i of the gradient path γ are of dimension $p+1$ and it has length $r\frac{1}{2}$, the gradient path γ is called upper $\mathcal{V}$path or upper gradient path. For any cells a and b, let $\Gamma(a,b)$ denote the set of gradient paths from a to b (of any length), i.e., such that the first cell in the sequence is a and the last cell in the sequence is b. A $\mathcal{V}$path is non trivial and closed if $r \geq 1$ and the first and last cells in the sequence are the same.

Definition 4.3.4 A discrete gradient vector field is a combinatorial vector field with non trivial closed $\mathcal{V}$paths. In this way, it can be seen as an acyclic cells pairing. A cell α is a critical cell of $\mathcal{V}$ if it is not paired with any other cell in $\mathcal{V}$.

Definition 4.3.5 A combinatorial integral operator defined on a cell complex K is a collection of disjoint pairs of (not necessary incident) cells $\{\alpha^{(p)}, \beta^{(p+1)}\}$ of the same connected component.

Therefore, a discrete gradient vector field is a special kind of combinatorial integral operator.

Forman ([30], [31]) proved that the topology of a discrete manifold is related to the critical cells of a discrete function defined on it, mimicking the results of Morse in the smooth case. The number of critical cell depends on the discrete gradient vector field considered (see Figure 4.4). In ([63]), the problem of the optimality (that is minimizing the number of critical cells for combinatorial vector fields) on a 2-manifold is analyzed using Hasse diagram and hypergraph tools. However, this problem has not been solved for the general case.

It is not always possible to obtain a number of critical cells that coincides with the Betti numbers of the complex. This is the case of the Bing's house and the Dunce hat complexes, that are contractible but not collapsible ([11]).

The next results will show that by using integral operators for chain complexes, we can solve this problem, and always reduce the initial complex to the minimum number of critical cells, that corresponds with the Betti numbers. This means that we are able to guarantee homological optimality (what is called perfection in the Discrete Morse Theory context, ([11]).

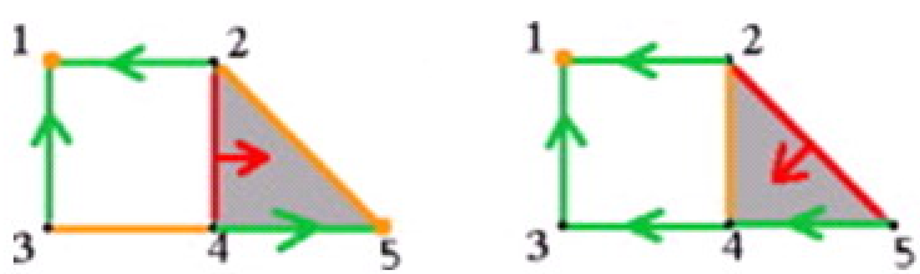


Figure 4.4. A cell pairing on the left ($\langle 1\rangle$, $\langle 5\rangle$, $\langle 3,4\rangle$ and $\langle 2,5\rangle$ are critical) and an optimal one on the right ($\langle 1\rangle$ and $\langle 2,4\rangle$ are critical). The pairing is represented with an arrow from the cell of lower dimension to its paired cell of higher dimension

Definition 4.3.6 Given an upper gradient path $\gamma \in \Gamma(a,b)$ formed by the alternating sequence $\gamma : a_0^{(p)}, b_0^{(p+1)}, a_1^{(p)}, b_1^{(p+1)}, \ldots, a_t^{(p)}, b_t^{(p+1)}$, its corresponding chain homotopy path is defined by the sequence $b_0^{(p+1)}, b_1^{(p+1)}, \ldots, b_t^{(p+1)}$.

First, the combinatorial integral operators derived from combinatorial vector fields are ϕpointwise nilpotent.

Proposition 4.3.7 *A discrete gradient vector field $\mathcal{V}$ gives rise to a ϕpointwise nilpotent integral-chain complex $(C(K), d, V)$.*

Combining Propositions 4.2.4, 4.2.7, 4.2.8, 4.2.12, 4.3.7, we assert the following result which is the key for reinterpreting Discrete Morse Theory in terms of an integral-chain complex:

Proposition 4.3.8 *If (C, d, V) is a ϕ-pointwise nilpotent integral-chain complex being V a gradient vector field, then there is an integral-chain equivalent ϕ-pure complex $(C, d, \widetilde{V})$ such that its harmonic complex*

$$\pi(C, d, \widetilde{V}) = (\mathrm{Ker}\, \widetilde{V} \setminus \widetilde{V}(C), d_\pi, 0).$$

This last integral-chain complex is constituted by finite linear combinations of the different critical cells of V and d_π can be seen as the boundary operator of the corresponding cell complex determined by the critical cells, also called harmonic Morse cell complex $M(C, d, V)$ associated to (C, d, V). Analogously, the Laplacian complex $\Delta(C, d, \widetilde{V})$ can be seen as the acyclic chain complex of the cell complex $M(C, d, V)$, also called Laplacian Morse complex associated to (C, d, V). Moreover, its boundary operator ∂_M is determined by

$$\partial_M(\Delta(\sigma^{(p)})) = d \circ \widetilde{V} \circ d(\sigma^{(p)}),\ \forall\ \sigma^{(p)} \in C.$$

Let us note that $H_*(M(C, d, V)) \cong H_*(K, \Lambda)$, where Λ is the considered commutative field. Moreover, the boundary operator d_π os the Morse cell complex $M(C, d, V)$ has a clear interpretation in terms of gradient paths of $\widetilde{V}$.

Proposition 4.3.9 *In the conditions of Proposition 4.3.8, and given a pcell α, $\widetilde{V}(\alpha)$ is a chain homotopy path.*

In ([74]), an integral operator ϕ giving rise to a homology integral chain complex is determined from a filtered cell complex by using an incremental technique. Given a p-cell r, $\phi(\sigma)$ is a sum of $(p+1)$-cells in which at least one cell τ satisfies that $\sigma \in \partial(\tau)$. This operator ϕ gives rise in a natural way to a combinatorial integral operator on K.

Due to the fact that $\widetilde{V}(C)$ admits a combinatorial basis, and the chain complex $(\widetilde{V}(C) \oplus (d \circ \widetilde{V}(C)), d)$ is acyclic, we can assume that the sum ω of the elements in the combinatorial basis of $\widetilde{V}(C)$ satisfies that $d(\omega) = 0$. That means that ω can be represented in terms of graphs using trees. In these trees, the nodes are p-cells and $(p+1)$-cells for all $p \geq 0$ of the complex. The neighbors of a p-cell are $(p+1)$-cells and vice versa (see Figure 4.5).

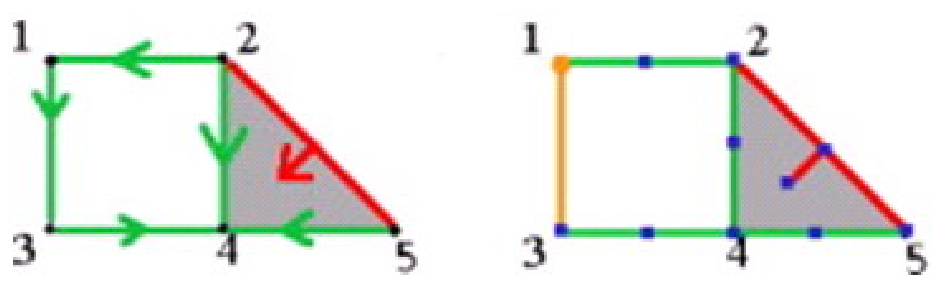


Figure 4.5. A combinatorial vector field (on the left).
On the right a gradient set of trees where cells $\langle 1\rangle$ and $\langle 1, 3\rangle$ do not belong to the forest, $\langle 2, 5\rangle$ and $\langle 2, 4, 5\rangle$ belong to the tree of dimension 1 and 2 and the rest of cells belong to the three of dimension 0 and 1

This forest, is a representation in homological terms of the cell complex K, and it is called homological spanning forest ([73], [74]).

Given a homological spanning forest, it is possible to distinguish two kind of trees: homologically essential and inessential trees. In a homologically inessential tree the number of $p$cells is the same as the number of $p+1$-cells. In a homologically $c$essential tree, the difference between the number of $p$cells and $p+1$-cells is a positive integer c. In this last case there exist c p-cells within this tree that represent a critical cell, that is, a homology generator. Therefore, given a homological spanning forest, a combinatorial integral operator can be directly deduced by maximally pairing each $p$cell with a $(p+1)$-cell using some specific strategy (eventually, allowing the pairing of non-incident cells) for each homologically essential or inessential tree. In this process, only c p-cells (critical cells) of a homologically c-essential tree will remain unpaired.

Let us emphasize that the notion of optimality here is guaranteed in terms of finding a combinatorial integral operator. Therefore, the minimum number of critical cells will always coincide with the Betti numbers. In the pairing process, we might find some pairs of non-incident cells $\{\alpha, \beta\}$. In order to obtain optimality in the sense of Forman (pairing of incident cells), classical cancellation results ([31]) involving the single path joining a and b can be applied.

Example 4.3.10 In Figure 4.6a) we can see the cell complex K of a torus T^2.

Figure 4.6b) and c) represent the tree structure of a chain homotopy operator ϕ described over the $\{0-\text{cells}, 1\text{-cells}\}$ and $\{1-\text{cells}, 2-\text{cells}\}$ of the initial complex respectively.

The corresponding optimal combinatorial pairing, is shown in Figure 4.6d) and e). The different colors in Figure 4.6e) represent the different trees of the homological spanning forest of 1-cells and 2-cells. The triple $(C(K), \partial, \phi)$ is a homology integral-chain complex.

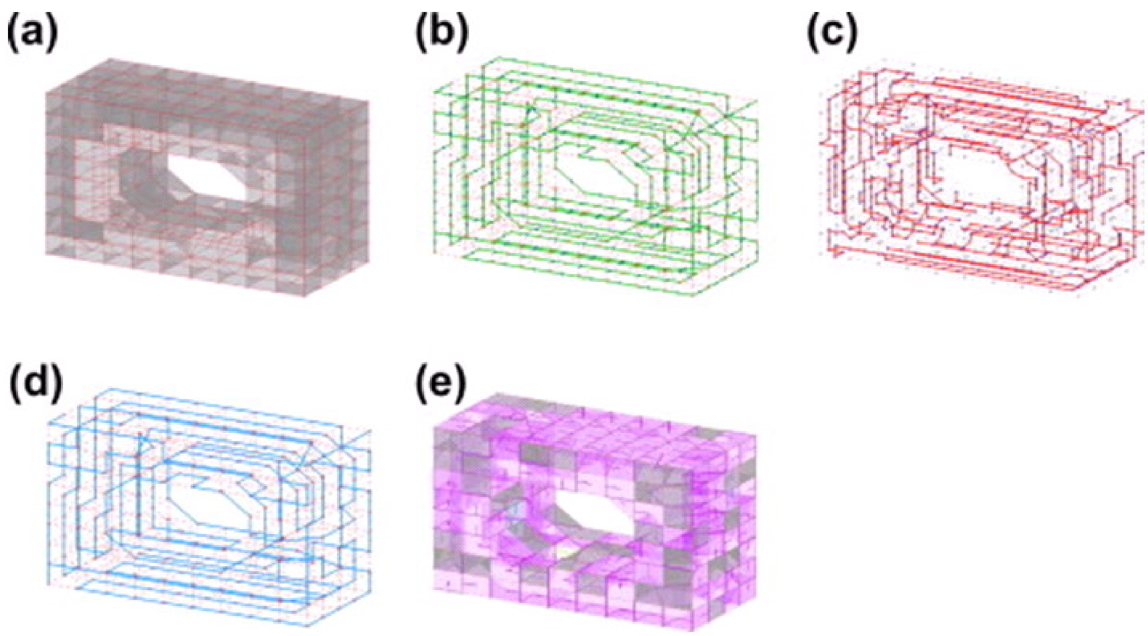


Figure 4.6. A torus cell complex T^2, part of its homological spanning forest representation, and the obtained optimal combinatorial pairing

Example 4.3.11 In Figure 4.7 the results obtained for an example of the Bing's house with two rooms is shown. Due to the fact that this example is a contractible complex which is not collapsible, it is not possible to get an optimal discrete gradient vector field. Nevertheless, we obtain a combinatorial integral operator involving all the cells from its associated homological spanning forest structure, guaranteeing homological optimality.

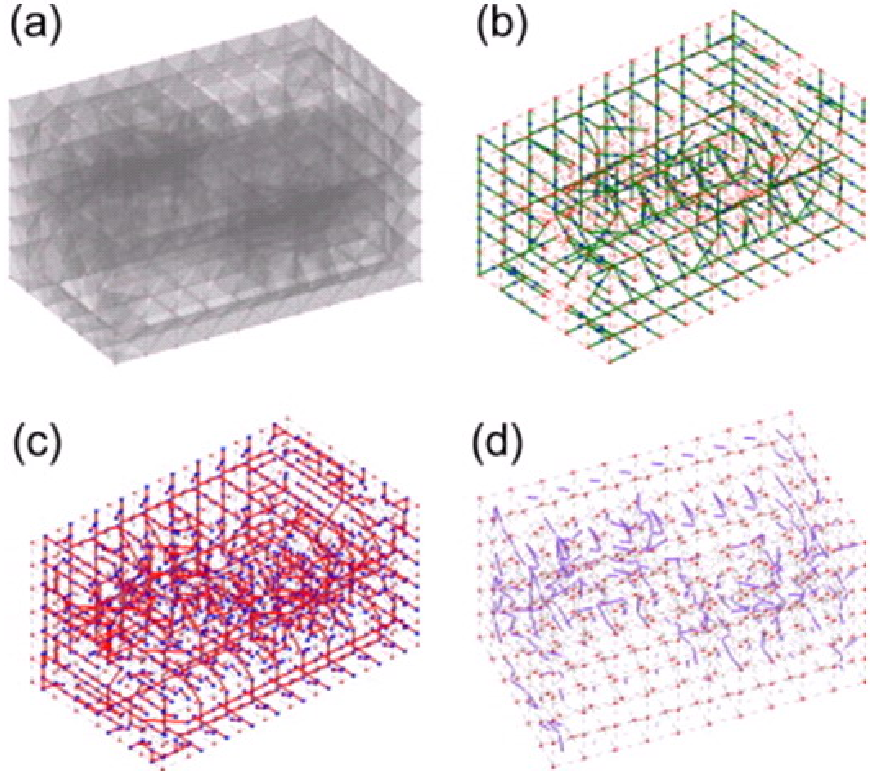


Figure 4.7. A Bing's house cell complex (a) and its homological spanning forest representation (b) 01, (c) 12 and (d) 23 cells).

4.4 The topological invariance of Morse-Smale numbers for 3-complexes

Discrete Morse numbers are linked to simple homotopy. To prove their invariance, we could prove that topologically equivalent cell complexes are simple homotopic, and that simple homotopic spaces have the same discrete Morse numbers. Unfortunately, the first affirmation is not true in the general case. We will use the following theorems, which demonstration can be found respectively in [77] and [21, 25.1].

Theorem 4.4.1 *Any two triangulations of a topological 3-manifold have a common subdivision.*

Theorem 4.4.2 *If K_* is a subdivision of K, then K and K^* are simple homotopy equivalent.*

Theorem 4.4.3 *Let K and L be homeomorphic 3-manifolds. Then for all p, we have $\gamma_p(K) = \gamma_p(L)$.*

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